

LECTURE 8

Theorem 1. For a compact angled 2-complex X , $2\pi\chi(X) = \sum \kappa(v) + \sum \kappa(f)$

PROOF. Tautology.

$$\text{Let } C = \sum_{c \in \text{corners}(X)} \angle(c).$$

Let $P = \sum_{e \in 1\text{-cells}(X)} \#e$, where $\#e$ = total number of 2-cells appearing locally along e = times e appears in attaching maps of 2-cells = the number of pages in the book spined by e .

(figure: a book)

EXAMPLE. $\langle a, b | ab^{-1}aab \rangle$ — $\#a = 3, \#b = 2$.

We will show:

$$\begin{aligned} (1) \quad \sum \kappa(f) &= C - \pi P + 2\pi F, \\ (2) \quad \sum \kappa(v) &= 2\pi V - 2\pi E + \pi P - C, \\ \text{so } \sum \kappa(f) + \sum \kappa(v) &= 2\pi V - 2\pi E + 2\pi F = 2\pi\chi(X), \end{aligned}$$

where V is number of 0-cells, E is number of 1-cells, F is number of 2-cells in X .

Proof of (1).

$$\begin{aligned} \kappa(f) &= \sum_{c \in \text{corners}(f)} \angle(c) - |f|\pi + 2\pi \\ \sum_{f \in X} \kappa(f) &= \sum_f \sum_{c \in \text{corners}(f)} \angle(c) - \sum_f |f|2\pi + \sum_f 2\pi \\ \sum_{f \in X} \kappa(f) &= C - \pi P + 2\pi F \end{aligned}$$

Proof of (2).

$$\begin{aligned} \kappa(v) &= 2\pi - \pi\chi(\text{link}(v)) - \sum_{c \in \text{corners}(v)} \angle(c) \\ \sum_{v \in X} \kappa(v) &= \sum_{v \in X} 2\pi - \sum_{v \in X} \pi\chi(\text{link}(v)) - \sum_{v \in X} \sum_{c \in \text{corners}(v)} \angle(c) = \\ &= 2\pi V - \sum_{v \in X} \pi \cdot \left(\frac{\text{vertices}}{\text{in link}(v)} \right) + \sum_{v \in X} \pi \cdot \left(\frac{\text{edges}}{\text{in link}(v)} \right) - C = \\ &= 2\pi V - 2\pi E + \pi P - C \end{aligned}$$

□

EXAMPLE. (figure: a graph)

If 2-complex is, actually, a graph (i.e. has no 2-cells), we have (for graph in the figure)

$$\begin{aligned} \kappa(v) &= 2\pi - \pi \deg(v) \\ 2\pi\chi(X) &= -4\pi \\ \chi(X) &= -2 = 7 - 9 \end{aligned}$$

EXAMPLE. (figure: an earring squared (with right angles))

We have one 0-cell, four 1-cells, four 2-cells. $\chi(X) = \chi(\text{earring}) \cdot \chi(\text{earring}) = (-1) \cdot (-1) = 1$.

Compute curvatures: $\kappa(f) = 0$,

$$\kappa(v) = 2\pi - \pi\chi(\text{link}(v)) - \sum_1 \angle = 2\pi + 8\pi - \frac{16}{2}\pi = 2\pi.$$

(We have $\text{link}(v) = K(4, 4)$, so $\chi(\text{link}(v)) = 8 - 16 = -8$.) So $2\pi\chi(X) = 2\pi$, i.e. $\chi(X) = 1$.

Nonpositive and negative (conformal) curvature.

Definition 1. (Gersten-Pride) *An angled 2-complex satisfies the nonpositive weight test if*

- 1) $\kappa(f) \leq 0$ for each 2-cell f ,
- 2) for each 0-cell $v \in X^0$ and each closed nontrivial immersed path σ in $\text{link}(v)$ with $\sigma = e_1 e_2 \dots e_n$,

$$|\sigma|_{\angle} = \sum_{i=1}^n \angle(e_i) \geq 2\pi,$$

and the negative weight test if both inequalities are strict.

(figure: example)

Remark. If all angles in X are nonnegative, it suffices to consider simple cycles in the (2) of definition. Indeed, any closed immersed path can be expressed as $\sigma = \sigma_1 \sigma_2 \dots \sigma_k$, where each σ_i is simple and $|\sigma|_{\angle} = \sum_{i=1}^k |\sigma_i|_{\angle} \geq k2\pi \geq 2\pi$.

(figure: a dumbbell) This example shows that requirement of nonnegativeness is essential.

Definition 2. *The girth of a graph Γ is the length $|\sigma|$ of the shortest simple cycle $|\sigma|$. If Γ is a forest, $\text{girth}(\Gamma) = \infty$.*

EXAMPLE. $\text{girth}(\text{cube}) = 4$, $\text{girth}(\text{figure: triangle with a double side}) = 2$, $\text{girth}(\text{figure: dumbbell}) = 1$.

Definition 3. *A (p, q) -complex X is a 2-complex satisfying*

- 1) each 2-cell has $|f| \geq p$,
- 2) for each 0-cell $v \in X^0$, $\text{link}(v)$ has girth $\geq q$.

(figure: square tiling) — (4,4), also (3,3), (1,4),...

(figure: hexagon tiling) — (6,3)

(figure: trigon tiling) — (6,3)

(figure: cube) — (4,3)

In a sense, the (p, q) condition is a combinatorial comparison condition with a regular tiling of type (p, q) .

Lemma 1. *If $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$, then the (p, q) -complex has an angle assignment satisfying nonpositive weight test. Moreover, if $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, then it satisfies negative weight test.*

PROOF. Assign angle of $\frac{p-2}{2}\pi$ to each corner. Then

$$\kappa(f) = |f| \cdot \frac{p-2}{p}\pi - (|f| - 2)\pi = 2\pi - \frac{|f|}{p}2\pi \leq 0,$$

and < 0 if $|f| > p$. Since $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$ implies $\frac{2}{q} \leq 1 - \frac{2}{p}$, for each simple cycle in the $\text{link}(v)$ we have

$$|\sigma|_{\angle} \geq q \cdot \frac{p-2}{p} \cdot \pi = q \left(1 - \frac{2}{p}\right) \pi \geq 2\pi,$$

and $> 2\pi$ if $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$.

Angle assignment $\frac{2\pi}{q}$ works equally well. If we take average of these angle assignments, then $\kappa(f) < 0$, $\kappa(v) < 0$, so we obtain negative weight test.

□

Now we have an enormous supply of (p, q) -complexes.

REMARK. If X is compact, then there are finitely many real numbers arising as angles, then

if always $\kappa(f) < 0$ and $|\sigma|_{\angle} \geq 2\pi$, increasing all angles slightly,
 if always $\kappa(f) \leq 0$ and $|\sigma|_{\angle} > 2\pi$, decreasing all angles slightly,

we obtain negative weight test.

EXAMPLE. $\langle a, t|a^t = a^2 \rangle$

(figure: standard 2-complex and link)

EXERCISE. $G = \langle a_1, a_2, \dots, a_n, t|(A_1)^t = A_2 \rangle$ where A_1, A_2 are freely reduced nonempty words in $a_i^{\pm 1}$. Verify that standard 2-complex satisfies nonpositive weight test.

EXAMPLE. $\langle a, b|aaabbbabab^{-1}ab^{-1} \rangle$.

(figure: link)

Standard 2-complex doesn't satisfy the weight test, since relator contains each pair of generators twice and therefore the link is a full 4-graph with double edges and $\kappa(f) \geq 6 \cdot 2\pi - 10\pi \geq 2\pi > 0$.

EXERCISE. Find a 2-complex with exactly one 2-cell such that it has a unique angle assignment satisfying the weight test.