Lecture 6

Isoperimetric Function. Two finite presentations P_1, P_n yield a isomorphic groups $\Leftrightarrow$ there is a finite sequence of Tietze transformations from one to another

$$P_1 \to P_2 \to \ldots \to P_n.$$

Tietze transformations:

I. Addition of a new generator:

$$\langle a_1, \ldots, a_k | R_1, \ldots, R_s \rangle \to \langle a_1, \ldots, a_k, b | R_1, \ldots, R_s, b = A \rangle,$$

where A is word in a_i.

Exercise. Prove that obvious map is isomorphism.

II. Addition of a new generator:

$$\langle a_1, \ldots, a_k | R_1, \ldots, R_s \rangle \to \langle a_1, \ldots, a_k | R_1, \ldots, R_s, V \rangle,$$

where $V \in \langle\langle R_i \rangle\rangle$. Topologically (see figure):

¡figure¿

Exercise. Prove that.

Definition 1. *Functions $f, g : \mathbb{N} \to \mathbb{N}$ are equivalent $f \sim g$, if $\exists K$ such that*

$$f(n) \leq Kg(Kn) + Kn,$$
$$g(n) \leq Kf(Kn) + Kn.$$

Theorem 1. *Let X, Y be finite 2-complexes with $\pi_1 X \cong \pi_1 Y$. Then $f_X \sim f_Y$.*

Proof. It suffices to prove the theorem in case of Tietze transformation.

1) Suppose $X \xrightarrow{I} Y$.

a) $f_X(n) \overset{?}{\leq} f_Y(n)$. Let $P \to X$ with $|P| \leq n$, let $D \to Y$ be a disc diagram for P in Y.

¡figure¿

How to remove 2-cell $b = A$?

¡figure¿

Since P doesn't pass through b, any minimal area diagram $D \to Y$ for P cannot contain b, or $b = A$, thus D maps to X, so $f_X(n) \leq f_Y(n)$.

b) $f_X(n) \overset{?}{\geq} f_Y(n)$. Let $P \to Y$ be a null-homotopic path. Replace each b and P by A to obtain $P' \to X$. Note that $|P'| \leq |A||P|$. Let D' be a minimal disc diagram for P' in X. Extend this to diagram $D \to Y$ by attaching at most $|P|$ copies of 2-cell $b = A$.

$$\begin{aligned} \text{Area}\,(D) &\leq \text{Area}\,(D') + |P| \leq \\ &\leq f_X(|P'|) + |P| \leq \\ &\leq f_X(|A| \cdot |P|) + |P|. \end{aligned}$$

¡figure¿

2) Suppose $X \xrightarrow{II} Y$.

¡figure¿

a) $f_X(n) \overset{?}{\leq} f_Y(n)$. Let E be a disc diagram in X whose boundary path is ∂V.

¡figure¿

Claim: $f_X(n) \leq \text{Area}\,(E)\, f_Y(n)$.

Proof. Let $P \to X$ be null-homotopic. Let $D \to Y$ be a minimal disc diagram for P. Substitute E for each occurrence of V in D to obtain new diagram $D' \to Y$.

¡figure¿

Area $(D') \leq$ Area (E) Area (D) (what if it's singular?).

b) $f_X(n) \geq f_Y(n)$ obviously.

□

Now we have a kind of a *group invariant*.

Definition 2. *Group word-hyperbolic if it has a finite presentation whose standard 2-complex X has linear isoperimetric function: $\exists K$ such that $f_X(n) \leq Kn$.*

NOTE. $f(n) \leq \frac{1}{100}n^2 \Rightarrow \exists K$ such that $f(n) \leq Kn$. That means there is a gap between quadratic and linear functions in the collection of all isoperimetric function.

Construction. Let X be a complex that is a graph of spaces (see figure) with each vertex space a subcomplex and each edge space a (subcomplex) $\times I$.

¡figure¿

Let $\Phi : Y \to X$ be combinatorial map of complexes. Then Y has an induced structure as a graph of spaces. Indeed, the vertex spaces of Y are are components of $\Phi^{-1}(X_V)$, where X_V is vertex space of X. Open edge spaces are components of $\Phi^{-1}(X_e \times (-1, 1))$.

EXAMPLE. Subcomplex of X is a graph of spaces. Cover of X decomposes as a graph of spaces. Universal cover of X is a tree of spaces.

¡figure¿

Covering space $K(2,3) \times \bigcirc$ (see figure)

¡figure¿

$\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} \cong \langle c, b | c^3 = b^2 \rangle \xleftarrow{\text{index } 6} \mathbb{F}_2 \times \mathbb{Z} \cong \pi_1(K(2,3) \times \bigcirc)$.

Universal cover (see figure):

¡figure¿ ¡figure¿

$\pi_1 X$ acts on $\widetilde{X}$ and thus acts on $\Gamma_{\widetilde{X}}$