MATH 579 - ASSIGNMENT 3

Posted Mar 18^{th} 2012
Due Mar 29^{th} 2012

Write the following assignment using LaTeX or the (very easy to use) LaTeX editor: LyX available for free at http://www.lyx.org/

1. Burger's Equation

Consider Burger's equation in conservative formulation (in 1D),

$$\begin{aligned} u_t + f(u)_x &= 0 \\ u(x, t=0) &= u_0(x) \\ f(u) &= \frac{1}{2}u^2 \end{aligned}$$

with $x \in [0, 2\pi]$ periodic, and $t \in]0, T]$.

(1) For the following Riemann problem,

$$u_0(x) = \begin{cases} 1, & x < \frac{\pi}{4} \\ 2, & \frac{\pi}{4} \le x < \frac{\pi}{2} \\ 1, & x \ge \frac{\pi}{2} \end{cases},$$

find the exact solution at $T = 1$. Note: the solution has two parts stitched together: a shock and a rarefaction.

(2) Consider now the differential formulation of the equation,

$$\begin{aligned} u_t + uu_x &= 0 \\ u(x, t=0) &= u_0(x), \end{aligned}$$

implement (e.g. using Matlab) each of the following finite-difference schemes. For each scheme, produce a convergence plot for the error in the l^1 norm. Also produce a plot of the l^1 norm of u (not the error), vs. time.
   (a) Upwind
   (b) RK3-TVD - WENO5

(3) Explain what you observed in (2) in a concise and synthetic way.

(4) Consider now the conservative formulation above. implement (e.g. using Matlab) each of the following finite-volume schemes. For each scheme, produce a convergence plot for the error in the l^1 norm. Also produce a plot of the l^1 norm of u (not the error), vs. time.
   (a) upwind
   (b) Lax-Friedrichs

(5) Explain what you observed in (4) in a concise and synthetic way.

(6) Explain the difference between the approaches in (2) and those in (4).

(7) Bonus question: implement and test the flux limiter approach to this problem. You are free to use either van Leer or Superbee limiter and whichever high order flux (e.g. Lax-Wendroff) in addition to the TVD (e.g. Upwind) flux.

2. Shallow water equations

The shallow water equations are (in conservation form),

$$\begin{bmatrix} h \\ u \end{bmatrix}_t + \begin{bmatrix} uh \\ \frac{1}{2}u^2 + gh \end{bmatrix}_x = 0$$

$$\begin{bmatrix} h \\ u \end{bmatrix}(x, t = 0) = \begin{bmatrix} h_0 \\ u_0 \end{bmatrix}(x),$$

with $x \in [0, L]$ periodic, and $t \in]0, T]$.

They describe the evolution of fluid surface, e.g. a shallow lake. $h(x,t)$ is the fluid depth, $u(x,t)$ is the fluid velocity, $g = 1$ is gravity.

(1) Assume, $L = 10$, and consider the following initial conditions,

$$h_0(x) = \begin{cases} 1, & x < 4 \\ 2, & 4 \leq x < 6 \\ 1, & x \geq 6 \end{cases}$$

$$u_0(x) = 0.$$

Implement a the finite volume method using Lax-Friedrichs fluxes, and plot the height of the fluid h, at $t = \{0, 1, 2, 3, 4\}$.

(2) Produce a plot of the l^1 norm of h (not the error), vs. time.

(3) Produce a plot of the l^1 norm of u (not the error), vs. time.

(4) Draw an overall conclusion about your findings.