Spectral theory, geometry and dynamical systems

Dmitry Jakobson

8th January 2010

- *M* is *n*-dimensional compact connected manifold, $n \geq 2$. *g* is a Riemannian metric on *M*: for any $U, V \in T_x M$, their inner product is $g(U, V)$. $g(\partial/\partial x_i, \partial/\partial x_j) := g_{ij}$.
- *g* defines analogs of div and grad. The *Laplacian* Δ of a function *f* is given by

$$\Delta f = \mathrm{div}(\mathrm{grad} f).$$

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- Example 1: $\mathbf{R}^2$.

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Periodic eigenfunctions on the 2-torus $\mathbf{T}^2$: $f(x \pm 2\pi, y \pm 2\pi) = f(x, y)$. They are

$$\sin(m \cdot x + n \cdot y), \cos(m \cdot x + n \cdot y), \;\; \lambda = m^2 + n^2.$$

- Fact: any square-integrable function $F(x, y)$ on $\mathbf{T}^2$ (s.t. $\int_{\mathbf{T}^2} |F(x, y)|^2 dxdy < \infty$), can be expanded into Fourier series,

$$F = \sum_{m,n=-\infty}^{+\infty} a_{m,n} \sin(mx + ny) + b_{m,n} \cos(mx + ny).$$

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- Example 2: sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Spherical coordinates: $(\phi, \theta) \in [0, \pi] \times [0, 2\pi]$, where $x = \sin\phi\cos\theta, y = \sin\phi\sin\theta, z = \cos\phi$.

$$\Delta f = \frac{1}{\sin^2\phi} \cdot \frac{\partial^2 f}{\partial\theta^2} + \frac{\cos\phi}{\sin\phi} \cdot \frac{\partial f}{\partial\phi} + \frac{\partial^2 f}{\partial\phi^2}$$

- Eigenfunctions are called *spherical harmonics*:

$$Y_l^m(\phi, \theta) = P_l^m(\cos\phi)(a\cos(m\theta) + b\sin(m\theta)).$$

Here $\lambda = l(l+1)$; $P_l^m, |m| \le l$ is *associated Legendre function*,

$$P_l^m(x) = \frac{(-1)^m}{2^l \cdot l!}(1 - x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}}\left((x^2 - 1)^l\right).$$

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- The same is true on any compact *M*. Eigenvalues of Δ:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

  Any square-integrable function *F* on *M* can be expanded in a series of eigenfunctions ϕ_i of Δ.
- Similar results hold for domains with boundary (you need to specify boundary conditions).
- Standard boundary conditions: *Dirichlet* (ϕ vanishes on the boundary); *Neumann* (normal derivative of ϕ vanishes on the boundary).
- Example: $M = [0, \pi]$. $\sin x$ satisfies Dirichlet b.c; $\cos x$ satisfies Neumann b.c.

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- Solving partial differential equations like *heat equation* $\partial u(x,t)/\partial t = c \cdot \Delta_x u(x,t)$ and *wave equation* $\partial^2 u(x,t)/\partial t^2 = c \cdot \Delta_x u(x,t)$.
- Stationary solutions of *Schrödinger equation* or "pure quantum states."
- *Inverse problems*: suppose you know some eigenvalues and eigenfunctions; describe the domain S (related problems appear in radar/remote sensing, x-ray/MRI, oil/gas/metal exploration etc).

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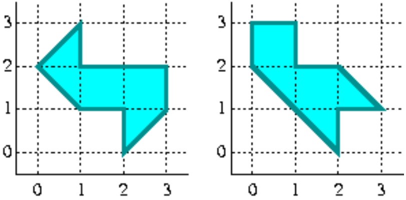
- Determine *the smallest* $\lambda > 0$ for a given surface S (its "bass note"), and other small eigenvalues.
- Mark Kac: "Can you hear the shape of a drum?" Can you determine the domain if you know its spectrum (all the λ_i-s)?
  Answer: No! Two different domains S can have the same spectrum (sound the same). Example below is due to Gordon, Webb and Wolpert.



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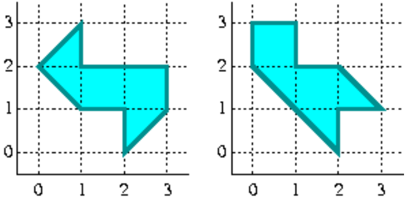

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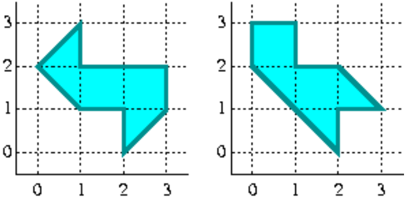

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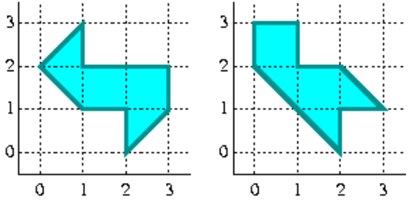

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$$N(t^2) = \#\{(m,n) : m^2 + n^2 < t^2\} =$$

$$\#\{(m,n) : \sqrt{m^2 + n^2} < t\}.$$

How many lattice points are inside the circle of radius t? Leading term is given by the *area*:

$$N(t^2) = \pi t^2 + R(t), \tag{1}$$

where $R(t)$ is the *remainder*.

- **Question:** How big is $R(t)$? Conjecture (Hardy): for any $\delta > 0$,

$$R(t) < C(\delta) \cdot t^{1/2+\delta}, \qquad as\ t \to \infty.$$

Best known estimate (Huxley, 2003):

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- An analogue of (1) holds for very general domains; it is called *Weyl's law* (Weyl, 1911).
  M is n-dimensional:

$$N(\lambda) = c_n \cdot \mathrm{vol}(M)\lambda^{n/2} + R(\lambda), \quad \textit{R is a remainder}.$$

- It is known (Avakumovic, Levitan, Hörmander) that

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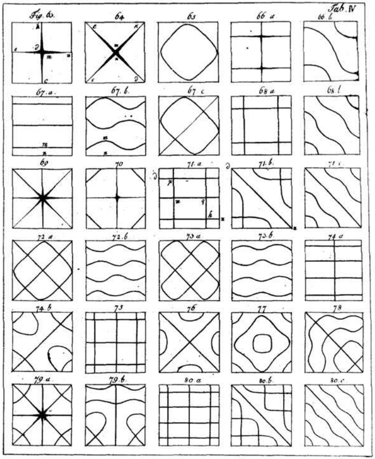
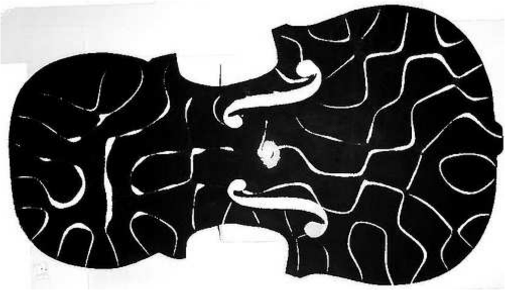
- **Nodal set** $\mathcal{N}(\phi_\lambda) = \{x \in M : \phi_\lambda(x) = 0\}$, codimension 1 is M. On a surface, it's a union of curves.
  First pictures: *Chladni plates.* E. Chladni, 18th century. He put sand on a plate and played with a violin bow to make it vibrate.



- Chladni patterns are still used to tune violins.



- How large are nodal sets, i.e. how large is $\mathrm{vol}_{n-1}(\mathcal{N}(\phi_\lambda))$?
  Dimension $n = 1$: eigenfunction $\sin(nx)$ has $\sim n = \sqrt{\lambda}$ zeros in $[0, 2\pi]$.
- For real-analytic metrics, Donnelly and Fefferman showed that there exists C_1, C_2 (independent of $\lambda > 0$) s.t.

$$C_1\sqrt{\lambda} \leq \mathrm{vol}_{n-1}(\mathcal{N}(\phi_\lambda)) \leq C_2\sqrt{\lambda}$$

  Similar bounds are conjectured for arbitrary smooth metrics in all dimensions (Cheng and Yau).
- In general, the best known result in dimension 2 for arbitrary metrics is

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- *Nodal domain* of ϕ is a connected component of $M \setminus \mathcal{N}(\phi)$. **Courant nodal domain theorem.** ϕ_k has $\leq k + 1$ nodal domains. The constant in the estimate was improved by Pleijel.
- Nazarov and Sodin showed that a *random* spherical harmonic ϕ_n has $\sim cn^2$ nodal domains.
- Eigenfunctions with large λ can have *few* nodal domains: $\phi_n(x, y) = \sin(nx + y)$ on $\mathbf{T}^2$ has *two* nodal domains for all n; $\lambda_n = n^2 + 1$.
- John Toth (McGill) and Zelditch recently studied number of open nodal lines (intersecting the boundary) in certain planar domains, and got a bound $\leq C \cdot \sqrt{n}$.
- Any collection of disjoint closed curves on S^2, invariant under $(x, y, z) \to (-x, -y, -z)$ is equivalent to a nodal set of a spherical harmonic (Eremenko, J, Nadirashvili)

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- **Answer** (Jakobson, Nadirashvili): Not always! For example, on $\mathbf{T}^2$ with a "metric of revolution"

$$(100 + \cos(4x))(dx^2 + dy^2)$$

there exists a sequence ϕ_i such that $\lambda_i \to \infty$, and each ϕ_i has exactly 16 critical points.
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- *Geodesic* is a curve that locally minimizes distance between points lying on it. Examples: straight lines in $\mathbf{R}^n$, great circles on S^n (that's how planes fly on S^2).
- *Geodesic flow* G^t is defined as follows: let $x \in M, v \in T_x M, g(v,v) = 1$. Consider a unique geodesic $\gamma_v(t)$ s.t. $\gamma(0) = x, \gamma'(0) = v$. Then

$$G^t(v) := \gamma_v'(t)$$

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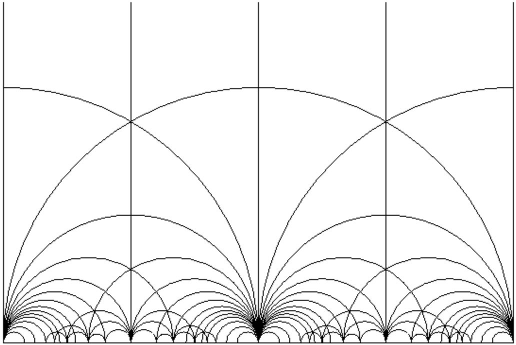
- Question: how G^t behaves for $t \to \infty$: do close trajectories converge (focusing), or diverge (de-focusing)?
- Focusing: S^n. All geodesics (meridians) starting at the north pole at $t = 0$ focus at the south pole at $t = \pi$ (N and S are called *conjugate points*).
- De-focusing: $\mathbf{H}^n$, the n-dimensional hyperbolic space. Geodesics diverge exponentially fast; no conjugate points.



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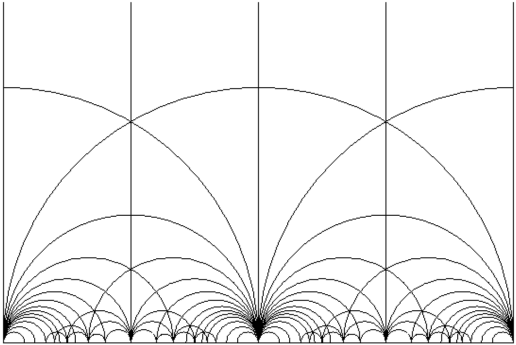

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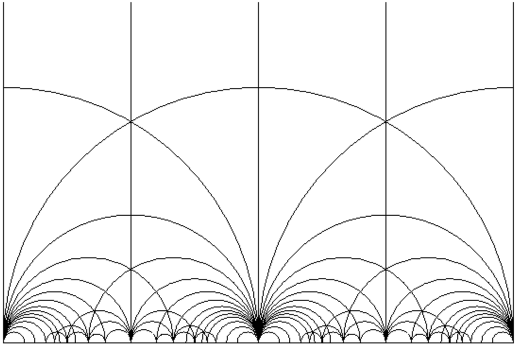

- **Curvature:** S surface in $\mathbf{R}^3$, given by $z = f(x, y)$. Let $\mathrm{grad} f(p) = 0$, then $K = \det(\partial^2 f/\partial x \partial y)$. If $K > 0$, then S is convex or concave at p; if $K < 0$, then S looks like a saddle at p. Also, $\mathrm{vol}(B_S(x_0, r)) = \mathrm{vol}(B_{\mathbf{R}^2}(r)) \left[1 - \frac{K(x_0)r^2}{12} + O(r^4)\right]$.
- *Positive curvature* $\Rightarrow$ focusing; $K = +1$ on S^2.
- Examples of regular geodesic flows: *flat torus* (move along straight lines); and *surfaces of revolution*. The flow on a 2-dimensional surface has 2 *first integrals*. Such flows are called *integrable*.
- *Negative curvature* $\Rightarrow$ de-focusing; $K = -1$ on $\mathbf{H}^2$. If all sectional curvatures on M are negative, then geodesic flow on M is *ergodic:* all invariant sets have measure 0 or 1, almost every geodesic $\gamma(t)$ becomes uniformly distributed (on the unit sphere bundle in TM) as $t \to \infty$.

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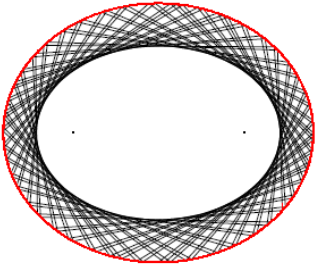
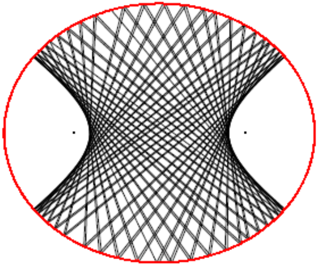

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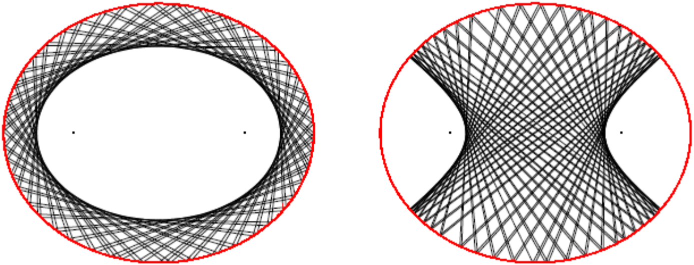

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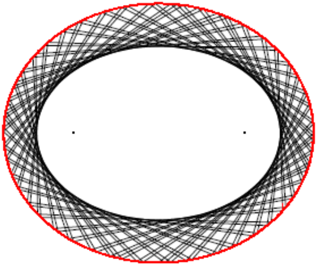
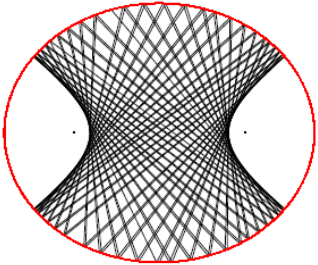
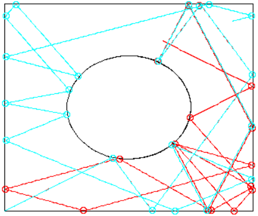
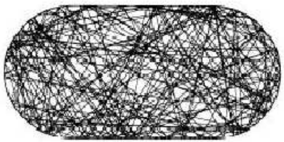


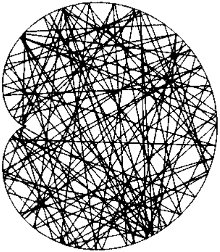


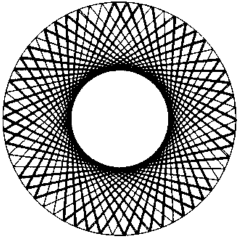
- Ergodic planar billiards: Sinai billiard and Bunimovich stadium





- Other examples of ergodic and integrable billiards: cardioid and circular billiards





- **Question:** Where do eigenfunctions concentrate?
- **Answer:** "Quantum ergodicity" theorem (Shnirelman, Zelditch, Colin de Verdiere): If the geodesic flow is erogdic ("almost all" trajectories become uniformly distributed), then "almost all" eigenfunctions become uniformly distributed.
- Billiard version: Gerard-Leichtnam, Zelditch-Zworski.
- What is the precise meaning? Eigenfunction ϕ_λ describes a quantum particle; $|\phi|^2$ - probability density of that particle. Let $A \subset M$; then $\int_A |\phi|^2$ - probability of finding the particle in A. For almost all ϕ_λ,

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$$\int_0^{2\pi} f(x)(\phi_n(x))^2 \, dx = \frac{1}{2\pi} \int_0^{2\pi} f(x)(1 - \cos(2nx)) \, dx$$

which by Riemann-Lebesgue lemma converges to

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- Conjecture: QUE should hold on compact negatively-curved manifolds (Rudnick-Sarnak). Proved for *arithmetic* hyperbolic manifolds (Lindenstrauss).

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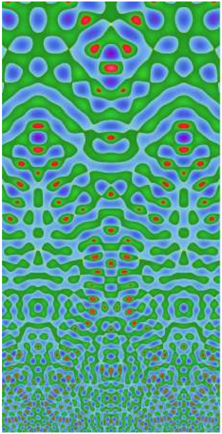
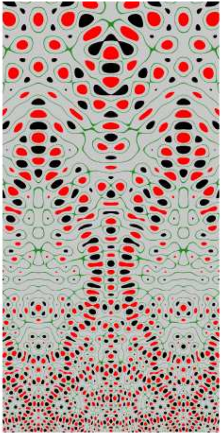
Eigenfunctions of the hyperbolic Laplacian on $\mathbf{H}^2/\mathrm{PSL}(2,\mathbf{Z})$, Hejhal:





- Eigenfunctions on $S^2 = \{x^2 + y^2 + z^2 = 1\}$. Let $\phi_n(x, y, z) = (x + iy)^n$. Then $|\phi_n|^2 = (1 - z^2)^n$. That expression is $= 1$ on the equator $\{z = 0\}$, and decays exponentially fast for $z > 0$ as $n \to \infty$. Therefore, $\phi_n^2 \to \delta_{equator}$ as $n \to \infty$.
- In fact, for any *even* nonnegative function f on S^2, there exists a sequence of spherical harmonics ϕ_n s.t. $\phi_n^2 \to f$ as $n \to \infty$ (Jakobson, Zelditch).

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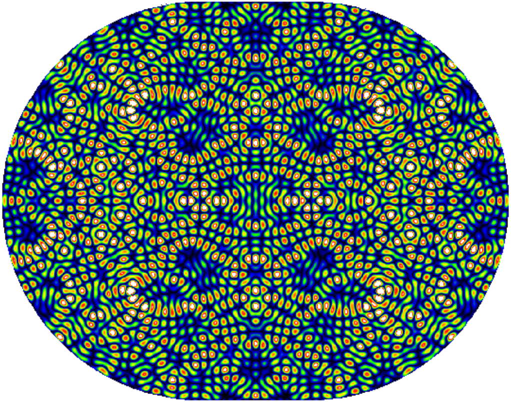
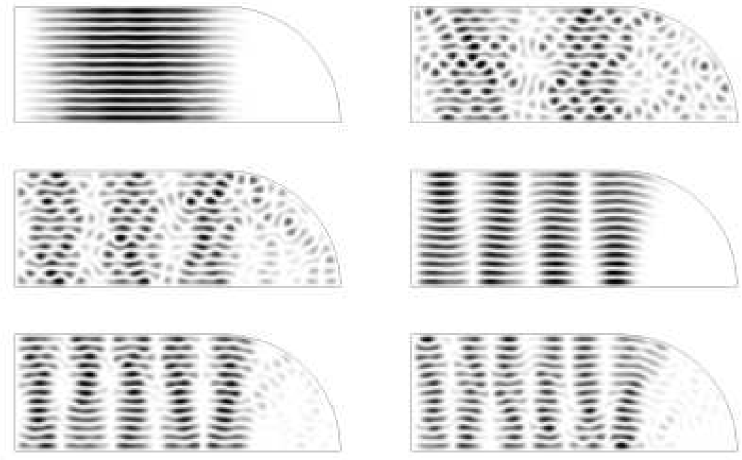
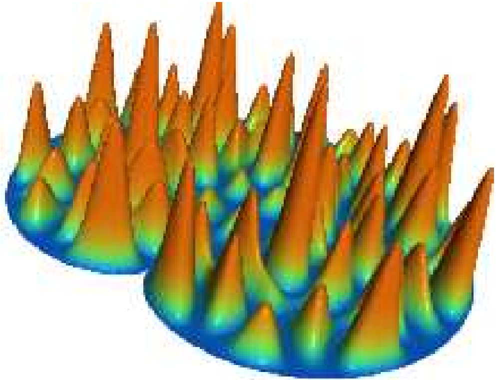
- Billiards: QUE conjectures *does not* hold for the Bunimovich stadium (Hassell). Ergodic eigenfunction:



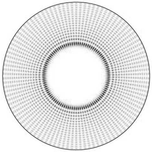
- Other stadium eigenfunctions, including "bouncing ball" eigenfunctions, for which QUE fails (they have density 0 among all eigenfunctions, so QE still holds).

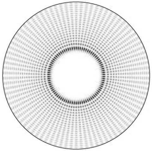


- Ergodic eigenfunction on a cardioid billiard.



- Billiards with caustics: there exist eigenfunctions that concentrate in the region between the caustic and the boundary ("whispering gallery") eigenfunctions. Example: circular billiard.





Another eigenfunction for a circular billiard.