

LECTURE 1

Free Product of Groups. Let A, B be groups, such that $A \cap B = \emptyset$.

Definition 1. The free product of groups A, B is a group $A * B$ given by the following:

Elements of $A * B$ are equivalence classes of finite sequences in $A \sqcup B$.

Equivalence relation is generated by two “elementary” equivalences.

“Removing trivial element”: $V1_A W \sim V1_B W \sim VW$.

“Combining together”: $Vg_1 g_2 W \sim V(g_1 g_2)W$, provided g_1 and g_2 both in A or both in B .

Multiplication is concatenating representatives:

$$[g_1 g_2 \cdots g_m][h_1 h_2 \cdots h_n] = [g_1 g_2 \cdots g_m h_1 h_2 \cdots h_n].$$

Examples of elements of $A * B$: $g_1 g_2 g_3$, 1 (empty sequence), g_1 .

Lemma 1. Multiplication is well-defined.

Proof. An exercise. □

One can easily see that $A * B$ is a group indeed:

multiplication is obviously associative;

identity is [1];

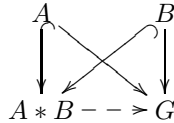
inverse element is given by $[g_1 g_2 \cdots g_m]^{-1} = [g_m^{-1} \cdots g_2^{-1} g_1^{-1}]$.

Lemma 2. Both A and B embed in $A * B$.

Proof. An exercise. □

Let A be presented by $\langle a_1, \dots | R_1, \dots \rangle$, and B be presented by $\langle b_1, \dots | S_1, \dots \rangle$. Then $A * B$ is presented by $\langle a_1, \dots, b_1, \dots | R_1, \dots, S_1, \dots \rangle$.

Universal Property. Free product $A * B$ satisfies the following universal property: for any group G and homomorphisms $A \rightarrow G, B \rightarrow G$ there exists a unique homomorphism $A * B \rightarrow G$ such that the diagram commutes.



In other words, $A * B$ is a universal sender in the category “pairs of homomorphisms $A \rightarrow G, B \rightarrow G$ ”.

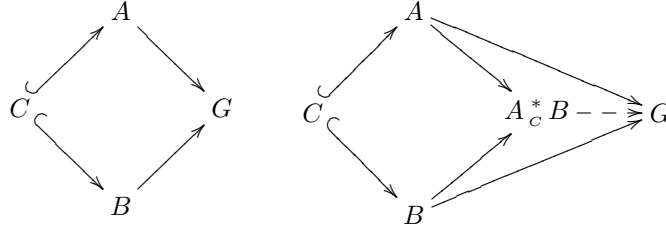
Therefore, we effectively can define a free product in four ways: constructively, in terms of presentations, by a universal property, and in terms of theory of categories.

Notable examples of free products: $F_2 \cong Z * Z$, or $\langle a, b | - \rangle \cong \langle a | - \rangle * \langle b | - \rangle$; $Z_2 * Z_3$.

Amalgamated Product of Groups. Let A, B, C be groups such that $A \cap B = \emptyset$. Let $\Phi_A : C \rightarrow A, \Phi_B : C \rightarrow B$ be isomorphic embeddings.

Definition 2. The amalgamated product of groups A, B along C is a group $A *_C B$ containing natural copies of A, B , characterized by the following property: for each

commutative diagram of homomorphisms (see left) there exists a unique homomorphism $A *_C B \rightarrow G$ such that diagram commutes (see right).



In other words, an amalgamated product is something like a free product of groups which share a common subgroup.

Let A be presented by $\langle a_1, \dots | R_1, \dots \rangle$, B be presented by $\langle b_1, \dots | S_1, \dots \rangle$, and C be generated by $\langle c_1, \dots \rangle$. Then $A *_C B$ is presented by

$$\langle a_1, \dots, b_1, \dots | R_1, \dots, S_1, \dots, \Phi_A(c_i) = \Phi_B(c_i) \forall i \rangle.$$

EXERCISE. Prove that.

Alternate Construction of Amalgamated Product. One can provide an alternate construction of $A *_C B$.

Definition 3. Let A, B be groups such that $A \cap B = \emptyset$. Let $C_A \subseteq A$, Let $C_B \subseteq B$, and $C_A \cong C_B \cong C$. Then $A *_C B = A_{C_A} *_C B$ is given by the following:

Elements of $A *_C B$ are equivalence classes of finite sequences in $A \sqcup B$.

Equivalence relation is generated by two “elementary” equivalences.

“Combining together”: $Vg_1g_2W \sim V(g_1g_2)W$, provided g_1 and g_2 both in A or both in B .

“Sharing a subgroup”: $V\Phi_A(c)W \sim V\Phi_B(c)W$.

Multiplication is concatenating representatives:

$$[g_1g_2 \cdots g_m][h_1h_2 \cdots h_n] = [g_1g_2 \cdots g_mh_1h_2 \cdots h_n].$$

An example: $g_1g_2 = (g_11_B)g_2 = g_11_Bg_2 = g_11_Ag_2$.

Lemma 3. Multiplication is well-defined.

Proof. An exercise. □

One can easily see that $A *_C B$ is really a group:

multiplication is obviously associative;

identity is $[1_B] = [1_A]$;

inverse element is given by $[g_1g_2 \cdots g_m]^{-1} = [g_m^{-1} \cdots g_2^{-1}g_1^{-1}]$.

Note that the empty sequence is not considered an element.

Normal Form Theorem. A normal form is a particular type of sequence (representing some element) $g_0g_1g_2 \cdots g_k$:

- g_0 is either in A or B ;
- ($i \geq 0$) g_i, g_{i+1} are not both in A or both in B ;
- ($i \geq 1$) g_i is not in $\Phi_A(C)$ or $\Phi_B(C)$.

The number k is called length of the sequence.

Theorem 1. 1) *Two normal forms representing the same element have the same length.*

2) 1_A and 1_B are the only normal forms for 1.

3) *If g_0 and h_0 are normal forms for some element, then either $g_0 = h_0$, or $g_0 = \Phi_A(c)$ and $h_0 = \Phi_B(c)$ for some $c \in C$ (or vice versa).*

REMARK. The theorem does not seem true. More exactly, the NFT is fine, but something is wrong with the definition of the normal form. For example, any element $a \in A - \Phi_A(C)$ has the normal form of the length 0: a , and one of the length 1: $\Phi_B(c) \cdot (\Phi_A(c^{-1})a)$. Maybe it should be said explicitly that the normal form starts with an element either of $A - \Phi_A(C)$ or of $B - \Phi_B(C)$, then the numeration of g 's starts with 1 instead of 0.

Corollary 1. *A, B imbed in $A_C^* B$. Moreover, $A \cap B \cong C$.*