MATH 578, Assignment 1
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Problem 1

**Part A, using an equidistant grid:**

Method: In order to test the numerical order of accuracy of the Lagrange interpolation technique, we must be able to compute the $L_\infty$-norm of the error for various $h$, *while keeping the degree of the interpolating polynomial constant.* Since the degree of the polynomial ($= N$) is directly related to the number of points on the grid ($= n$), via the relation $N = n - 1$, this means that the only way to change $h$ is to vary the size of the interval $[a, b]$.

For example: Let $N = 2$, so the grid must have $n = 3$ points. If we want to get 10 points on the final log-log plot, we can choose $h = \{2^{-0}, 2^{-1}, 2^{-2}, \ldots, 2^{-10}\}$, and make the code iterate over the size of the interval, ie:
- When $h = 2^{-0} = 1$, the grid comprises the points: $\{0, 1, 2\}$ (So $a = 0$ and $b = 2$).
- When $h = 2^{-1}$, the grid comprises the points: $\{0, \frac{1}{2}, 1\}$ (So $a = 0$ and $b = 1$).
- etc.

**1.** When $f(x) = \sin(x)$, the theorem tells us that the theoretical upper bound is:

$$\frac{h^n}{4n}||f^{(n)}||_\infty \leq \frac{h^n}{4n} \cdot 1 = O(h^n) \tag{1}$$

since $f^{(n)}(x) = \pm\cos(x)$ or $\pm\sin(x)$.

Results: The same behaviour is observed on each graph (see Figures 1, 2 and 3):

- when $h$ is large, the error is $O(10^0)$: the approximation is really poor. However that error cannot exceed 2, since $-1 \leq \sin(x) \leq +1$.
- as $h$ decreases, the approximation becomes better. The measured error lies close to the theoretical error, revealing that using an equidistant grid is not an optimal choice of abscissa points.
- for $N = 7, 16$, as $h$ becomes very small, the measured error exceeds the theoretical error, and they agree for $N = 2$. In this range, the results are no longer reliable, since the numbers involved are of $O(\epsilon_{\text{machine}}) \approx 10^{-16}$.

To determine the order of convergence, we refer to Table 1. The third column computes the slope between two consecutive points, ie:

$$\text{slope} = \frac{\log(\frac{||E(i-1)||_\infty}{||E(i)||_\infty})}{\log(\frac{h(i-1)}{h(i)})} = \frac{\log(\frac{||E(i-1)||_\infty}{||E(i)||_\infty}}{\log(2)} \tag{2}$$

Based on the graphs, we see that not all points are reliable: In general, one should only consider those values as the error approaches $\epsilon_{\text{machine}}$ (in boldface characters in the table).

Overall, those values seem to agree with the theorem, since we expect: $\log(||E||_\infty) \leq \log(\frac{h^n}{4n}) = C + n \cdot \log(h)$. And indeed, for
- $n = 3$, the convergence is $O(h^3)$,
- $n = 8$, the convergence is roughly of $O(h^8)$,
- $n = 17$, there are not enough points to assert what the order of convergence is. To make it clear that it is of $O(h^{17})$, one would have to sample the part between $h = 0.791 \ldots 0.391$ with more points, and perform a similar analysis.

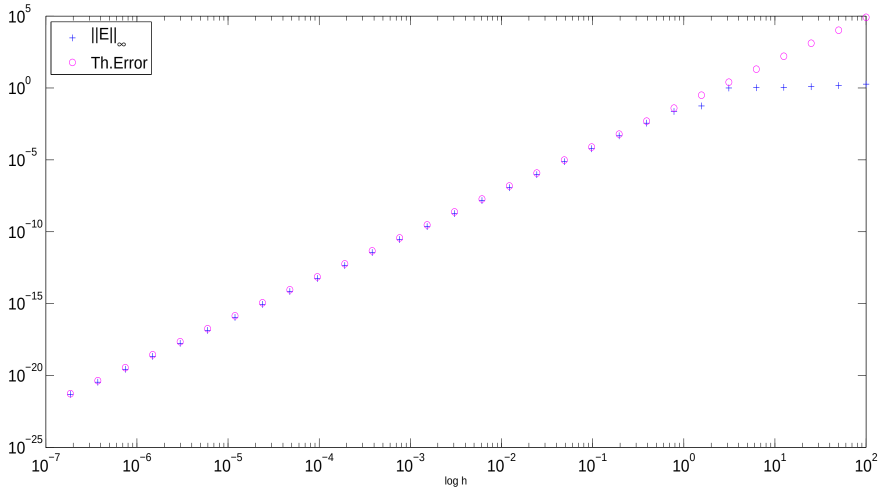


Figure 1: LOG-LOG PLOT OF $||E||_\infty$ AND THE THEORETICAL ERROR VS. $h$, FOR $f(x) = \sin(x)$ WITH $N = 2$.

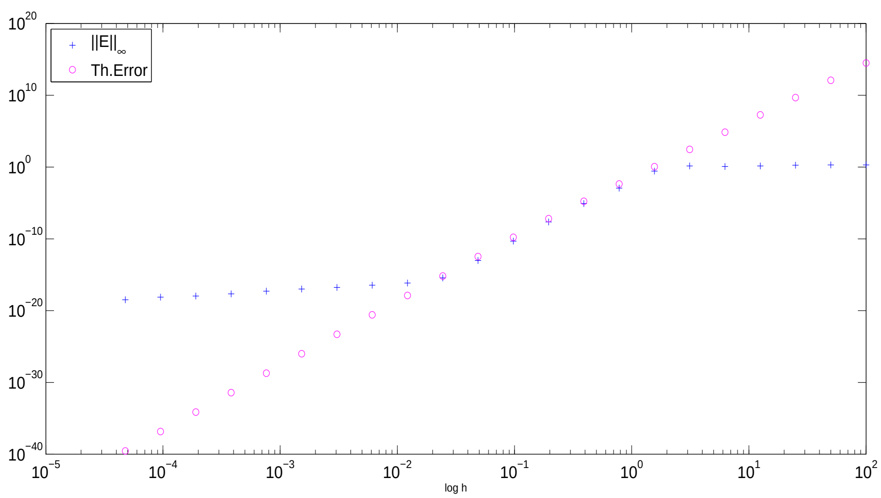


Figure 2: LOG-LOG PLOT OF $||E||_\infty$ AND THE THEORETICAL ERROR VS. $h$, FOR $f(x) = \sin(x)$ WITH $N = 7$.

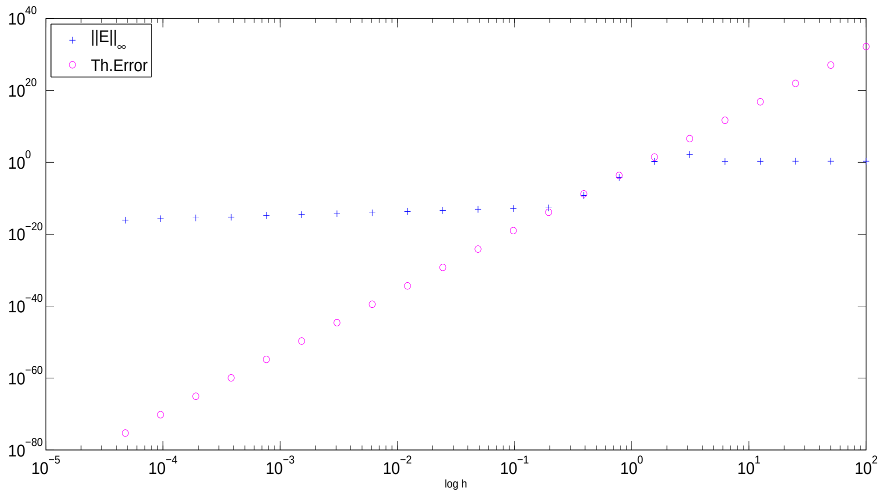


Figure 3: Log-log plot of $||E||_\infty$ and the Theoretical Error vs. $h$, for $f(x) = \sin(x)$ with $N = 16$.

| | $N=2$ | | | $N=7$ | | | $N=16$ | |
|---|---|---|---|---|---|---|---|---|
| $h(i)$ | $\|\|E(i)\|\|_\infty$ | slope | $h(i)$ | $\|\|E(i)\|\|_\infty$ | slope | $h(i)$ | $\|\|E(i)\|\|_\infty$ | slope |
| 1.00E+02 | 1.86E+00 | 1.0000 | 1.00E+02 | 1.98E+00 | 1.0000 | 1.00E+02 | 2.00E+00 | 1.0000 |
| 5.00E+01 | 1.49E+00 | 0.3247 | 5.00E+01 | 1.99E+00 | -0.0064 | 5.00E+01 | 2.00E+00 | 0.0007 |
| 2.50E+01 | 1.24E+00 | 0.2621 | 2.50E+01 | 1.79E+00 | 0.1585 | 2.50E+01 | 2.00E+00 | 0.0000 |
| 1.25E+01 | 1.11E+00 | 0.1614 | 1.25E+01 | 1.43E+00 | 0.3239 | 1.25E+01 | 1.86E+00 | 0.1009 |
| 6.25E+00 | 1.04E+00 | 0.0893 | 6.25E+00 | 1.21E+00 | 0.2416 | 6.25E+00 | 1.49E+00 | 0.3253 |
| 3.13E+00 | 1.00E+00 | 0.0593 | 3.13E+00 | 1.42E+00 | -0.2356 | 3.13E+00 | 1.34E+02 | -6.4925 |
| 1.56E+00 | 5.65E-02 | 4.1459 | 1.56E+00 | 2.74E-01 | 2.3755 | 1.56E+00 | 1.74E+00 | 6.2691 |
| 7.81E-01 | 2.32E-02 | 1.2810 | 7.81E-01 | 1.18E-03 | 7.8630 | 7.81E-01 | 5.64E-05 | **14.9090** |
| 3.91E-01 | 3.58E-03 | 2.6979 | 3.91E-01 | 8.23E-06 | **7.1591** | 3.91E-01 | 6.10E-10 | **16.4961** |
| 1.95E-01 | 4.70E-04 | 2.9289 | 1.95E-01 | 2.28E-08 | **8.4985** | 1.95E-01 | 2.04E-13 | 11.5447 |
| 9.77E-02 | 5.95E-05 | **2.9825** | 9.77E-02 | 4.82E-11 | **8.8836** | 9.77E-02 | 1.23E-13 | 0.7347 |
| 4.88E-02 | 7.46E-06 | **2.9956** | 4.88E-02 | 9.61E-14 | **8.9698** | 4.88E-02 | 8.59E-14 | 0.5149 |
| 2.44E-02 | 9.33E-07 | **2.9989** | 2.44E-02 | 3.61E-16 | **8.0570** | 2.44E-02 | 4.33E-14 | 0.9889 |
| 1.22E-02 | 1.17E-07 | **2.9997** | 1.22E-02 | 6.94E-17 | 2.3785 | 1.22E-02 | 2.27E-14 | 0.9331 |
| 6.10E-03 | 1.46E-08 | **2.9999** | 6.10E-03 | 3.47E-17 | 1.0000 | 6.10E-03 | 8.51E-15 | 1.4144 |
| 3.05E-03 | 1.82E-09 | **3.0000** | 3.05E-03 | 1.73E-17 | 1.0000 | 3.05E-03 | 4.77E-15 | 0.8335 |
| 1.53E-03 | 2.28E-10 | **3.0000** | 1.53E-03 | 1.04E-17 | 0.7370 | 1.53E-03 | 2.70E-15 | 0.8245 |
| 7.63E-04 | 2.85E-11 | **3.0000** | 7.63E-04 | 5.20E-18 | 1.0000 | 7.63E-04 | 1.49E-15 | 0.8536 |
| 3.81E-04 | 3.56E-12 | **3.0000** | 3.81E-04 | 2.17E-18 | 1.2630 | 3.81E-04 | 5.53E-16 | 1.4330 |
| 1.91E-04 | 4.45E-13 | **3.0000** | 1.91E-04 | 1.08E-18 | 1.0000 | 1.91E-04 | 3.36E-16 | 0.7171 |
| 9.54E-05 | 5.56E-14 | **3.0000** | 9.54E-05 | 7.59E-19 | 0.5146 | 9.54E-05 | 1.90E-16 | 0.8216 |
| 4.77E-05 | 6.96E-15 | **3.0000** | 4.77E-05 | 3.25E-19 | 1.2224 | 4.77E-05 | 8.29E-17 | 1.1971 |

Table 1: Order of accuracy of the method, for $f(x) = \sin(x)$

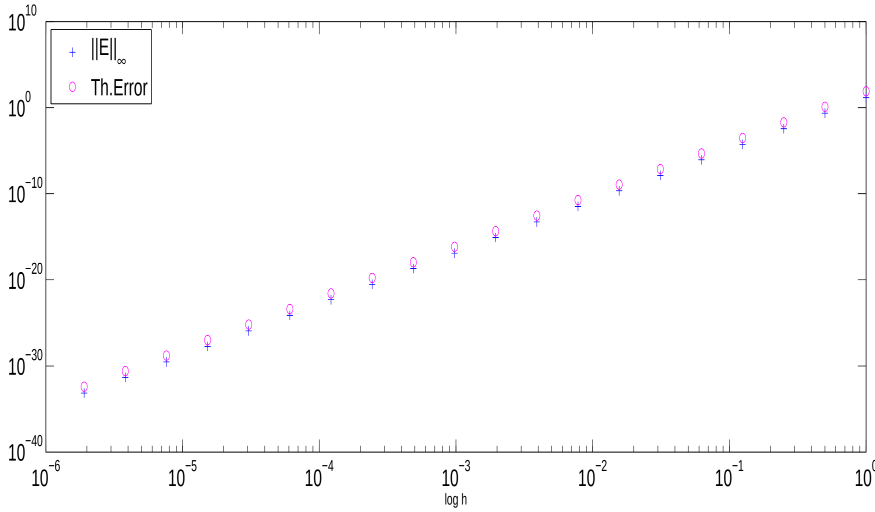


Figure 4: LOG-LOG PLOT OF $||E||_\infty$ AND THE THEORETICAL ERROR VS. $h$, FOR $f(x) = x^6$ WITH $N = 2$.

| $h(i)$ | $N=2$ $\vert\vert E(i)\vert\vert_\infty$ | slope |
|---|---|---|
| 1.00E+00 | 1.47E+01 | 1.00 |
| 5.00E-01 | 2.30E-01 | 6.00 |
| 2.50E-01 | 3.59E-03 | 6.00 |
| 1.25E-01 | 5.62E-05 | 6.00 |
| 6.25E-02 | 8.77E-07 | 6.00 |
| 3.13E-02 | 1.37E-08 | 6.00 |
| 1.56E-02 | 2.14E-10 | 6.00 |
| 7.81E-03 | 3.35E-12 | 6.00 |
| 3.91E-03 | 5.23E-14 | 6.00 |
| 1.95E-03 | 8.17E-16 | 6.00 |
| 9.77E-04 | 1.28E-17 | 6.00 |

Table 2: ORDER OF ACCURACY OF THE METHOD, FOR $f(x) = x^6$

**2.** When $f(x) = x^6$, if (like me) you decreased the size of the interval while maintaining $a$ constant (I chose $a = 0$), then the theorem tells us that the theoretical upper bound is:

$$\frac{h^n}{4n}||f^{(n)}||_\infty \leq \frac{h^n}{4n} \cdot (||\frac{6! \cdot x^{6-n}}{(6-n)!}||_\infty) = \frac{h^n}{4n} \cdot (\frac{6! \cdot b^{6-n}}{(6-n)!}) = \frac{h^n}{4n} \cdot (\frac{6! \cdot (a + (n-1) \cdot h)^{6-n}}{(6-n)!}) = O(h^6) \tag{3}$$

ie: the convergence is expected to be $O(h^6)$ no matter what the degree of the polynomial is!

<u>Results:</u> All 3 graphs display a straight line for the measured error, which has to be interpreted differently depending on $N$:

- For $N = 2$, Figure 4 and Table 2 reveal that $||E||_\infty$ follows exactly the predictions of the theorem. Data beyond $||E||_\infty \approx \epsilon_{\text{machine}}$ should be ignored.
- For $N = 7, 16$, the estimated error does not appear on the graph. Indeed, it is identically 0, since for $N > 6$, $f^{(N)} \equiv 0$. Equivalently, since we're trying to approximate $x^6$ with a polynomial of degree less or equal to 7 or 16, the Lagrange interpolation formula naturally returns $x^6$ itself. Numerically, the resulting error is thus expected to be very small, so small that it is below machine precision. Yet, for large values of $h$ the computer keeps returning values, which should therefore be discarded.

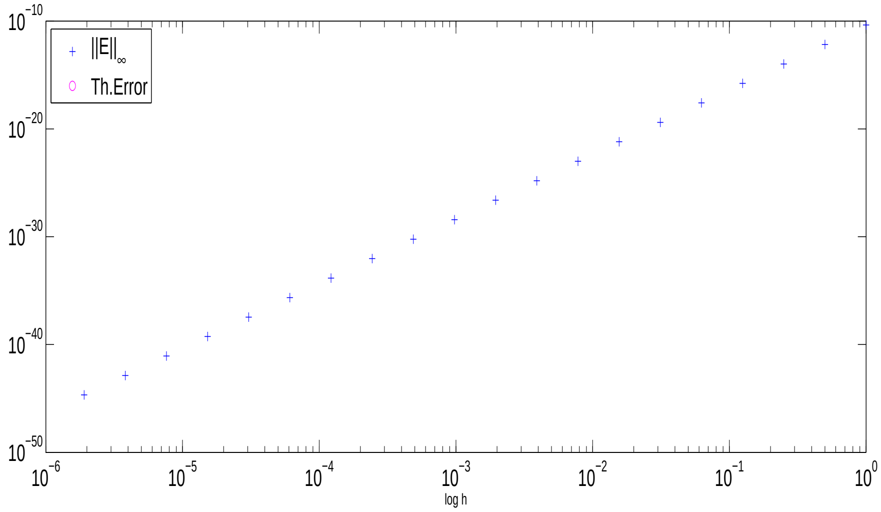


Figure 5: LOG-LOG PLOT OF $||E||_\infty$ AND THE THEORETICAL ERROR VS. $h$, FOR $f(x) = x^6$ WITH $N = 7$.

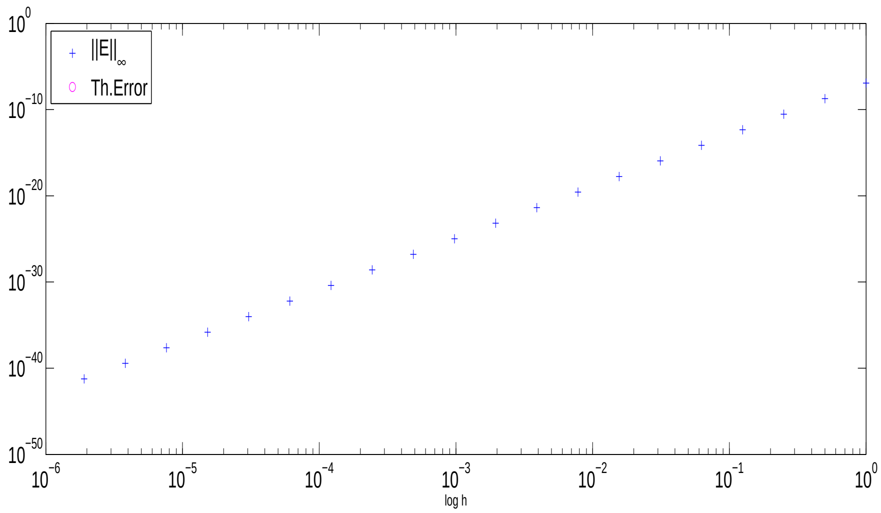


Figure 6: LOG-LOG PLOT OF $||E||_\infty$ AND THE THEORETICAL ERROR VS. $h$, FOR $f(x) = x^6$ WITH $N = 16$.

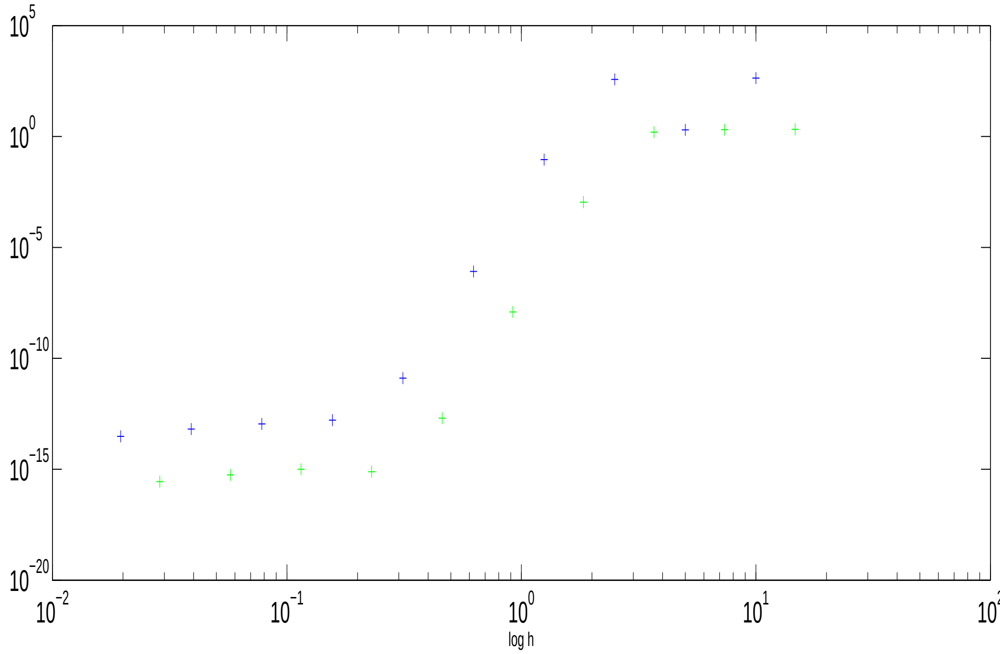


Figure 7: Log-log plot of $||E||_\infty$ for a Chebychev grid (in green) and $||E||_\infty$ for an equidistant grid (in blue) vs. $h$, for $f(x) = \sin(x)$ with $N = 16$.

**Part B, using a Chebyshev grid**

Method: Following the same procedure as in the Part A: $h$ is varied by shrinking the interval over which the function is approximated. The code is actually the same as for Part A, except for the building of the grid: the abscissa points are chosen to be the zeros of the appropriate Chebychev polynomial. Note that the $h$ used in the log-log plot is $h = \max_{i=2\ldots n} |x_i - x_{i-1}|$.

Results: Qualitatively, the resulting graphs and tables for $f(x) = \sin(x)$ and $f(x) = x^6$ are extremely similar to those obtained using an equidistant grid, and should be interpreted in the same way.

Quantitatively though, the remarkable difference is that $||E||_\infty$ for the Chebychev grid lies lower than the one for the equidistant grid. This becomes more appearant as the degree of the polynomial increases. Figure 7 compares the two errors for $N = 16$ with $f(x) = \sin(x)$. This agrees with the theory, which states that Chebychev points are an optimal choice of abscissa points for the Lagrange interpolation technique.

## Problem 2

•*Prove that if $u, v \in L^2$, then $(u * v)(x) = (v * u)(x)$.*

Solution:

$$(u * v)(x) = \int_{-\infty}^{+\infty} u(x-y)v(y)\mathrm{d}y \overbrace{=}^{z=x-y} \int_{+\infty}^{-\infty} u(z)v(x-z)(-\mathrm{d}z) = \int_{-\infty}^{+\infty} u(z)v(x-z)\mathrm{d}z = (v * u)(x)$$

•*Compute $\widehat{u_{(p)}}(\xi) \equiv F(u * u * \ldots * u)(\xi)$ for the function $u(x) = \begin{cases} \frac{1}{4} & -2 \leq x < 0 \\ -\frac{1}{4} & 0 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$.*

Solution: By induction,
$\underline{p = 1}$:

$$\begin{aligned}
\widehat{u}(\xi) &= \int_{\mathbb{R}} e^{-i\xi x}\, u(x)\mathrm{d}x = \frac{1}{4}\left(\int_{-2}^{0} e^{-i\xi x}\mathrm{d}x - \int_{0}^{2} e^{-i\xi x}\mathrm{d}x\right) = \frac{1}{4}\left(\frac{e^{-i\xi x}}{-i\xi}|_{-2}^{0} - \frac{e^{-i\xi x}}{-i\xi}|_{0}^{2}\right) \\
&= \frac{i}{4\xi}\left(1 - e^{+2i\xi} - e^{-2i\xi} + 1\right) = \frac{i}{4\xi}\left(1 - e^{+2i\xi} - e^{-2i\xi} + 1\right) = \frac{i}{\xi}\left(\frac{e^{+2i\xi} - 2 + e^{-2i\xi}}{-4}\right) \\
&= \frac{i}{\xi}\left(\frac{e^{+i\xi} - e^{-i\xi}}{2i}\right)^2 = \frac{i}{\xi}\sin^2(\xi)
\end{aligned}$$

$\underline{p = n}$: Assume that $\widehat{u_{(n)}}(\xi) = \left(\frac{i}{\xi}\sin^2(\xi)\right)^n$.

$\underline{p = n+1}$:

$$\begin{aligned}
\widehat{u_{(n+1)}}(\xi) &\equiv F(\underbrace{u * u * \ldots * u}_{n+1 \text{ times}})(\xi) \\
&= F(\underbrace{u * u * \ldots * u}_{n \text{ times}})(\xi) \cdot F(u)(\xi) && \text{by the Convolution Theorem} \\
&= \left(\frac{i}{\xi}\sin^2(\xi)\right)^n \cdot \frac{i}{\xi}\sin^2(\xi) && \text{by the previous steps} \\
&= \left(\frac{i}{\xi}\sin^2(\xi)\right)^{n+1}
\end{aligned}$$

Therefore, for all $p \in \mathbb{N}$, $\widehat{u_{(p)}}(\xi) = \left(\frac{i}{\xi}\sin^2(\xi)\right)^p$.

•*Compute the Fourier transform of the Gaussian function $u(x; s) \equiv \frac{1}{\sqrt{2\pi s^2}}\exp\left(\frac{-x^2}{2s^2}\right)$.*
Solution:

$$\widehat{u}(\xi) = \int_{-\infty}^{+\infty} \exp(-i\xi x) \cdot \frac{1}{\sqrt{2\pi s^2}}\exp\left(-\frac{x^2}{2s^2}\right)\mathrm{d}x = \frac{1}{\sqrt{2\pi s^2}}\int_{-\infty}^{+\infty}\exp\left(-\frac{x^2}{2s^2} - i\xi x\right)\mathrm{d}x$$

Rewriting the exponent

$$\begin{aligned}
-\frac{x^2}{2s^2} - i\xi x &= -\frac{1}{2}\left(\frac{x^2}{s^2} + 2i\xi x + (i\xi s)^2 - (i\xi s)^2\right) \\
&= -\frac{1}{2}\left(\frac{x}{s} + i\xi s\right)^2 - \frac{\xi^2 s^2}{2}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \qquad \widehat{u}(\xi) &= \frac{\exp\left(-\frac{\xi^2 s^2}{2}\right)}{\sqrt{2\pi s^2}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{x}{s}+i\xi s\right)^2\right) \mathrm{d}x && \text{let } y = \frac{x}{s}+i\xi s \quad \Leftrightarrow \quad s\mathrm{d}y = \mathrm{d}x \\
&= \frac{\exp\left(-\frac{\xi^2 s^2}{2}\right)}{\sqrt{2\pi s^2}} \cdot s \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}y^2\right) \mathrm{d}y && \text{using } \int_{-\infty}^{+\infty} \exp\left(-ax^2\right) \mathrm{d}x = \sqrt{\frac{\pi}{a}} \\
&= \frac{\exp\left(-\frac{\xi^2 s^2}{2}\right)}{\sqrt{2\pi s^2}} \cdot s \cdot \sqrt{2\pi} \\
&= \exp\left(-\frac{\xi^2 s^2}{2}\right)
\end{aligned}$$

•*Prove that the convolution of $p$ Gaussian functions (with variances $s_1^2, s_2^2, \ldots, s_p^2$) is a Gaussian function with $s = \sqrt{\sum s_i^2}$*

Solution: It is proved that the convolution of 2 Gaussian functions with variances $s_1^2, s_2^2$ is a Gaussian function with variance $s = \sqrt{s_1^2 + s_2^2}$. The claim for $n$ Gaussian functions follows by induction.

**Using the Convolution Theorem:**

$$\begin{aligned}
(u_{s_1} * u_{s_2})(x) &= F^{-1}\left(F(u_{s_1} * u_{s_2})(\xi)\right)(x) \\
&= F^{-1}\left(F(u_{s_1})(\xi) \cdot F(u_{s_2})(\xi)\right)(x) && \text{by the Convolution Theorem} \\
&= F^{-1}\left(\exp\left(-\frac{\xi^2 s_1^2}{2}\right) \cdot \exp\left(-\frac{\xi^2 s_2^2}{2}\right)\right)(x) && \text{by the previous question} \\
&= F^{-1}\left(\exp\left(-\frac{\xi^2\left(\sqrt{s_1^2+s_2^2}\right)^2}{2}\right)\right)(x) \\
&= \frac{1}{\sqrt{2\pi\left(s_1^2+s_2^2\right)}} \exp\left(\frac{-x^2}{2\left(s_1^2+s_2^2\right)}\right) && \text{by the previous question} \\
&=: u(x, \sqrt{s_1^2+s_2^2})
\end{aligned}$$

**Or working only in real space:**

$$\begin{aligned}
\int_{-\infty}^{+\infty} u(x-y; s_1)\, u(y; s_2) \mathrm{d}y &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi s_1^2}} \exp\left(-\frac{(x-y)^2}{2s_1^2}\right) \frac{1}{\sqrt{2\pi s_2^2}} \exp\left(-\frac{y^2}{2s_2^2}\right) \mathrm{d}y \\
&= \frac{1}{2\pi s_1 s_2} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{2s_1^2} - \frac{y^2}{2s_2^2}\right) \mathrm{d}y
\end{aligned}$$

Rewriting the exponent:

$$\begin{aligned}
-\frac{(x-y)^2}{2s_1^2} - \frac{y^2}{2s_2^2} &= -\frac{(x^2-2xy+y^2)s_2^2 + y^2 s_1^2}{2s_1^2 s_2^2} \\
&= -\frac{x^2 s_2^2 - 2xys_2^2 + y^2 s_2^2 + y^2 s_1^2}{2s_1^2 s_2^2} \\
&= -\frac{1}{2s_1^2 s_2^2} \cdot \left(y^2(s_1^2+s_2^2) - 2xys_2^2 + x^2 s_2^2\right) \\
&= -\frac{1}{2s_1^2 s_2^2} \cdot \left(y^2(s_1^2+s_2^2) + (2xys_1^2 - 2xys_1^2) - 2xys_2^2 + (x^2 s_1^2 - x^2 s_1^2) + x^2 s_2^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2s_1^2 s_2^2} \cdot \left(y^2(s_1^2+s_2^2) - 2xy(s_1^2+s_2^2) + x^2(s_1^2+s_2^2) + 2xys_1^2 - x^2 s_1^2\right) \\
&= -\frac{1}{2s_1^2 s_2^2} \cdot \left((y-x)^2(s_1^2+s_2^2) + 2xys_1^2 - x^2 s_1^2\right) \qquad\qquad \text{let } z = y - x \\
&= -\frac{1}{2s_1^2 s_2^2} \cdot \left(z^2(s_1^2+s_2^2) + 2x(z+x)s_1^2 - x^2 s_1^2\right) \\
&= -\frac{1}{2s_1^2 s_2^2} \cdot \left(z^2(s_1^2+s_2^2) + 2xzs_1^2 + x^2 s_1^2\right) \\
&= -\frac{1}{2s_1^2 s_2^2} \cdot \left(z^2(s_1^2+s_2^2) + 2\cdot z\sqrt{s_1^2+s_2^2} \cdot \frac{xs_1^2}{\sqrt{s_1^2+s_2^2}} + \left(\frac{x^2 s_1^4}{s_1^2+s_2^2} - \frac{x^2 s_1^4}{s_1^2+s_2^2}\right) + x^2 s_1^2\right) \\
&= -\frac{1}{2s_1^2 s_2^2} \cdot \left(\left(z^2(s_1^2+s_2^2) + 2\cdot z\sqrt{s_1^2+s_2^2} \cdot \frac{xs_1^2}{\sqrt{s_1^2+s_2^2}} + \frac{x^2 s_1^4}{s_1^2+s_2^2}\right) + \left(x^2 s_1^2 - \frac{x^2 s_1^4}{s_1^2+s_2^2}\right)\right) \\
&= -\frac{1}{2s_1^2 s_2^2} \cdot \left(\left(z\sqrt{s_1^2+s_2^2} + \frac{xs_1^2}{\sqrt{s_1^2+s_2^2}}\right)^2 + \frac{x^2 s_1^2 s_2^2}{s_1^2+s_2^2}\right) \qquad\qquad \text{let } a = z\sqrt{s_1^2+s_2^2} + \frac{xs_1^2}{\sqrt{s_1^2+s_2^2}} \\
&= -\frac{1}{2s_1^2 s_2^2}\left(a^2 + \frac{x^2 s_1^2 s_2^2}{s_1^2+s_2^2}\right) \\
&= -\frac{a^2}{2s_1^2 s_2^2} - \frac{x^2}{2(s_1^2+s_2^2)}
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \qquad \int_{-\infty}^{+\infty} u(x-y;s_1)u(y;s_2)\mathrm{d}y &= \frac{1}{2\pi s_1 s_2}\int_{-\infty}^{+\infty} \exp\left(-\frac{a^2}{2s_1^2 s_2^2} - \frac{x^2}{2(s_1^2+s_2^2)}\right)\frac{\mathrm{d}a}{\sqrt{s_1^2+s_2^2}} \\
&= \frac{1}{2\pi s_1 s_2\sqrt{s_1^2+s_2^2}}\exp\left(-\frac{x^2}{2(s_1^2+s_2^2)}\right)\int_{-\infty}^{+\infty}\exp\left(-\frac{a^2}{2s_1^2 s_2^2}\right)\mathrm{d}a \\
&= \frac{1}{2\pi s_1 s_2\sqrt{s_1^2+s_2^2}}\exp\left(-\frac{x^2}{2(s_1^2+s_2^2)}\right)\cdot\sqrt{2\pi s_1^2 s_2^2} \\
&= \frac{1}{\sqrt{2\pi(s_1^2+s_2^2)}}\exp\left(-\frac{x^2}{2(s_1^2+s_2^2)}\right) = u(x,\sqrt{s_1^2+s_2^2})
\end{aligned}
$$