Lecture 1

Free Product of Groups. Let A, B be groups, such that $A \cap B = \emptyset$.

Definition 1. *The free product of groups A, B is a group $A * B$ given by the following:*

Elements *of $A * B$ are equivalence classes of finite sequences in $A \sqcup B$.*

Equivalence *relation is generated by two "elementary" equivalences.*

"Removing trivial element": $V 1_A W \sim V 1_B W \sim VW$.

"Combining together": $V g_1 g_2 W \sim V(g_1 g_2)W$, provided g_1 and g_2 both in A or both in B.

Multiplication *is concatenating representatives:*

$$[g_1 g_2 \cdots g_m][h_1 h_2 \cdots h_n] = [g_1 g_2 \cdots g_m h_1 h_2 \cdots h_n].$$

Examples of elements of $A * B$: $g_1 g_2 g_3$, 1 (empty sequence), g_1.

Lemma 1. *Multiplication is well-defined.*

Proof. An exercise. $\square$

One can easily see that $A * B$ is a group indeed:

multiplication is obviously associative;
identity is $[1]$;
inverse element is given by $[g_1 g_2 \cdots g_m]^{-1} = [g_m^{-1} \cdots g_2^{-1} g_1^{-1}]$.

Lemma 2. *Both A and B embed in $A * B$.*

Proof. An exercise. $\square$

Let A be presented by $\langle a_1, \ldots | R_1, \ldots \rangle$, and B be presented by $\langle b_1, \ldots | S_1, \ldots \rangle$. Then $A * B$ is presented by $\langle a_1, \ldots, b_1, \ldots | R_1, \ldots, S_1, \ldots \rangle$.

Universal Property. Free product $A * B$ satisfies the following universal property: for any group G and homomorphisms $A \to G$, $B \to G$ there exists a unique homomorphism $A * B \to G$ such that the diagram commutes.

$$\begin{tikzcd}
A \arrow[d, hook] \arrow[dr] & B \arrow[dl, hook'] \arrow[d] \\
A * B \arrow[r, dashed] & G
\end{tikzcd}$$

In other words, $A * B$ is a universal sender in the category "pairs of homos $A \to G$, $B \to G$".

Therefore, we effectively can define a free product in four ways: constructively, in terms of presentations, by a universal property, and in terms of theory of categories.

Notable examples of free products: $F_2 \cong Z * Z$, or $\langle a, b | - \rangle \cong \langle a | - \rangle * \langle b | - \rangle$; $Z_2 * Z_3$.

Amalgamated Product of Groups. Let A, B, C be groups such that $A \cap B = \emptyset$. Let $\Phi_A : C \to A$, $\Phi_B : C \to B$ be isomorphic embeddings.

Definition 2. *The amalgamated product of groups A, B along C is a group $A \underset{C}{*} B$ containing natural copies of A, B, characterized by the following property: for each* *commutative diagram of homomorphisms (see left) there exists a unique homomorphism $A \ast_C B \to G$ such that diagram commutes (see right).*

$$\begin{tikzcd}
& A \arrow[rd] & \\
C \arrow[ur, hook] \arrow[dr, hook] & & G \\
& B \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[rd] \arrow[rrd] & & \\
C \arrow[ur, hook] \arrow[dr, hook] & & A \ast_C B \arrow[r, dashed] & G \\
& B \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

In other words, an amalgamated product is something like a free product of groups which share a common subgroup.

Let A be presented by $\langle a_1, \ldots | R_1, \ldots \rangle$, B be presented by $\langle b_1, \ldots | S_1, \ldots \rangle$, and C be generated by $\langle c_1, \ldots \rangle$. Then $A \ast_C B$ is presented by

$$\langle a_1, \ldots, b_1, \ldots | R_1, \ldots, S_1, \ldots, \Phi_A(c_i) = \Phi_B(c_i) \ \forall i \rangle.$$

Exercise. Prove that.

Alternate Construction of Amalgamated Product. One can provide an alternate construction of $A \ast_C B$.

Definition 3. *Let A, B be groups such that $A \cap B = \emptyset$. Let $C_A \subseteq A$, Let $C_B \subseteq B$, and $C_A \cong C_B \cong C$. Then $A \ast_C B = A \ast_{C_A = C_B} B$ is given by the following:*

Elements *of $A * B$ are equivalence classes of finite sequences in $A \sqcup B$.*

Equivalence *relation is generated by two "elementary" equivalences.*

"Combining together": $Vg_1g_2W \sim V(g_1g_2)W$, provided g_1 and g_2 both in A or both in B.

"Sharing a subgroup": $V\Phi_A(c)W \sim V\Phi_B(c)W$.

Multiplication *is concatenating representatives:*

$$[g_1g_2 \cdots g_m][h_1h_2 \cdots h_n] = [g_1g_2 \cdots g_m h_1 h_2 \cdots h_n].$$

An example: $g_1g_2 = (g_1 1_B)g_2 = g_1 1_B g_2 = g_1 1_A g_2$.

Lemma 3. *Multiplication is well-defined.*

Proof. An exercise. □

One can easily see that $A \ast_C B$ is really a group:

multiplication is obviously associative;
identity is $[1_B] = [1_A]$;
inverse element is given by $[g_1g_2 \cdots g_m]^{-1} = [g_m^{-1} \cdots g_2^{-1} g_1^{-1}]$.

Note that the empty sequence is not considered an element.

Normal Form Theorem. A normal form is a particular type of sequence (representing some element) $g_0g_1g_2 \cdots g_k$:

- g_0 is either in A or B;
- $(i \geq 0)$ g_i, g_{i+1} are not both in A or both in B;
- $(i \geq 1)$ g_i is not in $\Phi_A(C)$ or $\Phi_B(C)$.

The number k is called length of the sequence.

Theorem 1. *1) Two normal forms representing the same element have the same length.*
2) 1_A *and* 1_B *are the only normal forms for* 1.
3) If g_0 *and* h_0 *are normal forms for some element, then either* $g_0 = h_0$, *or* $g_0 = \Phi_A(c)$ *and* $h_0 = \Phi_B(c)$ *for some* $c \in C$ *(or vice versa).*

REMARK. The theorem does not seem true. More exactly, the NFT is fine, but something is wrong with the definition of the normal form. For example, any element $a \in A - \Phi_A(C)$ has the normal form of the length 0: a, and one of the length 1: $\Phi_B(c) \cdot (\Phi_A(c^{-1})a$. Maybe it should be said explicitly that the formal form starts with an element either of $A - \Phi_A(C)$ or of $B - \Phi_B(C)$, then the numeration of g's starts with 1 instead of 0.

Corollary 1. A, B *imbed in* $A \underset{C}{*} B$. *Moreover,* $A \cap B \cong C$.