Graph of Spaces.

Definition 1. *A graph of spaces X is a topological space with a structure dictated by a graph Γ and certain data.*

1) There is a topological space X_v for each vertex v in Γ. X_v is called a vertex space of X.
2) There is a topological space X_e for each edge e of Γ. X_e or $X_e \times I$ is edge space of X. ($I = [-1, 1]$)
3)For each edge e attached to vertices $i(e), \tau(e)$ there are continuous ("attaching") maps $\Phi_{e-} : X_e \to X_{i(e)}$, $\Phi_{e+} : X_e \to X_{\tau(e)}$.

From this data we can build the space X with quotient topology.

$$X = \left(\bigsqcup_{\substack{\text{vertices} \\ v}} X \right) \sqcup \left(\bigsqcup_{\substack{\text{edges} \\ e}} X_e \times I \right) \Bigg/ \begin{array}{l} \Phi_{e-}(a) \sim (a, -1) \, \forall a \in X_e \\ \Phi_{e+}(a) \sim (a, +1) \, \forall a \in X_e. \end{array}$$

Example (see figure).
¡figure¿

Theorem 1. *Assume all attaching maps of edge spaces in our graph of spaces are π_1-injective ($f : A \to B$ s.t. $f_* : \pi_1 A \to \pi_1 B$ is a monomorphism).*

Then $\pi_1 X$ "splits" as a graph of groups in the obvious way: use the same graph Γ, $G_v \cong \pi_1 X_v$, $G_e \cong \pi_1 X_e$, $\phi_{e_+} = (\Phi_{e_+})_* : \pi_1 X_e \to \pi_1 X_{\tau(e)}$, $\phi_{e_-} = (\Phi_{e_-})_* : \pi_1 X_e \to \pi_1 X_{i(e)}$.

(We can assume all spaces have a base point, and all the maps are base point preserving. In fact, it doesn't matter.)

Then $\pi_1 X \cong \pi_1$associated graph of groups.

Example (see figure "pair of pants").
¡figure¿
Example (see figure "BS(n, m)").
¡figure¿

Reminder. How to compute $\pi_1\Gamma$. Example (see figure "house").
¡figure¿

Pick a base point p in Γ. Pick a maximal tree $T \subset \Gamma$. Then there is a one-to-one correspondence "generators of $\pi_1\Gamma \leftrightarrow$ edges $\Gamma \setminus T$".

For each oriented edge e in $\Gamma \setminus T$ put $\overrightarrow{p\, i(e)} e \overleftarrow{p\, \tau(e)} = \sigma_e$. (Let $p \in T$ be a base point. For each $v \in T$ define $\overrightarrow{p\, v}$ to be the shortest edge path from p to v in T.)

Then, in our example, $\pi_1(\Gamma) = \langle \sigma_a, \sigma_b, \sigma_c \rangle$.

Arbitrary Group a Fundamental Group. Every group is a $\pi_1 X$ for some topological space X.

Theorem 2. *Let G be a group, let $P = \langle g_1, \dots | R_1, \dots \rangle$ be its presentation, let X be the standard 2-complex of P. Then $G = \pi_1 X$.*

(BTW, proof of the theorem is Van Kampen's theorem.)

In particular, let M be the multiplication table presentation. Let X_G be the standard 2-complex of M. (This is called canonical graph of spaces.)

Now, a monomorphism $H \subset G$ corresponds to a π_1-injective embedding $X_H \subset X_G$.

Using this we see that every graph of groups is induced (arises) from some graph of spaces (in fact, many graphs of spaces).

Example (see figure "non-canonical graph of spaces").