Lecture 5

Disc Diagrams.

Definition 1. *A disc diagram is a finite contractible planar* 2*-complex* D*.*

¡figure¿

We usually think of D as having a particular planar embedding $D \subset \mathbb{E}^2$. (Indeed, attaching a 2-cell along the border of a 2-complex we obtain such an embedding.)

A singular point of D is a point where is not locally a surface. Given an embedding $D \subset \mathbb{E}^2$, set of all singular points is a topological boundary of D.

Singular 1-cells are divided into two types:

isolated 1-cells, which have no 2-cells attached along, and
boundary 1-cells = singular\ isolated.

For example, the edge in the figure is not boundary, having one 2-cell attached twice.

¡figure¿

Choose a base point $b \in \partial D$. The boundary cycle path P of D based at b is a combinatorial path traveling "around D" starting and ending at b, that passes through each boundary edge and through each isolated edge twice.

P corresponds to attaching map of the 2-cell, whose interior is $\mathbb{E}^2 \backslash D$

Some notations:

Area (D) = number of 2-cells in D.
∂D is considered path in D.
D is none-singular, if D is homeomorphic to B^2, i.e. has no singular points.
$|\partial_P D|$ = length of boundary path.
$|P|$ = length of path P.

Definition 2. *A disc diagram in a* 2*-complex* X *is a combinatorial map* $D \to X$*.*

Combinatorial map $Y \to X$ between complexes sends open cells homeomorphically to open cells (of some dimension).

Convention: our 2-complexes are combinatorial, and attaching maps of 2-cells are combinatorial (after appropriate subdivision). That is, X is X' with polygons attached to it (in general case, attaching maps could be horrible, see figure). However, here we are missing "a disk attached to 0-cell".

¡figure¿

Theorem 1. *Let* $P \to X$ *be a closed combinatorial path. TFAE:*

1) P *is null-homotopic.*
2) $P \to X$ *factors as* $P \to D \to X$ *where* $D \to X$ *is a disc diagram and* $P \to D$ *is a boundary path.*

¡figure¿

In fact, this (see figure) can be made into proof.

Proof. 2⇒ 1 obviously.

$$\begin{array}{ccc} \pi_1 P & \longrightarrow & \pi_1 X \\ & \searrow \quad \nearrow & \\ & \pi_1 D & \end{array}$$

1⇒ 2

We assume $X^0 = \{\cdot\}$, so X is standard 2-complex of a presentation $\langle g_1, \dots | R_1, \dots \rangle$. Also, we assume $\pi_1 X \cong G$ of this presentation (isomorphism is induced by obvious map).

The path is null-homotopic ⇔
$P \sim 1$ in $\pi_1 X$ ⇔
the word P corresponding to path P represents 1 in G ⇔
P is freely equivalent to a product relators and inverses, $p \sim c_1^{-1} S_1^{\pm 1} c_1 c_2^{-1} S_2^{\pm 1} c_2 \cdots c_n^{-1} S_n^{\pm 1} c_n = P_0$ (c_i are words, S_i are some relators R_j).

A trick: ¡figure¿

$\partial_P D_0 = \prod c_i^{-1} S_i^{\pm 1} c_i \sim P$ D_0 obviously maps to X (according to labels on its edges).

Removal of backtracks. There is a sequence of words $P_0, P_1, \dots, P_t$, P_{i+1} is obtained from P_i by insertion or deletion of a backtrack. We follow this with a corresponding sequence of disc diagrams $D_0, D_1, \dots, D_t = D$, D_{i+1} is obtained from D_i by folding or removing spurs.

(A spur is "isolated" 1-cell with a "pendant" end.)

¡figure¿

□

Example. $\langle a, b | bab^{-1}a^{-1} \rangle$ Applying this argument we can see that (see figure):

$$bbaabab^{-1}a^{-1}b^{-1}b^{-1}a^{-1}a^{-1} \sim 1 \sim$$
$$b(bab^{-1}a^{-1})b^{-1} \cdot baba(bab^{-1}a^{-1})(baba)^{-1} \cdot ba(bab^{-1}a^{-1})(ba)^{-1} \cdot$$
$$bab^{-1}a^{-1} \cdot a(bab^{-1}a^{-1})a^{-1}$$

¡figure¿

This process can be inversed: given a disc diagram D for p (meaning with boundary graph p) we can easily express p as a freely equivalent to a product of conjugate relators.

¡figure¿

If we are careful, the length of conjugators c_i are bounded by $2 \sum |\partial R_i| + |\partial_p D|$, where $R_1, \dots, R_n$ are in the 2-skeleth of D.

¡figure¿

Isoperimetric Function.

Definition 3. *Given a null-homotopic path $p \to X$, define*

$$\text{Area}\,(p) = \inf\{\text{Area}\,(D)\,|p = \partial_p D\}$$

Definition 4. *Given a 2-complex X, its isoperimetric function $f : \mathbb{N} \to \mathbb{N}$ (here, $0, \infty \in \mathbb{N}$) is:*

$$f(n) = \sup\{\text{Area}\,(p)\,|p \to X \text{ nullhomotopic},\ |p| \le n\},$$

or $f(n) =$ largest area needed to fill in null-homotopic path of length $\le n$.

Define isoperimetric function of presentation in terms of standard 2-complex.

Example.

$\langle a, b | - \rangle$ $\quad f(n) = 0$

$\langle a, b | ab = ba \rangle$ $\quad f(n) \le \frac{n^2}{4}$, $f(n) \sim n^2$

$\langle a, b | ab = bba \rangle$ $\quad f(n) \le 2^n$, $f(n) \sim 2^n$

The quantitative nature of isoperimetric function of a finitely presented group

1) doesn't really depend on choice of finite presentation,
2) tells us a great deal about the group.

Theorem 2. $\langle a_1, \dots, a_n | R_1, \dots, R_n \rangle$ *has a solvable word problem* $\Leftrightarrow$ *isoperimetric function is computable (recursive).*

PROOF. Solvable word problem $\Rightarrow f(n)$ is recursive.

Enumerate all P such that $|P| \leq n$ and P represents 1_G; find smallest disc diagram, which represent path p.

Solvable word problem $\Leftarrow f(n)$ is recursive.

Given word P, compute $f(|P|)$. Build all disc diagrams in X with $\leq f(|P|)$ 2-cells and boundary path p. If $P = 1_G$, you will find some D_j, $p = \partial_p D_j$, otherwise $P \neq 1$.

□