Stronger NFT. Choose right coset representatives for $C\backslash A$, $Ca_1, Ca_2, \ldots$ and $C\backslash B$, $Cb_1, Cb_2, \ldots$ This gives a unique normal form $a_0b_1a_2b_3\cdots$ or $b_0a_1b_2a_3\cdots$ with the property that each a_i, b_i is one of the coset representatives.

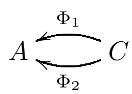
REMARK. This is not good either. It rather should be $ca_0b_1a_2b_3\cdots$

Existence of "ordinary" normal forms: each $g \in A \ast_C B$ equals $g_1g_2\cdots g_k$, $g_i \in A \sqcup B$. Continually "combine like terms" ($b_1a_2a_3b_4 \longrightarrow b_1(a_2a_3)b_4$) and combine elements of C with neighboring elements ($a_1\Phi_b(c)a_2 \to a_1\Phi_a(c)a_2 \to (a_1\Phi_a(c))a_2$).

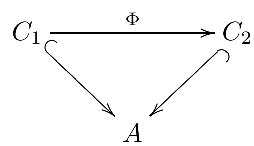
Existence of "stronger" normal forms: given an "ordinary" normal form $g_0g_1\cdots g_3g_4$ we continually replace g's with suitable coset representatives, starting with g_4:

$$g_0g_1\cdots g_3g_4 \xrightarrow{g_4\in Ca_4} g_0g_1\cdots(g_3c_4)a_4 \xrightarrow{g_3c_4\in Cb_3} g_0g_1\cdots c_3b_3a_4 \to \ldots \to ca_0b_1\cdots b_3a_4$$

HNN Extensions. The idea is to construct something given



or



Definition 1. *The HNN extension (HNN stands for Higman–Neumann-Neumann) associated to this data is the group $A\ast_C$ or $A\ast_\Phi$*

$$\langle A, t | c^t = \Phi(c)\, \forall c \in C_1\rangle \cong A \ast \langle t\rangle \Big/ \langle\langle c^t\Phi(c^{-1}), c \in C_1\rangle\rangle.$$

t is called stable letter.

NFT for HNN Extensions.

Theorem 1. *Every element in $A\ast_C$ can be expressed as $a_0t^{\varepsilon_1}a_1t^{\varepsilon_2}a_2t^{\varepsilon_3}\cdots$, each $\varepsilon_i = \pm 1$, each $a_i \in A$, and there is no appearance of $t^{-1}c_1t$, $c_1 \in C_1$ and tc_2t^{-1}, $c_2 \in C_2$.*

Length of g is number of t's in its normal form. Length is invariant.

A imbeds in $A\ast_C$.

EXAMPLE. Let $A = A_5$, $\beta = (12)(34)$, $\gamma = (123)$. Then $\langle A, t | \beta^t = \gamma\rangle \cong \mathbb{Z}$.

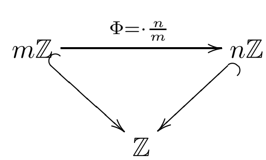
Hopfian groups.

Definition 2. *A group G is hopfian if every epimorphism $G \to G$ is a monomorphism.*

EXERCISE. Finitely generated abelian group is hopfian.

EXERCISE. $\mathbb{Z}^\infty, F_\infty$ are not hopfian. (Hint: kill one generator.)

Definition 3. $\mathrm{BS}\,(m,n) = \langle a, t | (a^m)^t = a^n\rangle$ *($m, n \neq 0$). (BS stands for Baumslag–Solitar.)*



$G = \mathrm{BS}\,(2,3)$ is not hopfian. Consider homomorphism $\psi : G \to G$ induced by $t \mapsto t$, $a \mapsto a^2$. Then the relation $[(a^2)^t = a^3] \mapsto [(a^4)^t = a^3]$, so ψ is really a homomorphism. Further, ψ is surjective since $t = \psi(t),\ a = \psi(a^{-1}(a)^t)$, but not injective, since $1 \neq [a^t, a]$ maps to 1. Show that.
$[a^t, a] = t^{-1}a^{-1}ta^{-1}t^{-1}ata$ — is a normal form of length > 0.
$[a^t, a] \mapsto [(a^2)^t, a^2] = [a^3, a^2] = 1$.

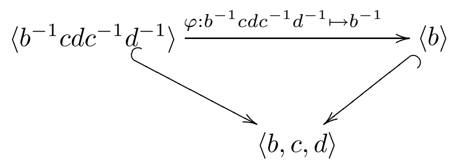
Reminder. $x^y = y^{-1}xy$, $[x, y] = x^{-1}y^{-1}xy$.

Exercise. $\mathrm{BS}\,(1, n)$ is hopfian, $\mathrm{BS}\,(n, n)$ is hopfian.

Constructions of amalgamated product and HNN extension are believed to be "equally strong". In the following example we present a group both as an amalgamated product and as a HNN extension.

Example. $G = \langle a, b, c, d | aba^{-1}b^{-1}cdc^{-1}d^{-1}\rangle$

- $G \cong F_2 \underset{\mathbb{Z}}{*} F_2 \cong \langle a, b\rangle \underset{\mathbb{Z}}{*} \langle c, d\rangle, \qquad (\mathbb{Z} = \langle [a, b]\rangle = \langle [c, d]\rangle)$.
- $G \cong F_3 *_{\mathbb{Z}}$, given by



and stable letter a, that is $(b^{-1}cdc^{-1}d^{-1})^a = b^{-1}$.

A Graph of Groups. Let Γ be a directed graph. For each vertex $v \in \Gamma$, we have a group G_v. For each edge $e \in \Gamma$, we have group G_e. An edge is attached at its initial vertex $i(e)$ and terminal vertex $t(e)$.

G_e has monomorphisms to $G_{i(e)}, G_{t(e)}$.

$$\varphi_{e+} : G_e \to G_{t(e)},\ \varphi_{e-} : G_e \to G_{i(e)}$$

From this data you can get a group $\pi_1\Gamma$, π_1(graph of groups).

Consider all combinatorial paths starting at some fixed vertex of the graph (see figure): $abc, abe^{-1}d^{-1}, \ldots$

Equivalence relation is generated by removal of backtracks: $abcc^{-1}e^{-1} \sim abe^{-1}$. So in our example, we get a subgroup $\langle a, b, c, d, e | -\rangle$ generated by $\langle abc, dec^{-1}\rangle$.

Label each edge a "stable" letter t_i, pick a base point.

Our group is the set of equivalence classes of "words" of the form $h_0s_1h_1s_2h_2 \cdots h_ks_kh_{k+1}$, where each $s_i = \text{some}\, t_j^{\pm 1}$.

If ht_ih' is in the word, then $h \in G_{i(e_i)}$, $h' \in G_{t(e_i)}$;
if $h't_i^{-1}h$ is in the word, then $h \in G_{i(e_i)}$, $h' \in G_{t(e_i)}$.

Equivalence relation is generated by "backtracks":

$$\varphi_{i-}(g_i)t_i \sim t_i\varphi_{i+}(g_i) \qquad \forall g_i \in G_i = G_{e_i}.$$