Lecture 3

Graph of Groups.

Definition 1. $\pi_1(\Gamma, p) \cong$ *combinatorial fundamental group of* Γ *based at* p.

> *Elements of group are all combinatorial edge paths (corresponding to certain words in* $a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, e^{\pm 1}, f^{\pm 1}$*).*
> *Equivalence relation is generated by removal/insertion of* ee^{-1}*,* $e^{-1}e$*.*
> *Multiply by concatenating.*

This gives well-defined, independent on the choice of vertex p group. This group can be viewed as a graph of trivial groups.
¡figure!¿

Define graph of groups in general case.
¡figure!¿

Definition 2. $\pi_1(\Gamma, p)$ *is a group defined by the following.*
Elements of $\pi_1(\Gamma, p)$ *consist of sequences (closed paths) of oriented edges and elements of vertex groups:* $g_1 e_1^{\epsilon_1} g_2 e_2^{\epsilon_2} \cdots g_{n-1} e_{n-1}^{\epsilon_{n-1}} g_n$*, where* e_i *is an edge (its label),* $\epsilon_i = \pm 1$ *(you might not have a choice),* g_i *in some vertex group, namely*
$g_i e_i^{+1} g_{i+1} \qquad g_i \in G_{i(e_i)},\ g_{i+1} \in G_{\tau(e_i)},\ g_i e_i^{-1} g_{i+1} \qquad g_i \in G_{\tau(e_i)},\ g_{i+1} \in G_{i(e_i)}.$

> Examples (see figure):
> 3_p – length 0,
> $4_p d 5_q d^{-1}$ – length 2 ($= 4_p d 5_q d^{-1} 0_p$),
> $4_p d 5_q d^{-1} 3_p$ – length 2,
> $3_p a^{-1} 2_s c^{-1} 2_r b^{-1} 5_p$ – length 3,
> $3_p a^{-1} 2_s c^{-1} 0_r b^{-1} 5_p$ – length 3
> (length = number of edges).

Two paths are equivalent if they differ by a sequence of insertions and removal backtracks. A backtrack is one of the following:

> $\phi_{a-}(g) \sim a\phi_{a+}(g)a^{-1}$*, where* $g \in G_a$
> $\phi_{a-}(g) \sim a\phi_{a+}(g)a^{-1}$*, where* $g \in G_a$
> *–or–*
> $a^{-1}\phi_{a-}(g)a\phi_{a+}(g^{-1}) \sim 1$
> $a\phi_{a+}(g^{-1}) \sim \phi_{a-}(g^{-1})a$
> $\phi_{a-}(g)a \sim a\phi_{a+}(g)$*, and so on (see diagram).*

With this equivalence relation all this becomes a well-defined group.

Example: $(3_p b)2_r b^{-1} \sim 1,\ (3_p b)1_r c 3_s a \sim (b 2_r) 1_r c 3_s a \sim b 3_r c 3_s a \sim b 1_r (c 3_s) 3_s a \sim b 1_r c 0_s a$.

Theorem 1. *Take a path and repeatedly apply removal backtracks till you can't remove anymore. Then your path represents the identity element iff you arrive at* Identity_p.

Resulting path is a "normal form", its length (number of edges) is length of the element represented by the path.

If we choose right coset representatives of edge groups in vertex groups and play like for HNN then obtain strong normal form.

¡figures. . . ¿

Complexes. Build topological space X from balls of various dimensions by gluing along together.

$B^n = \{\vec{x} \in \mathbb{R}^n : |\vec{x}| \leq 1\}$, $S^{n-1} = \{\vec{x} \in \mathbb{R}^n : |\vec{x}| = 1\}$.

Start with $X = X^0$ — disjoint union of 0-balls (see figure), attach a disjoint union of 1-balls ("1-cells") to X^0 along their boundary 0-spheres (see figure):

$$X^1 = \quad X^0 \sqcup B^1_i \Big/ \begin{array}{r} f_i(x) \sim x \,\forall x \in \partial B^1_i, \\ f_i : S^0_i \to X^0. \end{array}$$

Similarly, $X^2 = X^1 \underset{\substack{\text{along}\\ \text{boundaries}}}{\bigsqcup}$ bunch of 2-balls.

Examples (see figures):

Definition 3. *Given a presentation, say, $\langle a, b, c | aba^{-1}b^{-1}, bcbca\rangle$, build a 2-complex (see figure): 1-cell for each generator, 2-cell for each relator. This complex is called standard 2-complex of presentation.*

Theorem 2. *$\pi_1 X \cong G$, where G is a group, and X is its standard 2-complex.*