MATH 579, Solutions to Assignment 1,
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1. Fundamental solution of the Laplace equation

The radial part of the Laplacian in spherical coordinates reads:

$$\nabla^2 = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) \tag{1}$$

The ODE we wish to solve is:

$$\nabla^2 v(r) = \frac{n-1}{r}\frac{dv}{dr} + \frac{d^2 v}{dr^2} = 0 \tag{2}$$

Giving:

$$\implies \quad \frac{d^2v}{dr^2} = -\frac{n-1}{r}\frac{dv}{dr} \quad \implies \quad \int \frac{dv'}{v'} = -\int \frac{n-1}{r}dr \quad \implies \quad \ln v' = -(n-1)\ln r + A \quad \implies \quad v' = \frac{B}{r^{n-1}}$$

$$n = 1: \qquad v' = B \quad \implies \quad v(r) = Br + C \tag{3}$$

$$n = 2: \qquad v' = \frac{B}{r} \quad \implies \quad v(r) = B\ln r + C \tag{4}$$

$$n \geq 3: \qquad v' = \frac{B}{r^{n-1}} \quad \implies \quad v(r) = \frac{B}{(n-2)r^{n-2}} + C \tag{5}$$

The fundamental solution $\phi(r)$ is obtained by imposing the requirement $-\nabla^2\phi(r) = \delta(r)$ in the sense of distributions, ie: for any test function $f \in C_c^\infty(\mathbb{R})$:

$$\left\langle -\nabla^2\phi(r), f\right\rangle = \langle\delta(r), f\rangle = f(0) \quad \iff \quad -(-1)^{2n}\int_{\mathbb{R}^n}\phi(r)\nabla^2 f(\vec{x})d\vec{x} = f(0) \tag{6}$$

Using the radially symmetric harmonic functions just found, we solve for the coefficient B in each case.

- For $n = 1$:

$$\begin{aligned}
f(0) &= \left\langle -\nabla^2\phi(r), f\right\rangle \\
&= -\int_{\mathbb{R}} v(r)\nabla^2 f(x)dx \\
&= -\int_{-\infty}^0 v(-x)f''(x)dx - \int_0^\infty v(x)f''(x)dx \\
\int \text{ by parts}: \quad &= -f'(x)v(-x)|_{-\infty}^0 + \int_{-\infty}^0 f'(x)v'(-x)dx - f'(x)v(x)|_0^\infty + \int_0^\infty f'(x)v'(x)dx \\
&= -f'(0)v(0) + \underbrace{f'(x)}_{=0}v(-x)|_{x=-\infty} + \int_{-\infty}^0 f'(x)v'(-x)dx - \underbrace{f'(x)}_{=0}v(x)|_{x=\infty} + f'(0)v(0) + \int_0^\infty f'(x)v'(x)dx \\
&= \int_{-\infty}^0 f'(x)\frac{d}{dx}(-Bx+C)dx + \int_0^\infty f'(x)\frac{d}{dx}(Bx+C)dx \\
&= -B\int_{-\infty}^0 f'(x)dx + B\int_0^\infty f('x)dx \\
&= -B(f(0) - f(x)|_{x=-\infty}) + B(f(x)|_{x=\infty} - f(0)) \\
&= -2Bf(0)
\end{aligned} \tag{7}$$

Giving $B = -\frac{1}{2}$.

- Similarly, for $n = 2$:

$$\begin{aligned}
f(0) &= \left\langle -\nabla^2 \phi(r), f \right\rangle \\
&= -\int_{\mathbb{R}^2} v(r) \nabla^2 f(\vec{x}) d\vec{x} \\
&= -\int_0^{2\pi} \int_0^{\infty} v(r) \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) f(r,\theta) + + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} f(r,\theta) \right] r dr d\theta \\
&= -\int_0^{2\pi} \int_0^{\infty} v(r) \left[\frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) f(r,\theta) \right] dr d\theta - \int_0^{\infty} \frac{1}{r} \int_0^{2\pi} v(r) \left[\frac{\partial^2}{\partial \theta^2} f(r,\theta) \right] d\theta dr \\
\int \text{ by parts :} \quad &= -\int_0^{2\pi} \left[\underbrace{v(r) r \frac{\partial}{\partial r} f(r,\theta)|_0^{\infty}}_{=0-0} - \int_0^{\infty} v'(r) r \frac{\partial}{\partial r} f(r,\theta) dr \right] d\theta \\
& \quad - \int_0^{\infty} \frac{1}{r} \left[\underbrace{v(r) \frac{\partial}{\partial \theta} f(r,\theta)|_0^{2\pi}}_{=0 \text{ by periodic B.C.}} - \int_0^{2\pi} \underbrace{\frac{\partial}{\partial \theta} v(r)}_{=0} \left[\frac{\partial}{\partial \theta} f(r,\theta) \right] d\theta \right] dr \\
&= \int_0^{2\pi} \int_0^{\infty} \frac{\partial f}{\partial r} \; r \; v'(r) dr d\theta \\
&= \int_0^{2\pi} \int_0^{\infty} \frac{\partial f}{\partial r} \; r \; \frac{B}{r} dr d\theta \\
&= -2\pi B f(0)
\end{aligned} \tag{8}$$

Giving $B = -\frac{1}{2\pi}$.

- Finally, for $n \geq 3$: Since by the same reasoning as above, all derivatives with respect to the angles give 0-terms when integrating by parts, we only care about the radial component of the Laplacian. The solid angle is denoted by Ω.

$$\begin{aligned}
f(0) &= \left\langle -\nabla^2 \phi(r), f \right\rangle \\
&= -\int_{\mathbb{R}^n} v(r) \nabla^2 f(\vec{x}) d\vec{x} \\
&= -\int \int_0^{\infty} v(r) \left[\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} \right) f(r,\Omega) \right] r^{n-1} dr d\Omega \\
\int \text{ by parts :} \quad &= \int \int_0^{\infty} r^{n-1} v'(r) \frac{\partial}{\partial r} f(r,\Omega) dr d\Omega \\
&= \int \int_0^{\infty} r^{n-1} \left(-\frac{B}{r^{n-1}} \right) \frac{\partial}{\partial r} f(r,\Omega) dr d\Omega \\
&= -B \int \int_0^{\infty} \frac{\partial}{\partial r} f(r,\Omega) dr d\Omega \\
&= B \; \omega_n \; f(0)
\end{aligned} \tag{9}$$

Giving $B = \frac{1}{\omega_n}$, where ω_n is the area of the unit sphere S^n in $\mathbb{R}^n$.

Therefore:

$$\phi(r) = \begin{cases} -\frac{r}{2} & n = 1 \\ -\frac{1}{2\pi} \ln r & n = 2 \\ \frac{1}{(n-2)\omega_n r^{n-2}} & n = 3 \end{cases} \tag{10}$$

The LTE is obtained by plugging the true solution u into the scheme, using Taylor expansion, and verifying the order of the remainder. In the following, if the variable is not specified, it is then assumed to be x.

(1)

$$\tau(h) = \frac{1}{2h}\left(-u(x+2h) + 4u(x+h) - 3u\right) - u' \tag{11}$$

Since:

$$\begin{aligned} & -u(x+2h) + 4u(x+h) - 3u && (12)\\ = \;& -\left(u + (2h)u' + \frac{(2h)^2}{2}u'' + \frac{(2h)^3}{6}u''' + \dots\right) + 4\left(u + hu' + \frac{h^2}{2}u'' + \frac{h^3}{6}u''' + \dots\right) - 3u && (13)\\ = \;& 2hu' - \frac{2}{3}h^3u''' + \text{h.o.t.} && (14)\end{aligned}$$

we get:

$$\tau(h) = \frac{1}{2h}\left(2hu' - \frac{2}{3}h^3u''' + \text{h.o.t.}\right) - u' = -\frac{1}{3}h^2u''' + \text{h.o.t.} \tag{15}$$

and therefore $\alpha = 2$.

(2)

$$\tau(h) = \frac{1}{12h}\left(-u(x+2h) + 8u(x+h) - 8u(x-h) + u(x-2h)\right) - u' \tag{16}$$

Since:

$$\begin{aligned} & -u(x+2h) + 8u(x+h) - 8u(x-h) + u(x-2h) && (17)\\ = \;& -\left(u + (2h)u' + \frac{(2h)^2}{2}u'' + \frac{(2h)^3}{6}u''' + \frac{(2h)^4}{24}u^{(4)} + \frac{(2h)^5}{120}u^{(5)}\dots\right) && (18)\\ & +8\left(u + hu' + \frac{h^2}{2}u'' + \frac{h^3}{6}u''' + \frac{h^4}{24}u^{(4)} + \frac{h^5}{120}u^{(5)}\dots\right) && (19)\\ & -8\left(u - hu' + \frac{h^2}{2}u'' - \frac{h^3}{6}u''' + \frac{h^4}{24}u^{(4)} - \frac{h^5}{120}u^{(5)}\dots\right) && (20)\\ & +\left(u - (2h)u' + \frac{(2h)^2}{2}u'' - \frac{(2h)^3}{6}u''' + \frac{(2h)^4}{24}u^{(4)} - \frac{(2h)^5}{120}u^{(5)}\dots\right) && (21)\\ = \;& 12hu' - \frac{2}{5}h^5u^{(5)} + \text{h.o.t.} && (22)\end{aligned}$$

we get:

$$\tau(h) = \frac{1}{12h}\left(12hu' - \frac{2}{5}h^5u^{(5)} + \text{h.o.t.}\right) - u' = -\frac{1}{30}h^4u''' + \text{h.o.t.} \tag{23}$$

and therefore $\alpha = 4$.

(3)

$$\tau(h) = \frac{1}{2h^3}\left(-u(x-2h) + 2u(x-h) - 2u(x+h) + u(x+2h)\right) - u''' \tag{24}$$

Since:

$$\begin{aligned}
& -u(x-2h) + 2u(x-h) - 2u(x+h) + u(x+2h) && (25)\\
= & -\left(u - (2h)u' + \frac{(2h)^2}{2}u'' - \frac{(2h)^3}{6}u''' + \frac{(2h)^4}{24}u^{(4)} - \frac{(2h)^5}{120}u^{(5)} \dots\right) && (26)\\
& +2\left(u - hu' + \frac{h^2}{2}u'' - \frac{h^3}{6}u''' + \frac{h^4}{24}u^{(4)} - \frac{h^5}{120}u^{(5)} \dots\right) && (27)\\
& -2\left(u + hu' + \frac{h^2}{2}u'' + \frac{h^3}{6}u''' + \frac{h^4}{24}u^{(4)} + \frac{h^5}{120}u^{(5)} \dots\right) && (28)\\
& +\left(u + (2h)u' + \frac{(2h)^2}{2}u'' + \frac{(2h)^3}{6}u''' + \frac{(2h)^4}{24}u^{(4)} + \frac{(2h)^5}{120}u^{(5)} \dots\right) && (29)\\
= & \; 2h^3u''' + \frac{1}{2}h^5u^{(5)} + \text{h.o.t.} && (30)
\end{aligned}$$

we get:

$$\tau(h) = \frac{1}{2h^3}\left(2h^3u''' + \frac{1}{2}h^5u^{(5)} + \text{h.o.t.}\right) - u''' = \frac{1}{4}h^2u^{(5)} + \text{h.o.t.} \quad (31)$$

and therefore $\alpha = 2$.

(4)

$$\tau(h) = \frac{1}{h^4}\left(u - 4u(x+h) + 6u(x+2h) - 4u(x+3h) + u(x+4h)\right) - u'''' \quad (32)$$

Since:

$$u - 4u(x+h) + 6u(x+2h) - 4u(x+3h) + u(x+4h) = h^4u'''' + 2h^5u^{(5)} + \text{h.o.t.} \quad (33)$$

we get:

$$\tau(h) = \frac{1}{h^4}\left(h^4u'''' + 2h^5u^{(5)} + \text{h.o.t.}\right) - u'''' = 2hu^{(5)} + \text{h.o.t.} \quad (34)$$

and therefore $\alpha = 1$.

3. Finite-Difference Approximations of Derivatives (Applications)

In order to check the order of the LTE for each method, one can write a small program which iterates over the resolution h of the grid. For example, to verify that the LTE for the first derivative of $u_1(x) = \exp(x)$ using

$$u'(0) = \frac{u(h) - u(0)}{h} + O(h) \tag{35}$$

is indeed first order, the program computes:

$$\text{Error}(h) = \left| \frac{e^h - 1}{h} - 1 \right| \tag{36}$$

for $h = \{2^{-1}, 2^{-1.5}, \ldots, 2^{-10}\}$.

• $u_1(x) = e^x$: The results are shown on Figure (1) . The slope of each line was obtained using the polyfit function from MatLab, and gave as expected:
- 1.0196 for the $O(h)$ method (in blue) for the first derivative,
- 2.0010 for the $O(h^2)$ method (in black) for the first derivative,
- 2.0007 for the $O(h^2)$ method (in green) for the second derivative,
- 3.9935 for the $O(h^4)$ method (in red) for the second derivative. Note that for this method, the last 5 values were excluded from the data set. Since the method converges for those h's, the values returned are of $O(\epsilon_{\text{machine}})$, and are irrelevant to the convergence analysis.

The lines were made to pass through the last relevant point of the data set. Their slopes are as expected, since they match the order of the method in each case.

• $u_2(x) = x^2$: The results are shown on Figure (2) . The slope of the only line that appears is 1.0000, and therefore agrees with the order of the method, which is $O(h)$ for the first derivative.

Note that no data appear for the other methods, as the program returned values which were *exactly* 0 for those methods. This is due to the symmetry of the function: since the computation of the errors involve subtracting/dividing by numbers that are exactly equal (even as encoded in the machine), the computer is able to conclude that the answer is exactly 0 (and not some value of $O(\epsilon_{\text{machine}})$).

4. Heat Equation

Let Δx denote the gridsize in the spatial direction, and Δt the gridsize in the time direction. Using forward Euler in time, and the fourth order discretization from the previous problem in space, the heat equation reads:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = C\,\frac{-u_{i-2}^n + 16u_{i-1}^n - 30u_i^n + 16u_{i+1}^n - u_{i+2}^n}{12(\Delta x)^2} \tag{37}$$

We'll assume that the discretizations used near the boundaries have the same order.

(2) We start by computing the stability restriction one has to impose on Δt. We apply (quick) Von Neumann stability analysis to the scheme: Letting k denote the wavenumber, we get:

$$\begin{aligned}
\frac{G-1}{\Delta t} &= C\,\frac{-e^{-2ik\Delta x} + 16e^{-ik\Delta x} - 30 + 16e^{ik\Delta x} - e^{2ik\Delta x}}{12(\Delta x)^2} && (38)\\
&= \frac{C}{3(\Delta x)^2}\frac{1}{4}\left(-e^{-2ik\Delta x} + 16e^{-ik\Delta x} - 30 + 16e^{ik\Delta x} - e^{2ik\Delta x}\right) && (39)\\
&= -\frac{C}{3(\Delta x)^2}\frac{1}{4}\left(e^{-2ik\Delta x} - 16e^{-ik\Delta x} + 30 - 16e^{ik\Delta x} + e^{2ik\Delta x}\right) && (40)\\
&= -\frac{C}{3(\Delta x)^2}\frac{1}{4}\left(28 + e^{2ik\Delta x} + 2 + e^{-2ik\Delta x} - 16e^{-ik\Delta x} - 16e^{ik\Delta x}\right) && (41)\\
&= -\frac{C}{3(\Delta x)^2}\left(7 + \frac{e^{2ik\Delta x} + 2 + e^{-2ik\Delta x}}{4} - 8\frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2}\right) && (42)\\
&= -\frac{C}{3(\Delta x)^2}\left(7 + \cos^2(k\Delta x) - 8\cos(k\Delta x)\right) && (43)
\end{aligned}$$

$$\implies \quad G(k) = 1 - \frac{C\Delta t}{3(\Delta x)^2}\left(\cos^2(k\Delta x) - 8\cos(k\Delta x) + 7\right) \tag{44}$$

Let $y = \cos(k\Delta x)$, and

$$w(y) = y^2 - 8y + 7 = (y-1)(y-7) \tag{45}$$

So that $w \geq 0$ when $y = \cos(k\Delta x) \in [-1, 1]$. This guarantees that $G(k) < 1$. Now, in order to make sure that $|G(k)| \leq 1$, we must have:

$$\begin{aligned}
-1 \leq G(k) \quad &\iff \quad -1 \leq 1 - \frac{C\Delta t}{3(\Delta x)^2}\left(\cos^2(k\Delta x) - 8\cos(k\Delta x) + 7\right) && (46)\\
&\iff \quad \frac{C\Delta t}{3(\Delta x)^2}\left(\cos^2(k\Delta x) - 8\cos(k\Delta x) + 7\right) \leq 2 && (47)\\
&\impliedby \quad \frac{C\Delta t}{3(\Delta x)^2}\max\left(\cos^2(k\Delta x) - 8\cos(k\Delta x) + 7\right) \leq 2 && (48)\\
&\iff \quad \frac{C\Delta t}{3(\Delta x)^2}\left((-1)^2 - 8(-1) + 7\right) \leq 2 && (49)\\
&\iff \quad \frac{16C\Delta t}{3(\Delta x)^2} \leq 2 && (50)\\
&\iff \quad \Delta t \leq \frac{3}{8}\frac{(\Delta x)^2}{C} \qquad \textbf{stability criterion} && (51)
\end{aligned}$$

(3) A discretization scheme for a PDE is *consistent* provided that $\tau(\Delta x, \Delta t) \to 0$ as $\Delta x,\ \Delta t \to 0$, where τ is the LTE. We compute it by substituting the true solution in the scheme, and using Taylor expansions:

$$\begin{aligned}
&\tau(\Delta x, \Delta t)\\
= \;&\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} - C\,\frac{-u(x-2\Delta x, t) + 16u(x-\Delta x, t) - 30u(x,t) + 16u(x+\Delta x, t) - u(x+2\Delta x, t)}{12(\Delta x)^2}\\
= \;&u_t + \frac{\Delta t}{2}u_{tt} + \ldots - C(u_{xx} + O((\Delta x)^4)) && (52)
\end{aligned}$$

$$= u_t + \frac{\Delta t}{2} u_{tt} + \ldots - C(u_{xx} + O((\Delta x)^4)) \quad (53)$$

$$= \underbrace{u_t - C u_{xx}}_{=0} + O(\Delta t + (\Delta x)^4) = O(\Delta t + (\Delta x)^4) \quad (54)$$

Thus, τ obviously goes to 0 as Δx and Δt go to 0. Therefore, **the scheme is consistent**.

(1) There are 3 equivalent ways of computing the GTE.

- **Way 1:** One can always go back to the definition of the GTE. Let U^n be the true solution at stage n, and v^n be the solution returned by the scheme at stage n. Then

$$E^n = v^n - U^n \quad (55)$$

Consider the LTE computed above:

$$\tau = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} - C\,\frac{-u(x-2\Delta x,t) + 16u(x-\Delta x,t) - 30u(x,t) + 16u(x+\Delta x,t) - u(x+2\Delta x,t)}{12(\Delta x)^2}$$

$$\tau_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t} - C\,\frac{-u_{i-2}^n + 16u_{i-1}^n - 30u_i^n + 16u_{i+1}^n - u_{i+2}^n}{12(\Delta x)^2}$$

$$u_i^{n+1} = u_i^n + \frac{C\Delta t}{12(\Delta x)^2}(u_{i-2}^n + 16u_{i-1}^n - 30u_i^n + 16u_{i+1}^n - u_{i+2}^n) + \Delta t\ \tau_i^n$$

$$u_i^{n+1} = \frac{C\Delta t}{12(\Delta x)^2}\left(u_{i-2}^n + 16u_{i-1}^n + \left(\frac{12(\Delta x)^2}{C\Delta t} - 30\right)u_i^n + 16u_{i+1}^n - u_{i+2}^n\right) + \Delta t\ \tau_i^n$$

$$u_i^{n+1} = \frac{C\Delta t}{12(\Delta x)^2}(-1,\ 16,\ \frac{12(\Delta x)^2}{C\Delta t} - 30,\ 16,\ -1)\begin{pmatrix} u_{i-2}^n \\ u_{i-1}^n \\ u_i^n \\ u_{i+1}^n \\ u_{i+2}^n \end{pmatrix} + \Delta t\ \tau_i^n \quad (56)$$

So that at stage n, we have:

$$U^{n+1} = \frac{C\Delta t}{12(\Delta x)^2} B(\Delta t, \Delta x) U^n + \Delta t\ \tau^n + b^n \quad (57)$$

where

$$U^n = \begin{pmatrix} u_0^n \\ u_1^n \\ \vdots \\ u_{M-1}^n \\ u_M^n \end{pmatrix} \quad \text{and} \quad \tau^n = \begin{pmatrix} \tau_0^n \\ \tau_1^n \\ \vdots \\ \tau_{M-1}^n \\ \tau_M^n \end{pmatrix} \quad (58)$$

b^n is a vector taking care of the B.C., and $B(\Delta t, \Delta x)$ is a matrix (which I won't write explicitely, since we don't know the discretization near the boundaries). Since

$$v^{n+1} = \frac{C\Delta t}{12(\Delta x)^2} B(\Delta t, \Delta x) v^n + b^n \quad (59)$$

we get at stage N:

$$E^N = v^N - U^N \quad (60)$$

$$= \frac{C\Delta t}{12(\Delta x)^2} B(\Delta t, \Delta x)(v^{N-1} - U^{N-1}) - \Delta t\ \tau^N \quad (61)$$

$$= \frac{C\Delta t}{12(\Delta x)^2} B(\Delta t, \Delta x)\left(\frac{C\Delta t}{12(\Delta x)^2} B(\Delta t, \Delta x)(v^{N-2} - U^{N-2}) - \Delta t\ \tau^{N-2}\right) - \Delta t\ \tau^{N-1} \quad (62)$$

$$= \left(\frac{C\Delta t}{12(\Delta x)^2}\right)^2 B^2(\Delta t, \Delta x)(v^{N-2} - U^{N-2}) - \Delta t\left(\frac{C\Delta t}{12(\Delta x)^2} B(\Delta t, \Delta x)\ \tau^{N-2} + \tau^{N-1}\right) \quad (63)$$

$$\dots \tag{64}$$

$$= \left(\frac{C\Delta t}{12(\Delta x)^2}\right)^N B^N(\Delta t,\Delta x)(v^0 - U^0) - \Delta t \sum_{n=1}^{N} \left(\frac{C\Delta t}{12(\Delta x)^2}\right)^{N-n} B^{N-n}(\Delta t,\Delta x)\ \tau^{N-n} \tag{65}$$

$$= \left(\frac{C\Delta t}{12(\Delta x)^2}\right)^N B^N(\Delta t,\Delta x)E^0 - \Delta t \sum_{n=1}^{N} \left(\frac{C\Delta t}{12(\Delta x)^2}\right)^{N-n} B^{N-n}(\Delta t,\Delta x)\ \tau^{N-n} \tag{66}$$

We now wish to estimate this quantity: first using the triangle inequality, we get:

$$\text{GTE} = ||E^N|| \leq ||\left(\frac{C\Delta t}{12(\Delta x)^2}\right)^N B^N(\Delta t,\Delta x)E^0|| + \Delta t\ ||\sum_{n=1}^{N} \left(\frac{C\Delta t}{12(\Delta x)^2}\right)^{N-n} B^{N-n}(\Delta t,\Delta x)\ \tau^{N-n}|| \tag{67}$$

Now, taking stability into account, note that $||B((\Delta x)^2,\Delta x)|| = O(1)$. Letting $T = N\Delta t$, we get:

$$||E^N|| \leq ||\left(\frac{C\Delta t}{12(\Delta x)^2}\right)^N B^N(\Delta t,\Delta x)E^0|| + \Delta t\ ||\sum_{n=1}^{N} \left(\frac{C\Delta t}{12(\Delta x)^2}\right)^{N-n} B^{N-n}(\Delta t,\Delta x)\ \tau^{N-n}|| \tag{68}$$

$$= ||O(1)\cdot O(1)E^0|| + \Delta t\ ||\sum_{n=1}^{N} O(1)\cdot O(1)\ \tau^{N-n}|| \tag{69}$$

$$\leq ||E^0|| + \Delta t \sum_{n=1}^{N} ||\tau^{N-n}|| \leq ||E^0|| + \Delta t\cdot N||\tau^{N-n}|| \leq ||E^0|| + T||\tau^{N-n}|| \tag{70}$$

Assuming that the initial error is not too large, we then have:

$$\text{GTE} \leq ||\tau^{N-n}(\Delta t,\Delta x)|| \overbrace{=}^{\text{by stability}} ||\tau^{N-n}((\Delta x)^2,\Delta x)|| = O((\Delta x)^2 + (\Delta x)^4) = O((\Delta x)^2) \tag{71}$$

So that the **GTE** $\propto O((\Delta x)^2)$**.**

- **Way 2:** The above justifies the following shortcut: The GTE can be estimated by computing the LTE $\tau_i^n(\Delta t,\Delta x)$ and imposing stability to it:

$$\text{GTE} \leq \tau_i^n(\Delta t,\Delta x) \overbrace{=}^{\text{by stability}} \tau_i^n((\Delta x)^2,\Delta x) = O((\Delta x)^2 + (\Delta x)^4) = O((\Delta x)^2) \tag{72}$$

- **Way 3:** One can also compute the 'one-step error' for the scheme. This quantity is basically equal to $\Delta t\cdot\tau_i^n$ since it is computed as follows:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = C\ \frac{-u_{i-2}^n + 16u_{i-1}^n - 30u_i^n + 16u_{i+1}^n - u_{i+2}^n}{12(\Delta x)^2} \tag{73}$$

$$\iff u_i^{n+1} = u_i^n + C\ (\Delta t)\ \frac{-u_{i-2}^n + 16u_{i-1}^n - 30u_i^n + 16u_{i+1}^n - u_{i+2}^n}{12(\Delta x)^2} \tag{74}$$

then substitute the true solution and compute the difference of the two sides:

$$\begin{aligned}\alpha_i^n &= u(x,t) + C(\Delta t)\left(\frac{-u(x-2\Delta x) + 16u(x-\Delta x) - 30u(x) + 16u(x+\Delta x) - u(x+2\Delta x)}{12(\Delta x)^2}\right) - u(x,t+\Delta t) \\ &= O(\Delta t(\Delta x)^4 + (\Delta t)^2) \end{aligned} \tag{75}$$

One can then estimate the GTE by summing up the one-step error at each stage:

$$\text{GTE} \leq ||\sum_{n=1}^{N} \alpha^n|| \leq N||\alpha^n|| = \frac{T}{\Delta t}\ O(\Delta t(\Delta x)^4 + (\Delta t)^2)) \tag{76}$$

$$\overbrace{=}^{\text{by stability}} \frac{T}{(\Delta x)^2}\ O((\Delta x)^2(\Delta x)^4 + (\Delta x)^4)) = O((\Delta x)^2) \tag{77}$$

(4) Lastly, since we proved that the scheme is consistent and stable, by Lax Equivalence Theorem, we have that **the scheme is convergent**. (By the above, since the GTE is $O((\Delta x)^2)$, is goes to 0 as $\Delta x \to 0$.) Please note that Lax Equivalence Theorem for PDEs holds *provided the scheme is linear* (which is the case here). It may not hold for non-linear schemes.

Remark: Another way to get the one-step error for the scheme is to combine the LTE for the temporal and spatial discretizations, as follows:

- As proved in class, the LTE for forward Euler is $O((\Delta t)^2)$.
- The LTE for the spatial discretization is:

$$\begin{aligned} \tau(\Delta x) &= -u(x-2\Delta x) + 16u(x-\Delta x) - 30u(x) + 16u(x+\Delta x) - u(x+2\Delta x) - 12(\Delta x)^2 u_{xx} && (78) \\ &= 12(\Delta x)^2 \left(\frac{-u(x-2\Delta x) + 16u(x-\Delta x) - 30u(x) + 16u(x+\Delta x) - u(x+2\Delta x)}{12(\Delta x)^2} - u_{xx} \right) && (79) \\ &= 12(\Delta x)^2 \; O((\Delta x)^4) = O((\Delta x)^6) && (80) \end{aligned}$$

One you impose stability, this is equivalent to the previous method for getting the one-step error. △

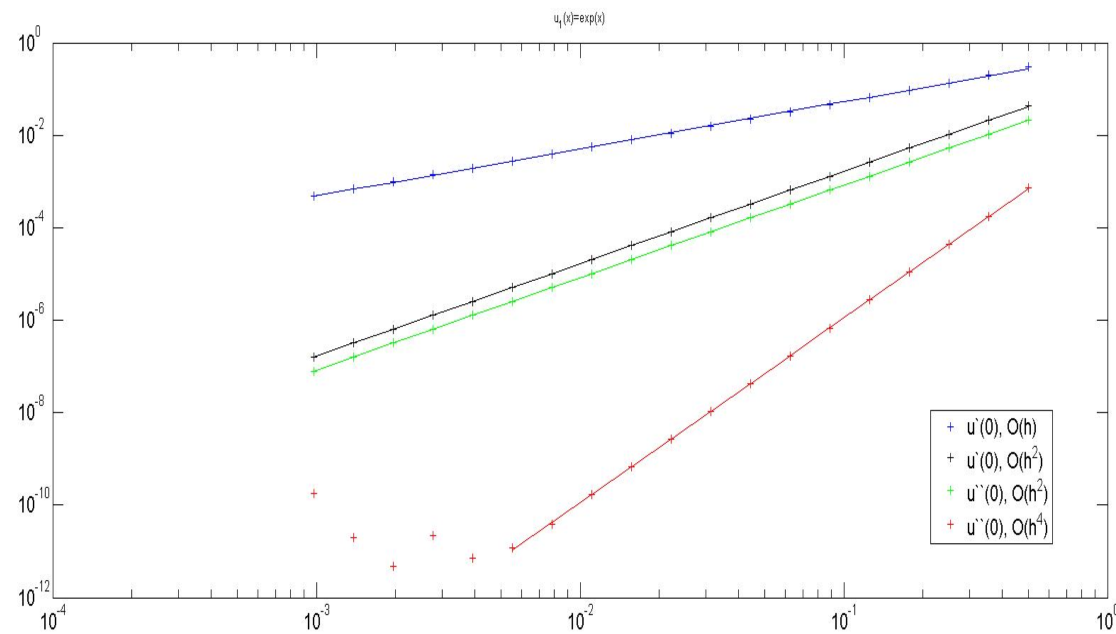

Figure 1: Convergence plot for $u_1 = e^x$ in log-log scale, where $\log(\text{LTE})$ is plotted against $\log(h)$.

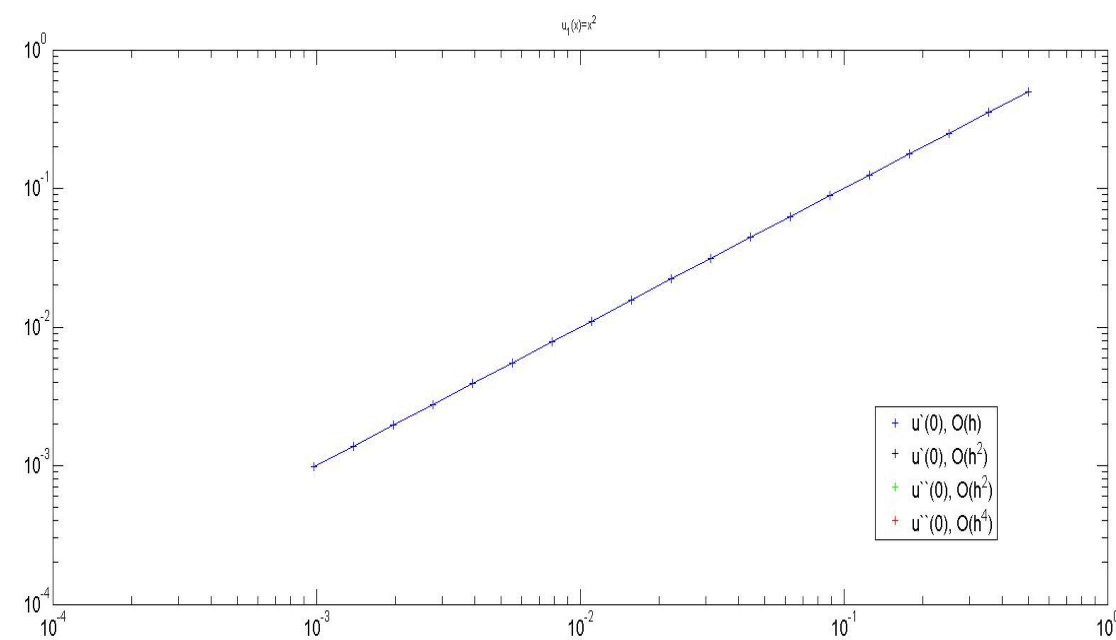

Figure 2: Convergence plot for $u_2 = x^2$ in log-log scale, where $\log(\text{LTE})$ is plotted against $\log(h)$.