Spectra, dynamical systems, and geometry

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$M = S^1 = \mathbf{R}/(2\pi\mathbf{Z})$ - a circle. $f(x)$ - a periodic function, $f(x + 2\pi) = f(x)$.

Laplacian Δ is the second derivative: $\Delta f = f''$. Eigenfunction $\phi = \phi_\lambda$ with eigenvalue $\lambda \geq 0$ satisfies $\Delta\phi + \lambda\phi = 0$. On the circle, such functions are constants (eigenvalue 0), $\sin(nx)$ and $\cos(nx)$, where $n \in \mathbf{N}$. Eiegvalues:

$$(\sin(nx))'' + n^2 \sin(nx) = 0, (\cos(nx))'' + n^2 \cos(nx) = 0.$$

Fact: every periodic (square-integrable) function can be expanded into *Fourier series*:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)).$$

Can use them to solve *heat* and *wave* equations:

Heat equation describes how heat propagates in a solid body. Temperature $u = u(x, t)$ depends on *position* x and *time* t.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0.$$

The initial temperature is

$$u(x, 0) = f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

One can check that

$$u_0(x,t) = a_0$$

and

$$u_n(x,t) = (a_n \cos(nx) + b_n \sin(nx)) \cdot e^{-n^2 t}, \qquad n \geq 1$$

are solutions to the heat equation. The general solution with initial temperature $f(x)$ is given by

$$\sum_{n=0}^{\infty} u_n(x,t).$$

One can also use Fourier series to solve the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

The "elementary" solutions will be

$$u_n(x,t) = (a_n \cos(nx) + b_n \sin(nx))(c_n \cos(nt) + d_n \sin(nt)).$$

This equation also describes the *vibrating string* (where u is the amplitude of vibration). Musicians playing string instruments (guitar, violin) knew some facts about eigenvalues a long time ago (that's how *music scale* was invented).

In *Quantum mechanics*, eigenfunctions $\sin(nx)$ and $\cos(nx)$ describe "pure states" of a quantum particle that lives on the circle S^1. Their squares $\sin^2(nx)$ and $\cos^2(nx)$ describe the "probability density" of the particle.
The probability $P_n([a, b])$ that the particle $\phi_n(x) = \sqrt{2}\sin(nx)$ lies in the interval $[a, b] \subset [0, 2\pi]$ is equal to

$$\frac{1}{2\pi}\int_a^b |\phi_n(x)|^2 dx.$$

Question: How does $P_n([a, b])$ behave as $n \to \infty$?
Answer:

$$P_n([a, b]) \to \frac{|b-a|}{2\pi}$$

The particle $\phi_n(x)$ becomes *uniformly distributed* in $[0, 2\pi]$, as $n \to \infty$.
This is the **Quantum Unique Ergodicity** theorem on the circle!

Proof: Let $h(x)$ be an *observable* (test function). To "observe" the particle ϕ_n, we compute the integral

$$P_n(h) := \frac{1}{2\pi}\int_0^{2\pi} h(x)\phi_n^2(x)dx = \frac{1}{2\pi}\int_0^{2\pi} h(x)\cdot 2\sin^2(nx)dx.$$

We know that $2\sin^2(nx) = 1 - \cos(2nx)$. The integral is therefore equal to

$$\frac{1}{2\pi}\int_0^{2\pi} h(x)dx - \frac{1}{2\pi}\int_0^{2\pi} h(x)\cos(2nx)dx.$$

The second integral is proportional to the $2n$-th *Fourier coefficient* of the function h, and goes to zero by *Riemann-Lebesgue lemma* in analysis, as $n \to \infty$. Therefore,

$$P_n(h) \to \frac{1}{2\pi}\int_0^{2\pi} h(x)dx, \qquad \textit{as } n \to \infty.$$

To complete the proof, take $h = \chi([a, b])$, the characteristic function of the interval $[a, b]$.
Q.E.D.
What happens in higher dimensions, for example if M is a surface?

Example 1: M is the flat 2-torus $\mathbf{T}^2 = \mathbf{R}^2/(2\pi\mathbf{Z})^2$.

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Periodic eigenfunctions on the 2-torus $\mathbf{T}^2$:
$\phi_\lambda(x \pm 2\pi, y \pm 2\pi) = \phi_\lambda(x, y)$. They are

$$\sin(mx)\sin(ny), \sin(mx)\cos(ny),$$
$$\cos(mx)\sin(ny), \cos(mx)\cos(ny), \quad \lambda = m^2 + n^2.$$

Example 2: *M* is a domain in the *hyperbolic plane* $\mathbf{H}^2$:

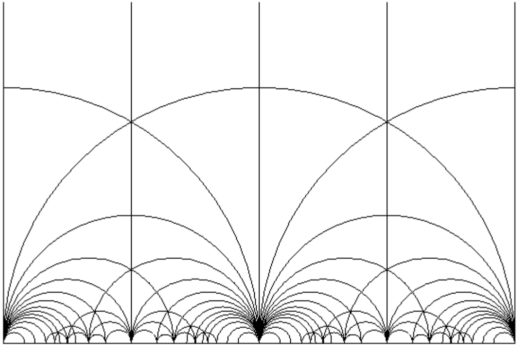
$$\{(x, y) : y > 0\}.$$

The Laplacian is given by

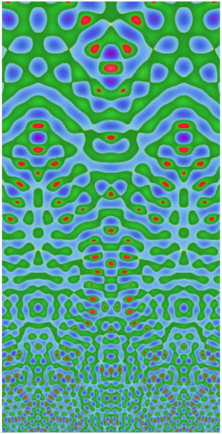
$$\Delta f = y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

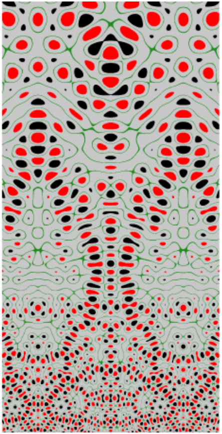
Eigenfunctions are functions on $\mathbf{H}^2$ periodic with respect to several *isometries* of $\mathbf{H}^2$ (motions that preserve lengths in the $\mathbf{H}^2$).

Geodesics are shortest paths from one point to another. They are *straight lines* in $\mathbf{R}^2$, and *vertical lines and semicircles with the diameter on the real axis* in $\mathbf{H}^2$.



M is a hyperbolic polygon whose sides are paired by isometries. Here are eigenfunctions of the hyperbolic Laplacian on the *modular surface* $\mathbf{H}^2/\mathrm{PSL}(2,\mathbf{Z})$, Hejhal:





- **Curvature:** Take a ball $B(x, r)$ centred at x of radius r in M. Then as $r \to 0$, its area satisfies

$$Area(B(x,r)) = \pi r^2 \left[1 - \frac{K(x) r^2}{12} + ... \right]$$

  The number $K(x)$ is called the *Gauss curvature* at $x \in M$.
- **Flat:** In $\mathbf{R}^2$, we have $K(x) = 0$ for every x.
- **Negative curvature:** In $\mathbf{H}^2$, $K(x) = -1$ for every x. So, in $\mathbf{H}^2$ circles are *bigger* than in $\mathbf{R}^2$.
- **Positive curvature:** On the *round sphere* S^2, $K(x) = +1$ for all x. So, in S^2 circles are *smaller* than in $\mathbf{R}^2$.

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- **Geodesic flow:** start at $x \in M$, go with unit speed along the unique geodesic γ_v in a direction v for time t; stop at a point y on γ_v. Let w be the tangent vector to γ_v at y. Then by definition the geodesic flow G^t is defined by

$$G^t(x, v) = (y, w).$$

- *Negative curvature:* geodesics never focus: if v_1, v_2 are two directions at x, and $G^t(x, v_1) = (y_1, w_1), G^t(x, v_2) = (y_2, w_2)$, then the distance between $w_1(t)$ and $w_2(t)$ grows exponentially in t.
- If $K < 0$ everywhere, then geodesic flow is *"chaotic:"* small changes in initial direction lead to very big changes after long time. It is *ergodic*: "almost all" trajectories become uniformly distributed.
- *Weather prediction* is difficult since the dynamical systems arising there are chaotic!
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- **Quantum ergodicity theorem** (Shnirelman, Zelditch, Colin de Verdiere): If $K < 0$ (and the geodesic flow is ergodic), then “almost all” eigenfunctions of Δ become uniformly distributed.
  There may be *exceptional* sequences of eigenfunctions that *do not* become uniformly distributed (“strong scars”), but these sequences are “thin.”
- If *all* eigenfunctions become uniformly distributed (no exceptions!), then *quantum unique ergodicity* (or QUE) holds. Example: S^1.
- **Conjecture** (Rudnick, Sarnak): QUE holds on negatively-curved manifolds; this includes hyperbolic surfaces.
- **Theorem** (Lindenstrauss; Soundararajan, Holowinsky): QUE holds for *arithmetic* hyperbolic surfaces.
  Arithmetic hyperbolic surfaces are very symmetric hyperbolic polygons M, coming from very special hyperbolic isometries.

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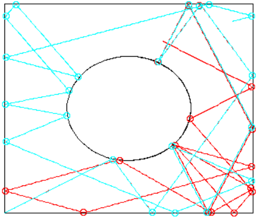
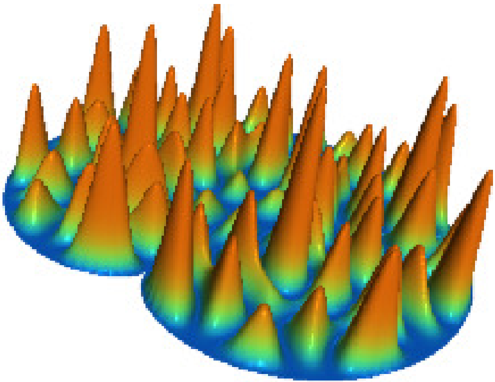
Billiards:
QE theorem holds for billiards (bounded domains in $\mathbf{R}^2$); proved by Gerard-Leichtnam, Zelditch-Zworski. Geodesic flow is replaced by the *billiard flow:* move along straight line until the boundary; at the boundary, angle of incidence equals angle of reflection.

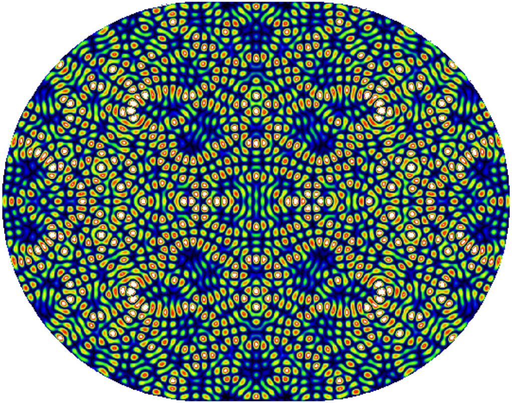
Ergodic planar billiards: Sinai billiard and Bunimovich stadium



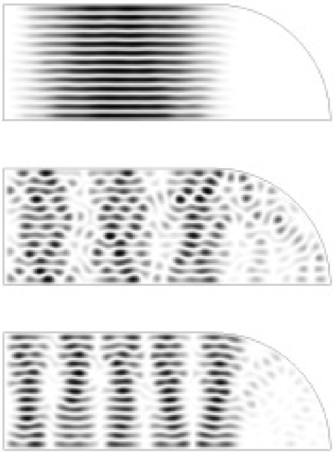
Ergodic eigenfunction on a cardioid billiard.



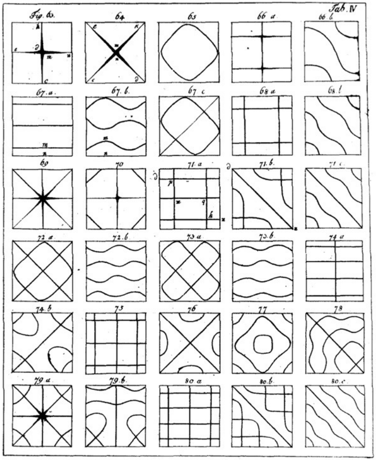
Ergodic eigenfunction on the stadium billiard:



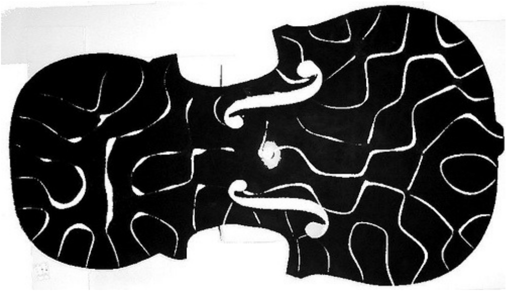
Theorem (Hassell): QUE conjecture *does not* hold for the (Bunimovich) stadium billiard.
Exceptions: "bouncing ball" eigenfunctions, (they have density 0 among all eigenfunctions, so QE still holds).



- **Nodal set** $\mathcal{N}(\phi_\lambda) = \{x \in M : \phi_\lambda(x) = 0\}$, codimension 1 is M. On a surface, it's a union of curves.
  First pictures: *Chladni plates.* E. Chladni, 18th century. He put sand on a plate and played with a violin bow to make it vibrate.



- Chladni patterns are still used to tune violins.



Rudnick showed that if certain *complex-valued* eigenfunctions become equidistributed on hyperbolic surfaces (as in QE), then the same holds for their nodal sets (which are *points*). The question about nodal sets of *real-valued* eigenfunctions (*lines*) is more difficult, and is unsolved.
We end with a movie showing nodal sets.