QUANTUM LIMITS ON FLAT TORI

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ABSTRACT. We classify all weak $*$ limits of squares of normalized eigenfunctions of the Laplacian on two-dimensional flat tori (we call these limits *quantum limits*). We also obtain several results about such limits in dimensions three and higher. Many of the results are a consequence of a Geometric Lemma which describes a property of simplices of codimension one in $\mathbb{R}^n$ whose vertices are lattice points on spheres. The Lemma follows from the finiteness of the number of solutions of a system of two Pell equations. A consequence of the Lemma is a generalization of the result of B. Connes. We also indicate a proof (communicated to us by J. Bourgain) of the absolute continuity of the quantum limits on a flat torus in any dimension. We generalize a two-dimensional result of Zygmund to three dimensions, we discuss various possible generalizations of that result to higher dimensions and the relation to L^p norms of densities of quantum limits and their Fourier series.

1. INTRODUCTION

In this paper we shall consider eigenfunctions of the Laplacian on the standard flat d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Let φ be an eigenfunction with the eigenvalue λ, and let $||\varphi||_2 = 1$. The multiplicity of λ is equal to the number of lattice points on a sphere of radius $\sqrt{\lambda}$ (or, equivalently, to the number of ways of writing λ as a sum of d squares). It is well-known that the multiplicity is unbounded for $d \geq 2$.

Many of the questions considered in this paper for $\mathbb{T}^d$ can be also considered for other flat tori $\mathbb{R}^d/\Lambda$, where Λ is a lattice (of rank d) in $\mathbb{R}^d$ different from $\mathbb{Z}^d$. The eigenvalues of the Laplacian for such tori are given by $\{|\xi|^2 : \xi \in \Lambda^*\}$ where Λ^* is the dual lattice. These numbers are given by the values (on $\mathbb{Z}^d$) of a positive-definite quadratic form Q in d variables. The multiplicity of an eigenvalue of the Laplacian is then equal to the number of points in $\mathbb{Z}^d$ on an ellipsoid which is the level set of Q. The multiplicities are unbounded when $\mathbb{R}^d/\Lambda$ is such a torus that Q is a multiple of a quadratic form with integer coefficients. Most of the results in the paper are also valid for such tori.

We are interested in all possible weak limits as $\lambda_j \to \infty$ of the (probability) measures

$$d\mu_j \ = \ |\varphi_j|^2 \, d\,\mathrm{vol} \tag{1}$$

(where $d\,\mathrm{vol}$ is the Riemannian volume form on $\mathbb{T}^d$), which are henceforth called *quantum limits*.

A. Zygmund proved in [Zyg] the following result:

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Proposition 1.1. *Let φ be an eigenfunction of the Laplacian on $\mathbb{T}^2$. Then*

$$||\varphi||_4/||\varphi||_2 \ \leq \ 5^{1/4}$$

It follows that for $d = 2$ any quantum limit $d\nu$ is absolutely continuous, and its density is a function $\psi \in L^2$ satisfying $||\psi||_2 \leq 5^{1/2}$. In fact, much more can be said about the densities of quantum limits on two-dimensional tori.

To formulate our result, denote the Fourier expansion of a quantum limit $d\nu$ on $\mathbb{T}^d$ by

$$d\nu \ \sim \ \sum_{\tau \in \mathbb{Z}^d} c_\tau e^{i\,(\tau, x)} \tag{2}$$

where (τ, x) is the usual scalar product in $\mathbb{R}^d$.

We prove

Theorem 1.2. *The density of every quantum limit $d\nu$ on $\mathbb{T}^2$ is a trigonometric polynomial all of whose non-zero frequencies lie on the union of (at most) two circles centered at the origin.*

In other words, for every $d\nu$ with the Fourier expansion (2) there exist two positive numbers r_1 and r_2 such that if $\tau \neq \mathbf{0}$ and $c_\tau \neq 0$ then either $|\tau| = r_1$ or $|\tau| = r_2$. We can also describe the combinations of frequencies τ (as above) that occur (together with the set of possible c_τ-s), as well as Wigner measures corresponding to $d\nu$-s.

The first question one asks for $d > 2$ is whether quantum limits are absolutely continuous with respect to Lebesgue measure or not. No analogue of 1.1 is known for $d > 2$, so the absolute continuity of quantum limits has to be proved differently than for $d = 2$. J. Bourgain has communicated to us the proof of

Theorem 1.3. *Quantum limits are absolutely continuous in any dimension.*

The proof involves repeated applications of a result due to B. Connes (see Proposition 1.16 below[1]); a property of sequences of nonnegative numbers uniformly bounded in l^1 is also used.

In the rest of the paper we obtain more precise information about the possible quantum limits in dimensions three and higher, although we are not able to classify all such limits as in dimension two.

The first result we prove is

Theorem 1.4. *For $3 \leq d \leq 5$, every quantum limit on $\mathbb{T}^d$ satisfies*

$$\sum_{\tau \in \mathbb{Z}^d} |c_\tau|^{d-2} \ < \ \infty \tag{3}$$

(where c_τ-s are defined by (2)).

We conjecture that (3) holds in any dimension $d > 5$, but are not able to prove it because of certain technical difficulties.

It follows from Theorem 1.4 that the density of any quantum limit on $\mathbb{T}^3$ is a function that has absolutely convergent Fourier series; for $d = 4$ it follows that any such function is in L^2. We also deduce from a certain conjecture about the distribution of numbers which are sums of two squares (conjecture 8.2) that on

[1] See also [Bou94, Lemma 25].

a three-dimensional torus there exist quantum limits which are not trigonometric polynomials; on a four-dimensional torus we can prove this unconditionally.

By Theorem 1.3, every quantum limit on $\mathbb{T}^d$ has a density which is a (nonnegative) function in $L^1(\mathbb{T}^d)$. The question arises whether quantum limits on $\mathbb{T}^d$ belong to any L^p for any $p > 1$ and $d \geq 5$ (the question for $d \leq 4$ being answered by Theorem 1.2 and Theorem 1.4). The answer turns out to be intimately related to the generalization of Zygmund's result 1.1 to higher dimensions.

To formulate the result, we give two definitions.

Definition 1.5. *The number $A(d,p) \leq \infty$ is defined by*

$$A(d,p) \;=\; \sup_{\Delta\varphi+\lambda\varphi=0} (||\varphi||_p/||\varphi||_2) \tag{4}$$

where the supremum is taken over the eigenfunctions of the Laplacian on $\mathbb{T}^d$.

The question of the rate of growth of $||\varphi||_p/||\varphi||_2$ with increasing eigenvalue is one of the basic questions in quantum chaos (cf. [Sa93]). Proposition 1.1 can now formulated as $A(2,4) \leq 5^{1/4} < \infty$. It is not known whether $A(d,p) < \infty$ for any $p > 2$ and $d \geq 3$. Bourgain showed in [Bou93] that $A(d,p) = \infty$ for $d \geq 4$ and $p \geq 2(d+1)/(d-3)$.

We next give

Definition 1.6. *The number $B(d,p) \leq \infty$ is defined by*

$$B(d,p) \;=\; \sup\left\{||\varphi||_p/||\varphi||_2 \;:\; \widehat{\varphi} \in (\Omega \cap \mathbb{Z}^{d+2})\right\} \tag{5}$$

where the supremum is taken over the trigonometric polynomials on $\mathbb{T}^{d+2}$ all of whose frequencies lie on an intersection of $\mathbb{Z}^{d+2}$ with a $(d-1)$-dimensional sphere Ω lying in a d-dimensional subspace of $\mathbb{R}^{d+2}$, and the L^p norms are for the functions on $\mathbb{T}^{d+2}$.

It is easy to see[2] that $B(d,p) \geq A(d,p)$. One can generalize Zygmund's argument and prove that $B(2,4) < \infty$. It may happen that if $A(d,p) < \infty$ then $B(d,p) < \infty$, but we cannot prove this.

We prove

Proposition 1.7. *If $B(d,p) < \infty$ for some $p > 2$, then the density of any quantum limit on $\mathbb{T}^{d+2}$ is in $L^{p/2}(\mathbb{T}^{d+2})$. Also, if $A(d,p) = \infty$ for some $p > 2$, then there exist quantum limits on $\mathbb{T}^{d+3}$ whose densities are not in $L^{p/2}(\mathbb{T}^{d+3})$.*

One can now use Bourgain's results from [Bou93] to conclude that for $d \geq 7$ there exist quantum limits on $\mathbb{T}^d$ whose densities are not in $L^{(d-2)/(d-6)}$; note that as $d \to \infty$, $(d-2)/(d-6)$ approaches 1.

We now discuss an analogue of (4) for the Fourier series of the function $|\varphi|^2$. Let φ be an eigenfunction of the Laplacian on $\mathbb{T}^d$ with the eigenvalue λ. Then

$$\varphi(x) \;=\; \sum_{\xi\in\mathbb{Z}^d;|\xi|^2=\lambda} a_\xi e^{i\,(\xi,x)}, \quad a_\xi \in \mathbb{C} \tag{6}$$

(if $||\varphi||_2 = 1$ then $\sum_\xi |a_\xi|^2 = 1$). It follows from the previous formula that

$$|\varphi|^2 = \sum_\tau b_\tau e^{i\,(\tau,x)}, \quad b_\tau \;=\; \sum_{\substack{\xi-\eta=\tau \\ |\xi|^2=|\eta|^2=\lambda}} a_\xi \bar{a}_\eta \tag{7}$$

[2] We can identify the eigenfunctions on $\mathbb{T}^d$ with the eigenfunctions on $\mathbb{T}^{d+2}$ all of whose frequencies lie in the subspace $\{x \in \mathbb{R}^{d+2} | x_{d+1} = x_{d+2} = 0\}$ of $\mathbb{R}^{d+2}$.

(denoting by $\mathbf{0}$ the zero vector in $\mathbb{Z}^d$, we remark that $b_{\mathbf{0}} = \sum_{|\xi|^2=\lambda} |a_\xi|^2 = ||\varphi||_2^2$, hence $b_{\mathbf{0}} = 1$ if $||\varphi||_2 = 1$).

If $A(d,p) < \infty$ for some $2 < p = p(d) \leq 4$, then by Hausdorff-Young Theorem

$$(\sum_{\tau \in \mathbb{Z}^d} |b_\tau|^q)^{1/q} \ll ||\varphi||_2^2 \tag{8}$$

uniformly on $\mathbb{T}^d$ for $q = q(d) = p(d)/(p(d)-2)$.

Accordingly, we give

Definition 1.8. *The number $C(d,q) \leq \infty$ is defined by*

$$C(d,q) = \sup_{\Delta\varphi+\lambda\varphi=0} (||\widehat{f}||_q/||\varphi||_2^2), \qquad f = |\varphi|^2 \tag{9}$$

where $\widehat{f}$ is the Fourier series of f, and the supremum is taken over the eigenfunctions of the Laplacian on $\mathbb{T}^d$.

We have just shown that for $2 \leq p \leq 4$, $(A(d,p))^2 \geq C(d, p/(p-2))$. Zygmund proves in [Zyg] that $C(2,2) < \infty$. Note that Bourgain's results in [Bou93] suggest that if (8) holds on $\mathbb{T}^d$ for $q = q(d)$, then $q(d) \to \infty$ as $d \to \infty$, and this can indeed be shown.

We prove

Proposition 1.9. *On a three-dimensional torus, $C(3,3) < \infty$.*

The proof is by a geometric argument in the spirit of [Zyg]. However, we cannot use Proposition 1.9 to bound $||\varphi||_p$ uniformly for any $p > 2$ because the exponent q in our bound is bigger than two. Proposition 1.9 is used in the proof of Theorem 1.4 for $d = 5$. The ideas of the proof of Proposition 1.9 should allow to prove that $C(d,d) < \infty$; however, for $d \geq 4$ certain technical difficulties arise which prevent us from extending Proposition 1.9 to dimensions four and higher.

We next give

Definition 1.10. *The number $D(d,q) \leq \infty$ is defined by*

$$D(d,q) = \sup\left\{||\widehat{f}||_q/||\varphi||_2^2 \ : \ \widehat{\varphi} \in (\Omega \cap \mathbb{Z}^{d+2})\right\}, \qquad f = |\varphi|^2 \tag{10}$$

where the supremum is taken over the same trigonometric polynomials on $\mathbb{T}^{d+2}$ as in the definition of $B(d,p)$.

It is easy to see that $D(d,q) \geq C(d,q)$ and that for $2 \leq p \leq 4$, $(B(d,p))^2 \geq D(d, p/(p-2))$.

We prove

Proposition 1.11. *If $D(d,q) < \infty$ for some $q > 1$, then for any quantum limit $d\nu$ on $\mathbb{T}^{d+2}$ given by (2)*

$$\sum_{\tau \in \mathbb{Z}^{d+2}} |c_\tau|^q < \infty \tag{11}$$

Also, if $C(d,q) = \infty$ for some $q > 1$, then there exist such quantum limits on $\mathbb{T}^{d+3}$ that the Fourier series of their densities are not in l^q.

One can generalize the proofs of Propositions 1.1 and 1.9 and show that $D(2,2) < \infty$ and $D(3,3) < \infty$. This and the above Proposition would prove Theorem 1.4 for

$d = 4$ and $d = 5$. It may happen that if $C(d, q) < \infty$ then $D(d, q) < \infty$, but we cannot prove it.

Proposition 1.11 shows that proving that all quantum limits on $\mathbb{T}^n$ are in L^p for some $p > 1$ requires extending[3] Zygmund's result (Proposition 1.1) to $\mathbb{T}^{n-3}$.

Finally, we discuss the rate of growth on $\mathbb{T}^d$ of the sum

$$\Sigma(\rho) \;=\; \sum_{\tau\in\mathbb{Z}^d;\,|\tau|<\rho} |c_\tau| \tag{12}$$

Conjecture 8.2 implies that on the four-dimensional torus there exist quantum limits for which $\Sigma(\rho) \to \infty$ as $\rho \to \infty$; we can prove this unconditionally for the five-dimensional torus. We prove

Proposition 1.12. *On $\mathbb{T}^d$, $\Sigma(\rho) \ll \rho^{d-3}$ for $d \geq 6$. For $d = 5$, $\Sigma(\rho) \ll \rho^{3/2+\varepsilon}$ and for $d = 4$, $\Sigma(\rho) \ll \rho^{\varepsilon}$ for every $\varepsilon > 0$.*

We expect that $\Sigma(\rho) \ll \rho^{d-4+\varepsilon}$ for any $\varepsilon > 0$. Conjecture 8.2 implies that for $d \geq 5$ and for any $\varepsilon > 0$,

$$\limsup_{\rho\to\infty} \frac{\Sigma(\rho)}{\rho^{d-4-\varepsilon}} \;=\; \infty$$

We can also prove unconditionally that for $d \geq 6$, $\overline{\lim}_{\rho\to\infty}(\Sigma(\rho)/\rho^{d-5-\varepsilon}) = \infty$.

Many of the above facts are consequences of a geometric Lemma which is of independent interest. To motivate the Lemma, note that it follows from (7) that the frequencies of the trigonometric polynomial $|\varphi|^2$ (where φ is the eigenfunction of the Laplacian on $\mathbb{T}^d$ with the eigenvalue λ) are given by the *chords* connecting the lattice points on the sphere of radius $\sqrt{\lambda}$ in $\mathbb{R}^d$ (centered at zero). Also, the size of the Fourier coefficient b_τ is determined by the pairs (ξ, η) of points in $\mathbb{Z}^d$ lying on the sphere S of radius $\sqrt{\lambda}$ whose "differences" $\xi - \eta$ are equal to τ. We will call such pairs *τ-resonances* (the terminology comes from mechanics, where it is usually assumed that $|\tau| \ll \sqrt{\lambda}$).

Given a measure $d\nu$ on $\mathbb{T}^d$ with the Fourier expansion (2), and a sequence φ_j of eigenfunctions of the Laplacian with (increasing) eigenvalues λ_j, by $|\varphi_j|^2 \to d\nu$ weak $*$ as $j \to \infty$ we shall mean that for every τ, $b_\tau(j) \to c_\tau$ as $j \to \infty$ (for $\tau = \mathbf{0}$, $b_\tau(j)$ is always equal to 1, so $c_{\mathbf{0}} = 1$). Thus we are interested in τ-resonances which appear on every sphere of radius $\sqrt{\lambda_j}$ for $j > N = N(\tau)$.

To state the Geometric Lemma, we give a definition. Let Δ be a d-dimensional simplex with $d+1$ vertices all lying in $\mathbb{Z}^n$ for some n (i.e. a line segment for $d = 1$, a triangle for $d = 2$ etc).

Definition 1.13. *The points $\xi_1, \xi_2, \ldots, \xi_{d+1} \in \mathbb{Z}^n$ lying on a sphere in $\mathbb{R}^n$ are called* Δ-resonant *if they form a d-dimensional simplex which can be translated into Δ.*

This definition is just a multi-dimensional generalization of τ-resonances defined above for $d = 1$. We now state the Lemma.

Lemma 1.14. *Let $\Delta_1, \ldots, \Delta_k$ be a collection of d-dimensional simplices with vertices in $\mathbb{Z}^{d+1}$, and let $S_1, \ldots, S_n, \ldots$ be an infinite sequence of d-dimensional spheres in $\mathbb{R}^{d+1}$ centered at the origin with increasing radii such that on each sphere*

[3] We expect that any "L^p" result for quantum limits on $\mathbb{T}^n$ is equivalent to the corresponding generalization of Proposition 1.1 to $\mathbb{T}^{n-2}$.

S_n there are Δ_j-resonant points for every $1 \le j \le k$. Then the radii of the circumscribed sphere Ω_j of Δ_j-s can take at most two different values.

For $d = 1$ this just means that the line segments $\Delta_1, \dots, \Delta_k$ can have at most two different lengths (hence at most two different circles in Theorem 1.2). The Lemma is proved by first reducing the problem to solving a system of Pell equations, and then using the (known) fact about the the finiteness of the number of solutions. This fact can be proved by either using an effective version of Siegel's Theorem about the finiteness of number of integer points on curves of genus one (cf. [Bak75, Ch. 4] or [B-D]); or by applying Baker's results on linear froms in logarithms (cf. [Bak73] and [Bak75]; also, cf. [Mi] for the proof of a similar statement).

The Geometric Lemma proves a certain finiteness statement (radius can take at most two different values) about simplices of codimension one in $\mathbb{R}^n$ whose vertices are lattice points on spheres. One can ask whether it is possible to prove an analogous finiteness statements for simplices of codimension two or more, perhaps with *two* (in "two different values") replaced by a larger number. The answer to the above question is negative. Namely, conjecture 8.2 implies that there exists an *infinite* sequence Δ_j of simplices with vertices in $\mathbb{Z}^n$ of codimension two, and a sequence S_k of spheres in $\mathbb{R}^n$ centered at the origin of increasing radii such that on S_k there are Δ_j-resonant points for $1 \le j \le k$ (this implies the existence of quantum limits on $\mathbb{T}^3$ which are not trigonometric polynomials); an analogous statement for simplices of codimension three can be proved unconditionally.

We can prove analogues of the Geometric Lemma for lattices in $SL(n, \mathbb{R})$; these results will be discussed elsewhere.

The main application of the Geometric Lemma is Proposition 1.17 given below. It is a generalization of a result of B. Connes (cf. [Con]). To state the result, we first define certain graphs on spheres.

Definition 1.15. *For a d-dimensional sphere S in $\mathbb{R}^n$ and for $\rho > 0$, define $G_\rho(S)$ to be the graph with vertices at the points of $\mathbb{Z}^n$ located on S obtained by connecting two such points whenever the distance between them is less than ρ. Also, for $\rho > \rho_1 > 0$ define $G_\rho^{\rho_1}(S)$ to be the graph with the same vertices, and with two vertices ξ and η connected whenever $\rho_1 < |\xi - \eta| < \rho$.*

Note that $G_\rho^{\rho_1}(S)$ coincides with $G_\rho(S)$ for $\rho_1 = 0$.

We now state Connes' result:

Proposition 1.16. *There exists a (positive) function $g = g_d(R) \asymp R^{2/(d+2)!}$ such that on any d-dimensional sphere S in $\mathbb{R}^n$ of radius R, for any $0 < \rho < g_d(R)$ (or, equivalently, $R \gg \rho^{(d+2)!/2}$) the connected components of $G_\rho(S)$ all lie on one of the $(d-1)$-dimensional spheres $\Omega_1, \Omega_2, \dots, \Omega_m$ formed by the intersection of S with a subspace of $\mathbb{R}^n$ of dimension d.*

Using Proposition 1.16 and the Geometric Lemma, we prove

Proposition 1.17. *Given a sequence of spheres S_j (of radius R_j) in $\mathbb{R}^{d+1}$ centered at the origin, there exist a subsequence $\{j_k\}$ of $\{j\}$, and a number $\rho_1 \ge 0$ with the following property: given $\rho > \rho_1$, there exists $N \in \mathbb{N}$ such that if we take any sphere $S = S_{j_k}$ with $j_k > N$ and consider the graph $G = G_\rho^{\rho_1}(S)$, then the connected components of G will lie on one of $(d-2)$-dimensional spheres $\Omega_1, \Omega_2, \dots, \Omega_m$, $m = m(j_k)$ (formed by the intersections of S with the subspaces of $\mathbb{R}^{d+1}$ of codimension two).*

Proposition 1.17 thus allows us to "reduce the dimension by two" when proving statements about the quantum limits.

The proofs of Theorem 1.2, Lemma 1.14 and Proposition 1.17, and of all the results about the quantum limits in dimensions three and four (except for Proposition 1.9) appear in [Jak1]. The results of this paper were announced in [Jak2].

The paper is organized as follows: in section 2 we discuss quantum limits on two-dimensional tori; we prove Theorem 1.2 and discuss microlocal quantum limits (liftings to $S^*\mathbb{T}^2$ of the usual quantum limits). In section 3 we indicate the proof of Theorem 1.3. In section 4 we prove Proposition 1.17. In section 5 we discuss quantum limits in dimensions three and higher; we prove Theorem 1.4 and Propositions 1.7 and 1.11. We also discuss the existence of quantum limits which are not trigonometric polynomials.

In section 6 we prove Proposition 1.9 and discuss possible ways of generalizing it to higher dimensions. In section 7 we prove Proposition 1.12 and discuss lower bounds for the sum $\Sigma(\rho)$ defined in (12). Finally, in section 8 we prove the geometric Lemma 1.14.

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2. Two-dimensional tori

Proof of Theorem 1.2. Given a measure $d\nu$ on $\mathbb{T}^2$ with the Fourier expansion (2), and a sequence of eigenfunctions φ_j converging weakly to $d\nu$, assume for contradiction that there exist three frequencies $\{\tau_1, \tau_2, \tau_3\}$, $\tau_i \neq \mathbf{0}$ in $\mathbb{Z}^2$ such that $|\tau_i| \neq |\tau_j|$ for $1 \leq i < j \leq 3$ and such that $c_{\tau_i} \neq 0$ in (2) for $1 \leq i \leq 3$. Let $|\varphi_j|^2$ have Fourier expansion (7). Since $b_\tau(j) \to c_\tau$ as $j \to \infty$, there exist $N \in \mathbb{N}$ such that $b_{\tau_i}(j) \neq 0$ for every $j > N$ and $1 \leq i \leq 3$. From the definition of $b_{\tau_i}(j)$ given in (7) it follows that there are τ_i-resonant points for every $1 \leq i \leq 3$ on the circle of radius $\sqrt{\lambda_j}$ for every $j > N$. This, however, contradicts Lemma 1.14 (proved in section 8), finishing the proof of Theorem 1.2.

□

We give a new definition.

Definition 2.1. *We call a set of vectors $\mathcal{F} = \{\tau_1, \ldots, \tau_n, \ldots\} \subseteq \mathbb{Z}^d$* admissible *if there exists a quantum limit $d\nu$ on $\mathbb{T}^d$ with $c_\tau \neq 0$ for every $\tau \in \mathcal{F}$.*

It follows form the proof of Lemma 1.14 that a necessary and sufficient condition for the set $\mathcal{F} = \{\tau_1, \ldots, \tau_k\} \in \mathbb{Z}^2$ to be admissible is the existence of infinitely many integer solutions of the system (52). It is easy to show that the set all lattice points lying on a circle centered at the origin is always admissible, and that if τ and $c \cdot \tau$ are both admissble for some $c \neq 0$, then $c = \pm 1$. Given two different circles centered at the origin, it may happen that no admissible set can have a nonempty intersection with both circles. It may also happen that the set $\mathcal{F}$ of *all* lattice points on both circles is not admissible while some subset of $\mathcal{F}$ is.

We next discuss the possible values of Fourier coefficients c_τ of a quantum limit on $\mathbb{T}^2$. Since

$$c_\tau = \lim_{j\to\infty} b_\tau(j)$$

where φ_j-s converge to $d\nu$ and where $|\varphi_j|^2$ has the Fourier expansion (7), it follows that $c_{\mathbf{0}} = 1$, $|c_\tau| \leq 1$ and $c_{-\tau} = \bar{c}_\tau$. Also, since τ-resonant pairs are "well-separated" on a large enough circle for different τ-s, it follows from (7) and Theorem 1.2 that

$$\sum_{\tau\neq\mathbf{0}} |c_\tau| \leq 1 \tag{13}$$

(guaranteeing the positivity of any quantum limit). It is easy to show that given a set $\mathcal{F} \in \mathbb{Z}^2$ of admissible frequencies, we can construct a quantum limit on $\mathbb{T}^2$ with any prescribed set of c_τ-s satisfying the above constraints.

We now turn to *Wigner measures* associated to the eigenfunctions of the Laplacian. Recall that for an eigenfunction φ, Wigner measure $d\omega_\varphi$ on $S^*\mathbb{T}^2 \simeq \mathbb{T}^3$ is defined by

$$\langle A\varphi, \varphi\rangle = \int_{S^*\mathbb{T}^2} \sigma\, d\omega_\varphi, \tag{14}$$

where A is a pseudo-differential operator of order zero with the symbol σ and $\langle,\rangle$ is the usual inner product in $L^2(\mathbb{T}^2)$.

Let (x,ξ) denote the coordinates in $T^*\mathbb{T}^2$, and let (x,θ) denote the coordinates in $S^*\mathbb{T}^2$ (here $\theta = \xi/|\xi|$). An easy calculation shows that the Wigner measure corresponding to φ having the Fourier expansion (6) can be written in the above coordinates as

$$d\omega_\varphi(x,\theta) = \sum_{\xi:|\xi|^2=\lambda} \delta_\xi(\theta) a_\xi \sum_{\eta:|\eta|^2=\lambda} \bar{a}_\eta e^{i(\eta-\xi,x)} dx d\theta \tag{15}$$

We are interested in the possible weak limits $d\omega$ of $d\omega_j = d\omega_{\varphi_j}$-s (which will be called *microlocal quantum limits*).

Write a microlocal limit $d\omega$ on $\mathbb{T}^d$ in the form

$$d\omega = \sum_{\tau\in\mathbb{Z}^d} f_\tau(\theta) e^{i(\tau,x)} dx d\theta \tag{16}$$

We first consider the possible f_τ for $\tau \neq \mathbf{0}$. Let ξ and η be a pair of τ-resonant points on a circle of radius $\sqrt{\lambda_j}$. As $j \to \infty$, it is easy to see that δ_ξ and δ_η both converge weakly to $\delta_{\tau/|\tau|\pm\pi/2}$ (the latter will be denoted $\delta_{\tau\pm\pi/2}$ for convenience). It is also a well-known fact (cf. [CC]) that there can be at most two points on any arc of the above circle of length $2\lambda_j^{1/6}$. Letting $\lambda_j \to \infty$, we conclude that if $|\varphi_j|^2$ converge to $d\nu$ with the Fourier expansion (2), then

$$f_\tau = c_{\tau,1}\delta_{\tau+\pi/2} + c_{\tau,2}\delta_{\tau-\pi/2}, \quad c_{\tau,1} + c_{\tau,2} = c_\tau \tag{17}$$

It is easy to show that $\overline{c_{\tau,1}} = c_{-\tau,2}$ and that

$$\sum_{\tau\neq\mathbf{0}} (|c_{\tau,1}| + |c_{\tau,2}|) \leq 1$$

Note that it can happen that $c_\tau = 0$ while $|c_{\tau,1}| + |c_{\tau,2}| > 0$. We can arrange for Wigner function $d\omega$ to have any of f_τ-s satisfying (17) and additional relations discussed above.

To complete our investigations, it remains to discuss $f_{\mathbf{0}}$ for the microlocal quantum limit $d\omega$ given by (16). It is clear from (15) that for an eigenfunction φ_j of the Laplacian on $\mathbb{T}^2$, $f_{\mathbf{0}}(j)$ is given by

$$\sum_{\xi\in\mathbb{Z}^2;|\xi|^2=\lambda_j} |a_\xi|^2\delta_\xi$$

Now, $d\omega$ determines (an admissible) set $\mathcal{F} = \{\tau : |c_{\tau,1}| + |c_{\tau,2}| > 0\}$. Let

$$\mathcal{G} = \{\theta : \theta \pm \pi/2 = \tau/|\tau|,\ \textit{some}\ \tau \in \mathcal{F}\}$$

The measure $f_{\mathbf{0}}\, d\theta$ can be written as a sum of two nonnegative measures $\nu_1 + \nu_2$, where

$$\nu_1 \ = \ \sum_{\theta\in\mathcal{G}} d_\theta\delta_\theta$$

It is easy to see that[4] $d_{\tau+\pi/2}+d_{\tau-\pi/2} \geq |c_{\tau,1}|+|c_{\tau,2}|+|c_{-\tau,1}|+|c_{-\tau,2}|$. Accordingly,

$$\sum_{\tau\neq\mathbf{0}}(|c_{\tau,1}| + |c_{\tau,2}|) \ \leq \ \sum_{\theta} d_\theta \ \leq \ 1$$

The measure ν_1 corresponds to pairs of τ-resonant points on circles of radius $\sqrt{\lambda_j}$ (for such τ that $|c_{\tau,1}| + |c_{\tau,2}| > 0$), while ν_2 corresponds to all other points.

ν_2 is a nonnegative measure on S^1 of total mass $1 - \sum_\theta d_\theta$. If the lattice points become equidistributed on circles of radius $\sqrt{\lambda_j}$ centered at the origin in $\mathbb{R}^2$ as $j \to \infty$, then the eigenfunctions φ_j-s can be chosen so that ν_2 is an arbitrary nonnegative measure of total mass $1 - \sum_\theta d_\theta$ (given f_τ-s for $\tau \neq \mathbf{0}$, and given ν_1). If the lattice points do not become equidistributed on the circles described above, we get some restrictions on the possible measures ν_2. The answer depends on the sequence of the eigenvalues λ_j.

3. Quantum limits are absolutely continuous in any dimension

In this section we prove Theorem 1.3. The proof was communicated to us by J. Bourgain (we remark that for $d = 2$ and $d = 3$ we prove much stronger statements than Theorem 1.3). The proof uses Proposition 1.16 and an elementary lemma from analysis (Lemma 3.1 below), sometimes called the *Sequence splitting* lemma.

In the notation of the introduction, let the sequence φ_j of eigenfunctions converge to a quantum limit $d\nu$. Let E be an arbitrary subset of $\mathbb{T}^{d+1}$ of Lebesgue measure zero. By the standard density argument, it suffices to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every nonnegative trigonometric polynomial $f(x)$ such that $||f||_2 < \delta$ (we can also assume without loss of generality that $||f||_\infty \leq 2$) there exists a subsequence j_k of j such that

$$\limsup_{k\to\infty} \int f(x)|\varphi_{j_k}(x)|^2\, dx \ \leq \ \varepsilon \tag{18}$$

Denote by S_j the sphere of radius $R_j = \sqrt{\lambda_j}$ in $\mathbb{R}^{d+1}$ centered at the origin. Then by the Proposition 1.16 there exists a function $g_d(R) \approx R^{2/(d+2)!}$ such that the connected components of the graph $G_{g_d(R)}(S_j)$ all lie in the hyperplanes. Denote

[4] Note that any admissible set in $\mathbb{Z}^2$ can contain at most two parallel vectors.

the $(d-1)$-dimensional spheres on which these components lie by $\Omega_\alpha(j;d-1)$ and their radii by $r_\alpha(j;d-1)$. Also, let

$$\sum_{\xi\in\mathbb{Z}^{d+1};\xi\in\Omega_\alpha(j;d-1)} |a_\xi|^2 \;=\; b_\alpha(j;d-1)$$

Define

$$\sum_{m-1<r_\alpha(j;d-1)\le m} b_\alpha(j;d-1) \;=\; B_m(j;d-1) \tag{19}$$

For each j we then get a (finite) sequence $B_m(j;d-1)$ with $\sum_{m=1}^\infty B_m(j;d-1)=1$.

To each of the $(d-1)$-dimensional spheres $\Omega_\alpha(j;d-1)$ we can again apply Proposition 1.16 and conclude that the connected components of the graph $G_{g_{d-1}(r_\alpha)}(\Omega_\alpha(j;d-1))$ lie on $(d-2)$-dimensional spheres, which we denote $\Omega_\alpha(j;d-2)$. Let $r_\alpha(j;d-2)$ be the radius of $\Omega_\alpha(j;d-2)$. Define $b_\alpha(j;d-2)$ and $B_m(j;d-2)$ as above. Again, for each j we get a (finite) sequence of nonnegative numbers whose sum is less than or equal to one.

To each of the $(d-2)$-dimensional spheres $\Omega_\alpha(j;d-2)$ we again apply Proposition 1.16 and repeat the whole process until the dimension l of the connected components $\Omega_\alpha(j;l)$ becomes equal to 1, i.e. until the components become circles. There were $(d-1)$ steps altogether, and we got $(d-2)$ sequences $B_m(j;l)$ for $2\le l\le d-1$ (we don't define any sequences for $l=1$).

We now state the "Sequence splitting" lemma:

Lemma 3.1. *Let $\{c_m(j)\}, j\in\mathbb{N}$ be a sequence of series of nonnegative numbers such that for every j, $\sum_{m=1}^\infty c_m(j)\le 1$. Then for every $\varepsilon>0$ there exists an integer n_0 and a subsequence j_k of j such that for every $n>n_0$*

$$\limsup_{k\to\infty}\left(\sum_{i=n_0}^{n} c_i(j_k)\right) \;\le\; \varepsilon$$

The proof is elementary and will be omitted. We now turn to the proof of the absolute continuity of quantum limits.

Proof of Theorem 1.3 for $d\ge 4$. Given $\varepsilon>0$ consider the sequences $B_m(j;d-1)$ defined above, and choose the subsequence $j_k^{(1)}$ of j and M_{d-1} such that Lemma 3.1 holds. Consider the sequence $B_m(j_k^{(1)};d-2)$ and choose the subsequence $j_k^{(2)}$ of $j_k^{(1)}$ and M_{d-2} such that Lemma 3.1 holds. Repeat the process for $B_m(j_k^{(2)};d-3)$ and so on, until we get a subsequence $j_k^{(d-2)}$ and the number M_2 such that Lemma 3.1 holds for $B_m(j_k^{(d-2)};2)$. Denote $j_k^{(d-2)}$ by j_k for convenience.

Let $M=\max\{M_{d-1},M_{d-2},\dots,M_2\}$. By Lemma 3.1 and by the choice of j_k, for every $2\le l\le d-1$ and for every $n>M$,

$$\limsup_{k\to\infty}\left(\sum_{m=M}^{n} B_m(j_k;l)\right) \;\le\; \varepsilon \tag{20}$$

Choose a trigonometric polynomial $f(x)$ such that $f(x)\ge 1$ in a small neighborhood of E, $f(x)\ge 0$ on $\mathbb{T}^{d+1}$, $||f||_\infty\le 2$, and that

$$||f||_2 \;<\; \varepsilon/M^d \tag{21}$$

Let $\widehat{f_n}(\xi)=0$ for $|\xi|>N$. For $2\le l\le d$, put $n_l=\max\{M,(g_l)^{-1}(N)\approx N^{(l+2)!/2}\}$.

Consider the connected component of $G_N(S_j)$. If $R_j > n_d$ ($\Leftrightarrow g_d(R_j) > N$), the connected components of $G_N(S_j)$ by Proposition 1.16 lie inside the connected components of $G_{g_d(R_j)}(S_j)$ which are $(d-1)$-dimensional spheres $\Omega_\alpha(j; d-1)$. Similarly, for any $2 \le l \le d-1$ and for any $\Omega_\alpha(j;l)$ if $r_\alpha(j;l) > n_l$, then the connected components of $G_N(\Omega_\alpha(j;l))$ lie inside the connected components of $G_{g_l(r_\alpha)}(\Omega_\alpha(j;l))$ which are $(l-1)$-dimensional spheres $\Omega_\alpha(j;l-1)$.

Now consider the expression (18) for the above choice of j_k and f. We claim that

$$\limsup_{k\to\infty} \int f(x)|\varphi_{j_k}(x)|^2\,dx \;\le\; 2d\varepsilon \tag{22}$$

This would clearly imply the Theorem.

Define the trigonometric polynomial $\varphi_{j_k}(\alpha;l)$ by

$$\varphi_{j_k}(\alpha;l)(x) \;=\; \sum_{\xi\in\mathbb{Z}^{d+1}\cap\Omega_\alpha(j_k;l)} a_\xi e^{i\,(\xi,x)}$$

where a_ξ-s are the coefficients in the expansion (6) of φ_{j_k}. We first note that since $\widehat{f}_n(\xi) = 0$ for $|\xi| > N$, formula (22) can be rewritten for $R_{j_k} > n_d$ by the above remarks as

$$\sum_{l=1}^{d-1}\sum_{\alpha}\int f|\varphi_{j_k}(\alpha;l)|^2 \tag{23}$$

where for $2 \le l \le d-1$ we can assume that $r_\alpha(j_k;l) \le n_l$. We shall estimate each term in (23) differently depending on the dimension l and radius $r_\alpha(j_k;l)$ of the sphere $\Omega_\alpha(j_k;l)$.

Consider first the case $l=1$. By Schwartz inequality and Zygmund's result 1.1,

$$\int f|\varphi_{j_k}(\alpha;1)|^2 \;\le\; ||f||_2\cdot||\varphi_{j_k}(\alpha;1)||_4^2 \;\le\; b_\alpha(j_k;1)\cdot 5^{1/2}\varepsilon$$

Since $\sum_\alpha b_\alpha(j_k;1) \le 1$, the contribution of the above terms to (23) doesn't exceed $5^{1/2}\varepsilon$.

Consider next the case $l \ge 2$, $r_\alpha(j_k;l) \le M$. We may estimate $||\varphi_{j_k}(\alpha;l)||_\infty$ trivially by

$$b_\alpha(j_k;l)[r_\alpha(j_k;l)]^l \;\le\; b_\alpha(j_k;l)M^l \le\; b_\alpha(j_k;l)M^d$$

Therefore,

$$\int f|\varphi_{j_k}(\alpha;l)|^2 \le ||\varphi_{j_k}(\alpha;l)||_\infty\cdot||f||_1 \le b_\alpha(j_k;l)M^d||f||_2 \le b_\alpha(j_k;l)\varepsilon$$

by the choice of f (cf. (21)). As before, we conclude that the contribution of the above terms to (23) is less than ε.

Finally, we consider the case $l \ge 2$, $M < r_\alpha(j_k;l) \le n_l$. Since $||f||_\infty \le 2$, we can estimate the terms in (23) as follows:

$$\int f|\varphi_{j_k}(\alpha;l)|^2 \;\le\; 2||\varphi_{j_k}(\alpha;l)||_2 \;=\; 2b_\alpha(j_k;l)$$

We then use (20) to conslude that the contribution of the above terms to (23) becomes less than or equal to $(2d-3)\varepsilon$ as $k\to\infty$.

After summing the contributions from the terms in the above three cases, (22) follows. This finishes the proof of Theorem 1.3.

$\square$

4. A Generalization of Connes' result

Proof of Proposition 1.17. Denote by R_j the radius of the sphere S_j. Consider all d-dimensional simplices formed by the points in $\mathbb{Z}^{d+1}$ lying on S_j, and let $\mathcal{R}(j) = \{r_1(j) < r_2(j) < \ldots < r_{m_j}(j)\}$ be the set of different radii of the circumscribed spheres of these simplices (in the increasing order). Suppose that $\liminf_{j\to\infty} r_1(j) = \infty$. Then Proposition 1.17 is true for $\rho_1 = 0$ and for $\{j_k\} = \{j\}$.

To prove it, fix $\rho > 0$. Let $R_j \gg \rho^{(d+2)!/2}$ for $j > N_1 \in \mathbb{N}$ so that Proposition 1.16 holds for S_j. Consider the $(d-1)$-dimensional spheres $\Omega_1, \Omega_2, \ldots, \Omega_{m(j)}$ on S_j whose existence is guaranteed by Proposition 1.16. It may happen that all points in $\mathbb{Z}^{d+1}$ on some Ω_i actually lie in a $(d-1)$-dimensional subspace of $\mathbb{R}^{d+1}$, hence on a $(d-2)$-dimensional sphere which we also denote by Ω_i. By the assumption there exists $N_2 \in \mathbb{N}$ such that $r_1(j) \gg \rho(d+1)!/2$ for $j > N_2$, so that Proposition 1.16 holds for each $(d-1)$-dimensional Ω_i for $n = d-1$. Take $N = \max\{N_1, N_2\}$. Then the Proposition follows for $j > N$, since $G^0_\rho(S_j) = G_\rho(S_j)$.

Therefore, we can assume that $c = \liminf_{j\to\infty} r_1(j) < \infty$. Since there are finitely many (up to translation) vectors with endpoints in $\mathbb{Z}^{d+1}$ which are shorter than $c+1$, there exists a subsequence $\{j_k\}$ of $\{j\}$ and a number $0 < r_1 < c+1$ such that $r_1 = r_{i_k}(j_k)$ for some $1 \le i_k \le m_{j_k}$ for all k (take r_1 to be the smallest possible). For convenience, denote the subsequence j_k by j again, and consider the sets $\mathcal{R}_1(j) = \mathcal{R}(j)\backslash\{r_1\}$. Suppose that $\liminf_{j\to\infty} \min\{r \in \mathcal{R}_1(j)\} = \infty$. Then Proposition 1.17 holds for $\rho_1 = 2r_1$ and $\{j_k\} = \{j\}$.

To prove it, let $\rho > 2r_1$. Choose N_1 as above, and choose N_2 so that for $j > N_2$, $r \gg \rho(d+1)!/2$ for each $r \in \mathcal{R}_1(j)$ so that Proposition 1.16 holds (for $n = d-1$) for each $(d-1)$-dimensional Ω_i whose radius is bigger than r_1. Consider the graph $G_\rho(S_j)$. Its connected components which are not $(d-2)$-dimensional spheres are $(d-1)$-dimensional spheres of radius r_1. The distance between any two vertices of such a component is less than or equal to $2r_1$. Any two vertices ξ and η of the graph $G^{\rho_1}_\rho(S_j)$ satisfy

$$\rho_1 < |\xi - \eta| < \rho \tag{24}$$

so such vertices cannot belong to a connected component of $G_\rho(S_j)$ which is a $(d-1)$-dimensional sphere. Therefore, Proposition 1.17 follows for $N = \max(N_1, N_2)$.

Accordingly, we can assume that $c_1 = \liminf_{j\to\infty} \min\{r \in \mathcal{R}_1(j)\} < \infty$. As before, we can prove that there exists $r_2 > r_1$ and a subsequence $\{j'_k\}$ of $\{j\}$ such that r_2 is in $\mathcal{R}_1(j'_k)$ for every k. We shall denote j'_k by j again for convenience. Define $\mathcal{R}_2(j) = \mathcal{R}_1(j)\backslash\{r_2\}$. We claim that $\liminf_{j\to\infty} \min\{r \in \mathcal{R}_2(j)\} = \infty$.

Suppose that our claim is not true. Then there is a number $r_3 > r_2$ and a subsequence j_l of j such $r_3 \in \mathcal{R}_2(j_l)$ for every l. In other words, r_1, r_2 and r_3 are in $\mathcal{R}(j_l)$ for every l. Since there are only finitely many different (up to translation) d-simplices with vertices in $\mathbb{Z}^{d+1}$ whose radius of a circumscribed sphere is less than or equal to r_3, there exists a subsequence j_n of j_l and three d-simplices, Δ_1, Δ_2 and Δ_3 such that the radius of the circumscribed sphere of Δ_i is equal to r_i and that there are Δ_i-resonant point on each S_{j_n} for each $1 \le i \le 3$ and for each n, which contradicts the Geometric Lemma.

We claim then that Proposition 1.17 holds for $\rho_1 = 2r_2$ and $\{j_k\} = \{j\}$. To prove it, let $\rho > 2r_2$. Choose N_1 as above, and choose N_2 so that for $j > N_2$, $r \gg \rho(d+1)!/2$ for each $r \in \mathcal{R}_2(j)$ so that Proposition 1.16 holds (for $n = d-1$) for each $(d-1)$-dimensional Ω_i whose radius is bigger than r_2. Consider the graph

$G_\rho(S_j)$. Its connected components which are not $(d-2)$-dimensional spheres are $(d-1)$-dimensional spheres of radius r_1 or r_2. The distance between any two vertices of such a component is less than or equal to $2r_2$, so such points cannot be the vertices of the graph $G_\rho^{\rho_1}(S_j)$. Accordingly, Proposition 1.17 follows for $N = \max(N_1, N_2)$. The proof is finished.

□

Corollary 4.1. *Proposition 1.17 for $d+1=3$ implies that the connected components of the graph $G_\rho^{\rho_1}(S_{j_k})$ are pairs of points or just single points.*

We finally remark that given $0 < r_1 < r_2$, a subsequence j_k of j such that $r_1 \in \mathcal{R}(j_k)$ and $r_2 \in \mathcal{R}(j_k)$ for each k, and a number $\rho > 2r_2$; the radius R_{j_k} of the sphere S_{j_k} for which Proposition 1.17 holds can be effectively bounded below, since the proof of the Geometric Lemma is effective.

5. Quantum limits in dimensions three and higher

We now turn to quantum limits on a three-dimensional torus.

Proof of Theorem 1.4 for $d=3$. Take a sequence φ_j of eigenfunctions and a quantum limit $d\nu = \lim_{j\to\infty} d\mu_j$, and let $d\nu$ have Fourier expansion (2). To prove the estimate (3), it suffices to show that there exists such $\rho_1 > 0$ that for every $\rho > \rho_1$

$$\sum_{\substack{\tau\in\mathbb{Z}^3\\ \rho_1<|\tau|\le\rho}} |c_\tau| \ \le\ 1, \tag{25}$$

since $|c_\tau| \le 1$ for every τ (since, in the notation of section 0, $|b_\tau(j)| \le 1$ for every j and τ and $c_\tau = \lim_{j\to\infty} b_\tau(j)$), and since there are finitely many τ with $|\tau| < \rho_1$. Thus letting $\rho \to \infty$ in (25) we obtain (3).

To prove (25), it suffices to show that there exists a subsequence j_k of j and $N \in \mathbb{N}$ such that for every $j_k > N$

$$\sum_{\substack{\tau\in\mathbb{Z}^3\\ \rho_1<|\tau|\le\rho}} |b_\tau(j_k)| \ \le\ 1, \tag{26}$$

since $c_\tau = \lim_{k\to\infty} b_\tau(j_k)$ for every subsequence j_k of j. We are going to prove (26).

Recall that if $|\varphi_j|^2$ has the Fourier expansion (7), then

$$b_\tau(j) \ =\ \sum_{\substack{\xi-\eta=\tau\\ |\xi|^2=|\eta|^2=\lambda_j}} a_\xi(j)\bar{a}_\eta(j)$$

So, we can estimate the sum in (26) by

$$\sum_{\rho_1<|\tau|\le\rho} \ \sum_{\substack{\xi-\eta=\tau\\ |\xi|^2=|\eta|^2=\lambda_{j_k}}} \frac{|a_\xi(j_k)|^2 + |a_\eta(j_k)|^2}{2} \tag{27}$$

We know that $\sum_{|\xi|^2=\lambda_j} |a_\xi(j)|^2 = 1$. So, to prove (26) it will suffice to show that after the change of order of summation in (27) each point ξ appears at most twice in (27) for $j_k > N$.

Now, choose ρ, ρ_1, $\{j_k\}$ and N as in Proposition 1.17. By corollary 4.1 then, every ξ has at most *one* neighbor η satisfying (24), hence it appears at most twice in (27) after the change of summation (for $\tau = \pm(\xi - \eta)$). This proves (26) and Theorem 1.4 in dimension three follows.

$\square$

To establish that there exist quantum limits on $\mathbb{T}^3$ which are not trigonometric polynomials, we remark that it follows from the discussion at the end of section 8 that conjecture 8.2 implies the existence of a sequence $\{\tau_j\}$ of vectors in $\mathbb{Z}^3$ and a sequence M_k of positive integers such that on a sphere of radius $\sqrt{M_k}$ in $\mathbb{R}^3$ centered at $\mathbf{0}$ there are τ_j-resonant pairs for $1 \leq j \leq k$ (in fact, we can take the frequencies τ_j to be $(2a_j, 0, 0)$). This proves the existence of the required quantum limits.

Unlike for $d = 3$, it is easy to show that on $\mathbb{T}^4$ (and hence on $\mathbb{T}^d$ for $d \geq 4$) there exist quantum limits that are not trigonometric polynomials. In fact, since vectors are simplices of codimension three in $\mathbb{R}^4$, the discussion at the end of section 8 proves the existence of a sequence S_k of spheres in $\mathbb{R}^4$ centered at the origin such that on S_k there are $(2j, 0, 0, 0)$-resonant points for $1 \leq j \leq k$. This establishes (unconditionally) the existence of the required quantum limits.

We next turn to proving Propositions 1.7 and 1.11. The first part of the proof is an application of Proposition 1.17, while the second part uses the ideas of the proof of the existence of quantum limits on $\mathbb{T}^4$ whose densities are not trigonometric polynomials.

Proof of Propositions 1.7 and 1.11. We first prove that $B(d,p) < \infty$ (respectively, $D(d,p) < \infty$) implies the corresponding L^p statement for the densities of quantum limits on $\mathbb{T}^{d+2}$ (respectively, the corresponding l^q statement for their Fourier series).

Given a sequence φ_j of eigenfunctions on $\mathbb{T}^{d+2}$ converging to $d\nu$, choose a subsequence j_k (henceforth called j for convenience) and a number ρ_1 as in Proposition 1.17 (where the sphere S_j is a sphere in $\mathbb{R}^{d+2}$ of radius $\sqrt{\lambda_j}$ centered at the origin).

The trigonometric polynomials are dense in $L^{p/2}(\mathbb{T}^{d+2})$. So, for proving the first part of Proposition 1.7 it suffices to prove that for any trigonometric polynomial $g \in L^r(\mathbb{T}^{d+2})$ (where $2/p + 1/r = 1$) with $||g||_r \leq 1$

$$\overline{\lim}_{j\to\infty} \int |\varphi_j|^2 \, g \ \leq \ C \tag{28}$$

where C does not depend on g. Moreover, one can assume without loss of generality that $\widehat{g}(\tau) = 0$ for $|\tau| \leq \rho_1$. Given such a trigonometric polynomial g, let $\widehat{g}(\tau) = 0$ for $|\tau| \geq \rho$ (where of course $\rho > \rho_1$).

To prove the first part of Proposition 1.11 it suffices to show that for any $\rho > \rho_1$, the sum (in the notation of (7))

$$\left(\sum_{\substack{\tau(j)\in\mathbb{Z}^{d+2} \\ \rho_1 < |\tau(j)| \leq \rho}} |c_\tau|^q \right)^{1/q} \tag{29}$$

is uniformly bounded above (and the bound is independent of ρ).

Given $\rho > \rho_1$, choose $N = N(\rho)$ as in Proposition 1.17, and take $j > N$. By Proposition 1.17 we conclude that the connected components of the graph $G_\rho^{\rho_1}(S_j)$ all lie on $(d-1)$-dimensional spheres $\Omega_1, \Omega_2, \ldots, \Omega_m$. Denote the sum in (29) by

$F(\rho)$. It is easy to see that

$$F(\rho) \;=\; \int |\varphi_j|^2 \,\psi$$

where ψ is a trigonometric polynomial with $\widehat{\psi}(\tau) = 0$ for $|\tau| \geq \rho$ and for $|\tau| \leq \rho_1$, and with $||\widehat{\psi}||_s = 1$. Here $s \leq 2$ is the conjugate exponent of q (i.e. $1/s + 1/q = 1$).

Let φ_j be given by (6). Define a function $\varphi_j(i)$ by

$$\varphi_j(i)(x) \;=\; \sum_{\xi \in \Omega_i \cap \mathbb{Z}^{d+2}} a_\xi \, e^{i\,(\xi,\,x)}$$

By the definition of the graph $G_\rho^{\rho_1}(S_j)$

$$F(\rho) \;=\; \sum_{i=1}^{m} \int |\varphi_j(i)|^2 \,\psi$$

Also,

$$\int |\varphi_j|^2 \, g \;=\; \sum_{i=1}^{m} \int |\varphi_j(i)|^2 \, g$$

By Hölder inequality

$$\int |\varphi_j|^2 \, g \;\leq\; \sum_{i=1}^{m} ||\varphi_j(i)||_p^2 \; ||g||_r$$

Also, let $f_i = |\varphi_j(i)|^2$. By Hölder inequality

$$F(\rho) \;\leq\; \sum_{i=1}^{m} ||\widehat{f_i}||_q ||\widehat{\psi}||_s$$

By the definition (5) of $B(d,p)$ and by the assumption of Proposition 1.7

$$||\varphi_j(i)||_p \;\leq\; B(d,p)||\varphi_j(i)||_2$$

Also, by the definition (10) of $D(d,q)$ and by the assumption of Proposition 1.11,

$$||\widehat{f_i}||_q \;\leq\; D(d,q)||\varphi_j(i)||_2^2$$

Accordingly, both $\int |\varphi_j|^2 g$ and $F(\rho)$ are uniformly bounded above by a multiple of

$$\sum_{i=1}^{m} ||\varphi_j(i)||_2^2 \;\leq\; ||\varphi_j||_2^2 \;=\; 1$$

This gives the required uniform bounds for (28) and (29) and finishes the proof of the first statements of Propositions 1.7 and 1.11.

We next turn to proving the second statements in the above Propositions. By the assumption of Proposition 1.7 $A(d,p) = \infty$, so there exists a sequence of eigenfunctions φ_j on $\mathbb{T}^d$ such that $||\varphi_j||_2 = 1$ and $||\varphi_j||_p = F(j)$ for some $2 < p$ where $F(j)$ is a rapidly increasing function (the rate of growth will be specified later). Let $r > 1$ be given by $2/p + 1/r = 1$. It is easy to show that for each j there exists a trigonometric polynomial g_j with $||g_j||_r \leq 1$ and

$$\int |\varphi_j|^2 \, g_j \;\geq\; F(j)/2 \tag{30}$$

Also, since $|\widehat{g}_j(\tau)| \leq 1$ for any $\tau \in \mathbb{Z}^d$ by Hölder inequality, we can assume without loss of generality that $\widehat{g}_j(\mathbf{0}) = 0$.

By the assumption of Proposition 1.11 $C(d,p)=\infty$, so there exists a sequence of eigenfunctions φ_j on $\mathbb{T}^d$ such that $||\varphi_j||_2=1$ and $||\widehat{f_j}||_q=G(j)$ for $q>1$, where $f_j=|\varphi_j|^2$ and $G(j)$ is a function with the same properties as $F(j)$ above. Define $s>1$ by $1/q+1/s=1$. It is easy to see that there for each j there exists a trigonometric polynomial ψ_j such that $||\widehat{\psi_j}||_s\le 1$, $\widehat{\psi_j}(\mathbf{0})=0$ and

$$\int |\varphi_j|^2\,\psi_j \;\ge\; F(j)/2 \tag{31}$$

Let $f_m(x)=f(mx)$ for $m\in\mathbb{N}$ and $x\in\mathbb{T}^d$. Since $\int f_m(x)dx=\int f(x)dx$ and $\widehat{f_m}(\tau)=\widehat{f}(m\tau)$ for every m, we can arrange inductively by the above remarks that

$$\int |\varphi_i|^2\,g_j \;=\; 0 \tag{32}$$

for $i\neq j$ in the assumptions of Proposition 1.7, and

$$\int |\varphi_i|^2\,\psi_j \;=\; 0 \tag{33}$$

for $i\neq j$ in the assumptions of Proposition 1.11. Also, we can clearly assume that the eigenvalues λ_j-s are increasing.

Also, if we denote by $\lambda_j\in\mathbb{N}$ the eigenvalues of φ_j, we can assume without loss of generality by the above remarks that $8|\lambda_j$ for every j (this will be used later in constructing the quantum limits on $\mathbb{T}^{d+3}$).

Define a function g on $\mathbb{T}^{d+3}$ by

$$g(x_1,\dots,x_{d+3}) \;=\; \sum_{i=1}^{\infty}\frac{g_i(x_1,\dots,x_d)}{2^j} \tag{34}$$

By construction, $g\in L^r(\mathbb{T}^{d+3})$ and $||g||_r\le 1$. Similarly, define a function ψ on $\mathbb{T}^{d+3}$ by

$$\psi(x_1,\dots,x_{d+3}) \;=\; \sum_{i=1}^{\infty}\frac{\psi_i(x_1,\dots,x_d)}{2^j} \tag{35}$$

By construction, $\widehat{\psi}\in l^s(\mathbb{Z}^{d+3})$ and $||\widehat{\psi}||_s\le 1$.

We shall now construct a sequence f_j of eigenfunctions on $\mathbb{T}^{d+3}$ converging to a quantum limit whose density is not in $L^{p/2}$. Let f_j have the eigenvalue $M_j\in\mathbb{N}$. Choose $M_k>\lambda_k$ and $M_k\not\equiv 0,4,7(\bmod 8)$. Consider the sphere S_k of radius $\sqrt{M_k}$ in $\mathbb{R}^{d+3}$ centered at the origin. Since $8|\lambda_j$ by construction, $M_k-\lambda_j$ is a sum of three squares for $1\le j\le k$, so among the $(d-1)$-dimensional spheres obtained by intersecting S_k with d-dimensional planes $\{x : x_{d+1}=c_1, x_{d+2}=c_2, x_{d+3}=c_3\}, c_i\in\mathbb{Z}$, there are spheres $\Omega_j(k)$ of radius $\sqrt{\lambda_j}$. Let $x_{d+i}=c_i(j,k),\ i=1,2,3$ for all $x\in\Omega_j(k)$.

Let φ_j be the sequence of eigenfunctions in the assumption of Propositions 1.7. Define $\tilde{\varphi}_j(k)$ for $1\le j\le k$ by

$$[\tilde{\varphi}_j(k)](x_1,\dots,x_{d+3}) \;=\; \varphi_j(x_1,\dots,x_d)\,e^{i\,(c_1x_{d+1}+c_2x_{d+2}+c_3x_{d+3})} \tag{36}$$

where $c_l=c_l(j,k)$ for $1\le l\le 3$. Also, let $\tilde{f}_k$ be a trigonometric polynomial with frequencies in

$$S_k\backslash(\Omega_1(k)\cup\ldots\cup\Omega_k(k))$$

and with $||\tilde{f}_k||_2^2=2^{-k}$.

Finally, define the eigenfunction f_k (with the eigenvalue M_k) on $\mathbb{T}^{d+3}$ by

$$f_k \; = \; \tilde{f}_k \; + \; \sum_{j=1}^{k} \frac{\tilde{\varphi}_j(k)}{2^{j/2}} \tag{37}$$

By construction, $||f_k||_2 = 1$. It is also easy to see that one can choose $\Omega_j(k)$-s and $\tilde{f}_k$-s so that $|f_k|^2$ converge weak $*$ to a quantum limit $d\nu$ whose density is equal to

$$f \; = \; \sum_{k=1}^{\infty} \frac{|\varphi_k|^2}{2^k} \tag{38}$$

Now, it follows from (30), (32), (34), (36) and (37) that

$$\int |f_k|^2 \, g \; \geq \; \sum_{j=1}^{k} \frac{F(j)}{2 \cdot 4^j}$$

Accordingly, if we choose the sequence φ_j on $\mathbb{T}^d$ so that the series $\sum_{j=1}^{\infty} F(j)/4^j$ diverges, we can conclude that $f \notin L^{p/2}(\mathbb{T}^{d+3})$, proving Proposition 1.7.

Proposition 1.11 is proved similarly; namely, one defines eigenfunctions h_j on $\mathbb{T}^{d+3}$ by (36) and (37) (which converges to a quantum limit with the density given by (38) as before), and uses the formulas (31), (33) and (35) to conclude that the Fourier transform of the above density is not in $l^q(\mathbb{Z}^{d+3})$. This finishes the proof of Proposition 1.11.

$\square$

We next turn to the

Proof of Theorem 1.4 for $d = 4$ and $d = 5$. We first remark that both Proposition 1.1 (cf. [Zyg]) and Proposition 1.9 (cf. section 6) are proved by a geometric argument that can be easily generalized to show that $D(2,2) < \infty$ and $D(3,3) < \infty$. The Theorem now follows from the first part of Proposition 1.11 proved above.[5]

$\square$

We now want to prove another statement about quantum limits on $\mathbb{T}^d$ for $d \geq 4$. We have proved the existence of quantum limits on $\mathbb{T}^d$ whose densities are not trigonometric polynomials. We shall now discuss the existence of quantum limits whose Fourier series are not in l^1 (i.e. for which the sum $\sum_\tau |c_\tau|$ diverges). We shall deduce from the conjecture 8.2 the existence of such limits for $d = 4$, while for $d \geq 5$ their existence will be proved unconditionally. The same proof can be generalized to establish the existence of quantum limits whose Fourier series are not in $l^{2-\delta}$ for any $\delta > 0$. The rate of possible divergence of the sum $\sum_\tau |c_\tau|$ will be estimated in section 7.

Recall that $r_2(n)$ denotes the number of representations of n as a sum of two squares. It suffices to show that there exists two increasing sequences of positive integers a_j and M_k such that $r_2(a_j) \to \infty$ as $j \to \infty$ and $M_k - a_j^2$ is representable as a sum of $d-2$ squares for $1 \leq j \leq k$ (and hence on a $(d-1)$-sphere S_k of radius $\sqrt{M_k}$ there exists a section of S_k by a plane parallel to (x_1, x_2)-plane which is a circle of radius a_j containing $r_2(a_j^2)$ lattice points). But that is precisely the statement of conjecture 8.2 for $d = 4$; for $d \geq 5$ this will be proved unconditionally in section 8.

[5] A slightly different proof of the above Theorem for $d = 4$ is given in [Jak1].

Finally, we discuss microlocal quantum limits on $\mathbb{T}^d$ for $d \geq 3$. Writing a microlocal quantum limit in the form (16), we see that for $\tau \neq \mathbf{0}$, the measure $f_\tau(\theta)\,d\theta$ is supported on a $(d-2)$-dimensional sphere[6] Ω_τ consisting of the unit vectors in $\mathbb{R}^d$ which are orthogonal to τ, and that

$$\int_{S^{d-1}} f_\tau(\theta)\,d\theta \;=\; c_\tau$$

Given the sequence of eigenvalues λ_j, the possible values of $f_\tau(\theta)$ depend on how well τ-resonant pairs become distributed (near the plane of Ω_τ) on the sphere of radius $\sqrt{\lambda_j}$ centered at the origin in $\mathbb{R}^d$.

$f_{\mathbf{0}}(\theta)\,d\theta$ is a nonnegative measure on S^{d-1} of mass 1. As in section 2, we can write this measure as $\nu_1+\nu_2$ where ν_1 comes from τ-resonant pairs (for such τ that $f_\tau(\theta)\not\equiv 0$), and ν_2 corresponds to the remaining points. Since the lattice points on the sphere S^n become equidistributed for $n \geq 2$ as the radius of the sphere increases (cf. [DSP]), we can arrange the eigenfunctions φ_j so that ν_2 is any nonnegative measure of total mass $1-\nu_1(S^{d-1})$ (if we are given f_τ-s for $\tau \neq \mathbf{0}$ and ν_1).

6. A generalization of Zygmund's result

We start this section by proving Proposition 1.9. To do that, we first consider the following question: given two points A and B on a two-sphere S in $\mathbb{R}^3$ which do not coincide and are not diametrically opposite, and given two parallel and equal chords $\mathbf{u}$ and $\mathbf{v}$ of S (not equal to AB) such that A is one of the endpoints of $\mathbf{u}$, and B is one of the endpoints of $\mathbf{v}$, where on S can other endpoints of $\mathbf{u}$ and $\mathbf{v}$ lie?

Denote the other endpoints of $\mathbf{u}$ and $\mathbf{v}$ by A_1 and B_1, and let BB' and B_1B_1' be diameters of S (note that $B' \neq A$). If $\overline{AA_1} = \overline{BB_1}$ then $A_1 = B_1 + \overline{BA}$. If $\overline{AA_1} = -\overline{BB_1}$ then $A_1 = B_1' + \overline{B'A}$. So either the point $A_1 + \overline{AB}$ or the point $A_1 + \overline{AB'}$ lies on S.

Now, suppose that we are given *three* different points A, B and C (no two of which are diametrically opposite), which are the endpoints of three parallel and equal nonzero chords $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ of S, respectively. Suppose also that none of the three vectors connects any two of the three points. Let A_1 be the other endpoint of $\mathbf{u}$. Then both points in one of the following four pairs: (i) $(A_1+\overline{AB}, A_1+\overline{AC})$; (ii) $(A_1+\overline{AB}, A_1+\overline{AC'})$; (iii) $(A_1+\overline{AB'}, A_1+\overline{AC})$; (iv) $(A_1+\overline{AB'}, A_1+\overline{AC'})$; lie on S. Suppose we are in the case (i). By a well-known property of (strictly) convex surfaces,[7] there are at most two points on S satisfying the conditions of (i); one of such points is A. Since $\mathbf{u} = \pm\overline{AA_1} \neq \mathbf{0}$, there is at most *one* point A_1 on S such that (i) holds. The cases (ii), (iii) and (iv) give us at most three other possibilities, so there are at most *four* possibilities altogether for A_1.

Proof of Proposition 1.9. Let φ be an eigenfunction of the Laplacian on $\mathbb{T}^3$, and let $|\varphi|^2$ have the Fourier expansion (7), and let S denote the sphere in $\mathbb{R}^3$ of radius $\sqrt{\lambda}$ centered at the origin. We are going to estimate $\sum_\tau |c_\tau|^3$ by

$$M \cdot \left(\sum_{\xi\in S} |a_\xi|^2\right)^3 \tag{39}$$

[6] On $\mathbb{T}^2$, this sphere has dimension zero and consists of two points $\tau/|\tau| \pm \pi/2$.

[7] Given a triangle $\Delta \in \mathbb{R}^3$, there are at most *two* Δ-resonant triples of points on S; this is a three-dimensional analogue of the property Zygmund used in proving Proposition 1.1.

with $M = 6$ independent of λ, thus proving that $C(3,3) \leq M^{1/3} < \infty$ (where C(3,3) is defined by (9)). It suffices to estimate

$$\sum_{0 \neq \tau \in \mathbb{Z}^3} \left(\sum_{\substack{\xi,\eta \in S \\ \xi-\eta=\tau}} |a_\xi| \cdot |a_\eta| \right)^3 \tag{40}$$

When we expand the above expression, we will get the sum of expressions of type

$$|a_{\xi_1}| \cdot |a_{\eta_1}| \cdot |a_{\xi_2}| \cdot |a_{\eta_2}| \cdot |a_{\xi_3}| \cdot |a_{\eta_3}| \tag{41}$$

for some $\tau = \xi_1 - \eta_1 = \xi_2 - \eta_2 = \xi_3 - \eta_3$. We first estimate (40) when at least two of ξ_i-s coincide and are equal to ξ_0, forcing η_i-s to coincide and be equal to some η_0. After changing the summation, the sum of the above terms in (40) for fixed ξ_0 and η_0 can be estimated by trivially by

$$|a_{\xi_0}|^2|a_{\eta_0}|^2 \left(\sum_{\xi-\eta=\xi_0-\eta_0} \frac{|a_\xi|^2+|a_\eta|^2}{2} \right) \leq |a_{\xi_0}|^2|a_{\eta_0}|^2 \sum_{\xi \in S} \frac{|a_\xi|^2}{2} = \frac{|a_{\xi_0}|^2|a_{\eta_0}|^2}{2}$$

and the sum of the above terms over different (ξ_0, η_0) can be estimated by $(\sum_{\xi \in S} |a_\xi|^2)^2$ which is equal to 1.

Accordingly, it remains to estimate the sum of the terms (41) with different ξ_i-s (and therefore different η_i-s). Estimate the product (41) by

$$\frac{1}{2}\left(|a_{\xi_1}|^2|a_{\xi_2}|^2|a_{\xi_3}|^2 + |a_{\eta_1}|^2|a_{\eta_2}|^2|a_{\eta_3}|^2\right) \tag{42}$$

We claim that the above sum can be estimated by (39) with $M = 4$, thus proving Proposition 1.9.

Let χ_1, χ_2 and χ_3 be such that the term $|a_{\chi_1}|^2|a_{\chi_2}|^2|a_{\chi_3}|^2$ appears in (39) more than 4 times. No two of these points can coincide by our assumption. Also if (say) χ_1 and χ_2 are diametrically opposite, then there are no points $\tilde{\chi}_1$ and $\tilde{\chi}_2$ on S such that $\chi_1 - \tilde{\chi}_1 = \chi_2 - \tilde{\chi}_2$, hence by (42) the term $|a_{\chi_1}|^2|a_{\chi_2}|^2|a_{\chi_3}|^2$ doesn't appear in (39), which is a contradiction.

Therefore, by the discussion in the beginning of this section, we get at most four possibilities for $\tilde{\chi}_1$, which contradicts the assumption that the term $|a_{\chi_1}|^2|a_{\chi_2}|^2|a_{\chi_3}|^2$ appears in (39) more than four times. This finishes the proof of Proposition 1.9. □

We remark that in the proof of Proposition 1.9 we never used the fact that the frequencies ξ of the trigonometric polynomial $\varphi(x)$ are in $\mathbb{Z}^3$. Accordingly, Proposition 1.9 holds for arbitrary trigonometric polynomial[8] with frequencies on a two-sphere in $\mathbb{R}^3$ (or on a two-sphere lying in a three-dimensional subspace of $\mathbb{R}^5$, thus proving that $D(3,3) < \infty$, a statement needed in the proof of Theorem 1.4 for $d = 5$).

It is also clear from the proof that Proposition 1.9 generalizes to trigonometric polynomials all of whose frequencies lie on any strictly convex symmetric surface in $\mathbb{R}^3$.

[8]We prove that $||\widehat{f}||_3 \ll ||\widehat{\varphi}||_2^2 = \sum_\tau |a_\tau|^2$.

7. Estimating sums of powers of normed Fourier coefficients in a ball

In this section we prove Proposition 1.12. We proved in this paper that the sum of absolute values of Fourier coefficients of any quantum limit on $\mathbb{T}^3$ converges. As usually, we let a quantum limit $d\nu$ have the Fourier expansion (2). We proved in section 5 that sum of normed Fourier coefficients of a quantum limit can diverge on $\mathbb{T}^d$ for $d \geq 4$. We shall estimate the rate of such divergence by estimating the sum $\Sigma(\rho)$ defined in (12).

Given a sequence φ_j of eigenfunctions of the Laplacian on $\mathbb{T}^d$ converging to a quantum limit $d\nu$, choose a subsequence $\{j_k\}$ of $\{j\}$ and $\rho_1 > 0$ such that Proposition 1.17 holds (we shall denote j_k by j for convenience). If we choose $\rho \gg \rho_1$, it suffices to estimate

$$\sum_{\rho_1 < |\tau| < \rho} |b_\tau(j)| \tag{43}$$

for $j > N$, where N is given by Proposition 1.17. Denote by S_j the sphere of radius $\sqrt{\lambda_j}$ centered at the origin in $\mathbb{R}^d$. Choose a point $\xi \in \mathbb{Z}^d$ on S_j, and let the connected component of $G_\rho^{\rho_1}(S)$ to which ξ belongs lie on a $(d-3)$-dimensional sphere Ω of radius R (see Proposition 1.17). By a straightforward modification of the proof of Theorem 1.4 for $d = 3$, we can bound (43) by the number of vertices of $G_\rho^{\rho_1}(S)$ connected to ξ. So, it suffices to estimate the number of these vertices.

We first consider the case $d \geq 5$. By the usual "sphere packing" argument, we can bound the number of the above vertices by the volume of a "polar cap" of radius ρ on Ω around ξ. This gives the bound ρ^{d-3} and proves Proposition 1.12 for $d \geq 6$. We now turn to estimating the bounds for $d = 4, 5$.

Proof of Proposition 1.12 for $d = 4$ and $d = 5$ We first consider $d = 4$. Then in the above notation Ω is just a circle in some two-dimensional subspace of $\mathbb{R}^4$. Suppose that the point ξ (defined above) is connected to at least two vertices of $G_\rho^{\rho_1}(S)$ lying on Ω. Then Ω is a circumscribed circle around a triangle with two sides shorter than ρ and with vertices in $\mathbb{Z}^4$. Using the well-known formula

$$abc = 4AR,$$

(where a, b, c are the sides of the triangle and A is its area), we conclude that $R = R(\Omega) \ll \rho^3$, and the number of such vertices lying on Ω is $\ll (\rho^3)^{\varepsilon/3} = \rho^\varepsilon$. Also, when $R \gg \rho^3$, there can be at most one vertex connected to ξ on Ω. This allows us to bound (43) by ρ^ε and finishes the proof of Proposition 1.12 for $d = 4$.[9]

For $d = 5$ we proceed differently. We remark that the sum (43) is equal to

$$\int_{\mathbb{T}^d} |\varphi_j(x)|^2 f_\rho(x; j)\, dx$$

where $f_\rho(x; j)$ is a trigonometric polynomial whose coefficients $d_\tau(j)$ are equal to zero for $|\tau| \geq \rho$ and for $\tau \leq \rho_1$ and have absolute value one for $\rho_1 < |\tau| < \rho$.

Apply Proposition 1.17 for $d = 5$ and let the connected components of $G_\rho^{\rho_1}(S_j)$ be located on two-dimensional spheres which we denote by $\Omega_\alpha = \Omega_\alpha(j)$. As in

[9]The argument above gives a bound of $\max_{n \ll \rho^3}\{r_2(n)\}$, where $r_2(n)$ denotes the number of ways of representing n as a sum of two squares. Using the results of [CC] we can bound (43) by $\max_{n \ll \rho^2}\{r_2(n)\}$. It is well-known that $r_2(n)$ grows slower than any power of n but can grow faster than any power of $\log n$.

the proof of Theorem 1.3, denote the radius of Ω_α by r_α, and $\sum_{\xi\in\mathbb{Z}^5\cap\Omega_\alpha}|a_\xi|^2$ by $c_\alpha = c_\alpha(j)$. Let

$$\varphi_j(\alpha)(x) \;=\; \sum_{\xi\in\mathbb{Z}^5\cap\Omega_\alpha} a_\xi\; e^{i\,(x,\xi)}$$

By the definition of $G^{\rho_1}_\rho(S_j)$ and f_ρ then,

$$\int |\varphi_j|^2 f_\rho(j) \;=\; \sum_\alpha \int |\varphi_j(\alpha)|^2 f_\rho(j) \tag{44}$$

It suffices to estimate (44). If we assume that the lattice points on Ω_α do not all lie in a single two-plane, we conclude that Ω_α is circumscribed around a tetrahedron whose vertices have integer coordinates and whose sides are $\ll \rho$. It is easy to see using Kramer's rule that for a d-dimensional sphere Ω_α circumscribed around a d-dimensional simplex with above properties, the radius r_α satisfies

$$r_\alpha \ll \rho^{d+2} \tag{45}$$

Therefore, by the formula (45) with $d=2$, if $r_\alpha \gg \rho^4$ then all the lattice points on Ω_α lie on a circle and we can estimate the corresponding term similarly to those for $\mathbb{T}^4$. So, it suffices to consider only the terms with $r_\alpha \ll \rho^4$.

The two-sphere Ω_α lies in some three-dimensional subspace of $\mathbb{R}^5$, and all the nonzero frequencies of $|\varphi_j(\alpha)|^2$ lie in the same subspace. Accordingly, the integrals in the right-hand side of (44) can be rewritten as

$$\int |\varphi(\alpha)|^2 \cdot f_\rho(\alpha)$$

where $f_\rho(\alpha) = f_\rho(j;\alpha)$ is a trigonometric polynomial with $||f_\rho(\alpha)||_2 \le 1$ all of whose frequencies lie in the same three-dimensional subspace of $\mathbb{R}^5$ in which Ω_α lies. These integrals can be estimated using Schwarz inequality by

$$||f_\rho(\alpha)||_2 \cdot ||\varphi_j(\alpha)||_4^2 \tag{46}$$

Now, $||f_\rho(\alpha)||_2 \ll \rho^{3/2}$. It remains to estimate $||\varphi_j(\alpha)||_4^2$.

Denote by $b_\tau(j;\alpha)$ the Fourier coefficients of $|\varphi_j(\alpha)|^2$. Since Ω_α is two-dimensional, the number of terms in the sum (7) defining $b_\tau(\alpha)$ is $\ll r_\alpha^\varepsilon$ for every ε, and one easily deduces that

$$||\varphi_j(\alpha)||_4^2 \;=\; \left(\sum_\tau |b_\tau(j;\alpha)|^2\right)^{1/2} \;\ll\; c_\alpha(j) r_\alpha(j)^\varepsilon$$

for every $\varepsilon > 0$. Since we are only considering $r_\alpha \ll \rho^4$ (see above), and since $\sum_\alpha c_\alpha(j) \le 1$, we can sum the estimates (46) for the terms of (44) and prove Proposition (1.12) for $d=5$. The proof is finished.

□

By interpolation we obtain the following

Corollary 7.1. *On the d-dimensional torus for $d \ge 6$ for any $1 \le p \le \infty$*

$$\left(\sum_{|\tau|<\rho} |c_\tau|^p\right)^{1/p} \;\ll\; \rho^{(d-3)/p} \tag{47}$$

For $d=5$, the above sum is $\ll \rho^{3/2p+\varepsilon}$ and for $d=4$ it is $\ll \rho^\varepsilon$ for every $\varepsilon > 0$.

We next obtain a better bound for the sum (47) for $p = 2$ and $d = 5$ (recall that (47) is uniformly bounded for $p = 2$ and $d = 4$). By the standard argument it suffices to estimate

$$\int_{\mathbb{T}^d} |\varphi_j(x)|^2 f_\rho(x;j)\,dx$$

where $f_\rho(x;j)$ is a trigonometric polynomial with $||f_\rho||_2 = 1$ whose coefficients $d_\tau(j)$ are equal to zero for $|\tau| \geq \rho$ and for $\tau \leq \rho_1$. Arguing as in the proof of Proposition 1.12, we can rewrite the above integral as (44) and make the estimate (46). By Proposition 1.16 applied to Ω_α and by Zygmund's Proposition 1.1, it suffices to consider the terms of (44) with $r_\alpha \ll \rho^{12}$, and by the argument of Proposition 1.12, this proves that (47) is bounded by ρ^ε for every $\varepsilon > 0$.

We now turn to studying the *lower* bounds for the sum (12). The discussion at the end of section 8 shows that conjecture 8.2 implies that for $d \geq 5$ there exists a sequence Ω_j of spheres in $\mathbb{R}^{d-2}$ (with increasing radii r_j, centered at the origin) with Ω_j containing $N_j \sim r_j^{d-4}$ lattice points; and a sequence S_j of spheres in $\mathbb{R}^d$ (with increasing radii, centered at the origin) such that if we consider sections of S_k by the subspaces $x_1 = c_1, x_2 = c_2$ where $c_i \in \mathbb{Z}$, spheres Ω_j will appear among these sections for $1 \leq j \leq k$. Also, the discussion at the end of section 8 proves a similar statement unconditionally for simplices of codimension three (and $d \geq 6$). This can be used in deriving lower bounds for the sum $\Sigma(\rho)$ defined in (12).

If we consider the eigenfunction φ on $\mathbb{T}^{d-2}$ with the eigenvalue r_j^2 on $\mathbb{T}^{d-2}$ such that $||\varphi||_2 = 1$ and such that all Fourier coefficients of φ are equal positive numbers, it is easy to see that (in the notation of (7))

$$\sum_{\tau \in \mathbb{Z}^{d-2}} |b_\tau| \ \sim \ r_j^{d-4}$$

(the corresponding sum for the eigenfunction on $\mathbb{T}^{d-3}$ is $\sim r_j^{d-5}$). It is now easy to see that conjecture 8.2 implies that for $d \geq 5$ there exist quantum limits on $\mathbb{T}^d$ with $\limsup_{\rho\to\infty}(\Sigma(\rho)/\rho^{d-4-\varepsilon}) = \infty$ for any $\varepsilon > 0$. Also, this proves unconditionally that for $d \geq 6$ there exist quantum limits on $\mathbb{T}^d$ with $\limsup_{\rho\to\infty}(\Sigma(\rho)/\rho^{d-5-\varepsilon}) = \infty$, as claimed in the introduction.

8. A Geometric Lemma

In this section we present the proof of Lemma 1.14. We start by proving the following (known) result:

Proposition 8.1. *Let $A_1, A_2, A_3 (A_i > 0)$ and C_1, C_2 be integers. Suppose that the system*

$$\begin{cases} A_1 u_1^2 - A_2 u_2^2 &= C_1, \\ A_1 u_1^2 - A_3 u_3^2 &= C_2. \end{cases} \tag{48}$$

has infinitely many integer solutions (u_1, u_2, u_3). Then $C_1 C_2 (C_1 - C_2) = 0$.

The proof we present below uses an effective version of Siegel's Theorem. The author is grateful to the referee for the comments which greatly improved the presentation of the results.

Proof of Proposition 8.1. Let $y = u_2 u_3$ and $x = u_1$. Then each solution (u_1, u_2, u_3) of (48) gives rise to the solution (x, y) of

$$A_2 A_3 y^2 \ = \ (A_1 x^2 - C_1)(A_1 x^2 - C_2) \tag{49}$$

If $C_1C_2(C_1-C_2)\neq 0$, (49) defines a curve of genus one, and so by [Bak75, Theorem 4.3] it has only finitely many integer solutions (x,y), and there exists an explicit upper bound on $|x|,|y|$ in terms of A_i-s and C_j-s, and the same holds for the solutions of (48). This finishes the proof.

□

An alternative proof (given in an earlier version of this paper) uses Baker's Theorem on lower bounds for linear forms in logarithms ([Bak73]), or the version given in [B-D] which is especially well adapted to the present situation. A proof of an equivalent statement can be found in [Mi].

The effectiveness of the proof of Proposition 8.1 implies that given three vectors $\tau_1,\tau_2,\tau_3\in\mathbb{Z}^2$ of different lengths, we can find effectively such $R>0$ (depending on τ_i-s) that any circle in $\mathbb{R}^2$ of radius bigger than R centered at $\mathbf{0}$ cannot contain τ_i-resonant points for every $1\leq i\leq 3$. For $\mathbb{T}^d, d>2$ this implies that given three d-simplices Δ_1,Δ_2 and Δ_3 with vertices in $\mathbb{Z}^d$, whose radii r_1,r_2,r_3 of circumscribed spheres are all different, there is an effectively computable $R=R(r_1,r_2,r_3)$ such that any sphere in $\mathbb{R}^d$ (centered at $\mathbf{0}$) of radius bigger than R, cannot contain Δ_i-resonant points for every $1\leq i\leq 3$.

We now turn to proving the Geometric Lemma for $d=1$; it is a consequence of Proposition 8.1 and the fact that the center of a circle lies on the perpendicular passing through the midpoint of any of its chords.

Proof of Lemma 1.14 for $d=1$. Let a vector $\tau\in\mathbb{Z}^2$ have coordinates (p,q). Let $\xi,\eta\in\mathbb{Z}^2$ be τ-resonant points with coordinates (m,n) and $(m-p,n-q)$ that lie on the same circle of radius R centered at the origin. From $R^2=m^2+n^2=(m-p)^2+(n-q)^2$ we can deduce that $2|(p^2+q^2)$, so $2|(p\pm q)$. Consider the points $P=2(m,n)$ and $Q=2(m-p,n-q)$ on the circle of radius $2R$ centered at zero. The midpoint $S=(2m-p,2n-q)$ of the chord PQ is a lattice point, and the origin lies on a perpendicular to PQ passing through S, so $\overline{OS}\in\mathbb{Z}^2$. Vector $(-q,p)$ is orthogonal to (p,q). Let s be the greatest common divisor of p and q and let

$$p=\tilde{p}s,\ \ q=\tilde{q}s \tag{50}$$

Then it is easy to see that $\overline{OS}$ has to be an integer multiple of $(-\tilde{q},\tilde{p})$, and so by the Pythagorean Theorem

$$4R^2\ =\ (2m)^2+(2n)^2\ =\ p^2+q^2+t^2(\tilde{p}^2+\tilde{q}^2)$$

where $t\in\mathbb{Z}$. The above formula can be written as

$$4R^2\ =\ (\tilde{p}^2+\tilde{q}^2)(s^2+u^2),\quad u=2t+s\Longleftrightarrow 2\,|\,(u-s) \tag{51}$$

All possible values of R are given by (51) with different $u\in\mathbb{Z},2|(u-s)$.

Let $\{\tau_1,\dots,\tau_k\},\tau_i\in\mathbb{Z}^2,\tau_i\neq\mathbf{0}$ with coordinates (p_i,q_i) be as in the assumption of Lemma 1.14 for $d=1$. We find from (51) that this is equivalent to the existence of infinitely many solutions $(u_1,\dots,u_k),u_i\in\mathbb{Z}$ of the system of equations

$$(\tilde{p}_1^2+\tilde{q}_1^2)(s_1^2+u_1^2)\ =\ \dots\ =(\tilde{p}_k^2+\tilde{q}_k^2)(s_k^2+u_k^2),\quad 2\,|\,(u_i-s_i) \tag{52}$$

It suffices to prove Lemma 1.14 for $k=3$, so we can assume that there are two equations in (52). If we denote $\tilde{p}_i^2+\tilde{q}_i^2$ by A_i, and $(p_i^2+q_i^2)=s_i^2(\tilde{p}_i^2+\tilde{q}_i^2)$ by B_i, then these equations can be rewritten as a system of two Pell equations:

$$\left\{\begin{array}{lcl} A_1u_1^2-A_2u_2^2 & = & B_2-B_1, \\ A_1u_1^2-A_3u_3^2 & = & B_3-B_1. \end{array}\right. \tag{53}$$

By Proposition 8.1, if (53) has infinitely many solutions then

$$(B_2 - B_1)(B_3 - B_1)(B_3 - B_2) = 0$$

and two of the lengths coincide. The proof is finished.

□

We next turn to proving the Geometric Lemma for $d > 1$; it is a consequence of Proposition 8.1 and the fact that the center of a d-sphere in $\mathbb{R}^{d+1}$ containing a given d-dimensional simplex lies on the line perpendicular to the hyperspace of the simplex passing through the center of its circumscribed ($(d-1)$-dimensional) sphere.

Proof of Lemma 1.14 for $d > 1$. Let Δ be a d-dimensional simplex with vertices in $\mathbb{Z}^{d+1}$, and let $A_0, A_1, \dots, A_d$ be its vertices. Denote by O_1 the center of the circumscribed sphere around Δ, and by r its radius. It is easy to show that if $A_i \in \mathbb{Z}^{d+1}$ then $O_1 \in \mathbb{Q}^{d+1}$, and the denominators of the coordinates of O_1 are all divisors of the number $2g$, where g is the (integer) determinant of the Gramm-Schmidt matrix of Δ.

Let O be the origin in $\mathbb{R}^{d+1}$, and let R be the radius of a sphere $\mathbb{R}^{d+1}$ containing the Δ-resonant points A_i. Denote by $\mathbf{w}$ the shortest vector with integer coordinates orthogonal to the hyperplane of Δ, and let $|\mathbf{w}|^2 = w$. By the discussion above, the vector $2g\overline{OO_1}$ must be an ineteger multiple of $\mathbf{w}$. So, by the Pythagorean Theorem,

$$R^2 \;=\; \frac{1}{4g^2}\left(4g^2r^2 + m^2 w\right), \quad m \in \mathbb{Z},\ 4g^2 \,|\, (4g^2r^2 + m^2 w) \tag{54}$$

(in the above formula, $4g^2r^2 \in \mathbb{Z}$, while r^2 may be in $\mathbb{Q}$ but not in $\mathbb{Z}$). Accordingly, the radius R of any sphere in $\mathbb{R}^{d+1}$ centered at the origin containing Δ-resonant points is given by the formula (54) for some appropriate $m \in \mathbb{Z}$.

Again, it suffices to prove Lemma 1.14 for $k = 3$. Let $\Delta_1, \Delta_2, \Delta_3$ be d-dimensional simplices as in the assumption of Lemma 1.14. By the above discussion then, there are infinitely many solutions (m_1, m_2, m_3) of the system

$$\frac{1}{4g_1^2}\left(4g_1^2r_1^2 + m_1^2 w_1\right) = \frac{1}{4g_2^2}\left(4g_2^2r_2^2 + m_2^2 w_2\right) = \frac{1}{4g_3^2}\left(4g_3^2r_3^2 + m_3^2 w_3\right) \tag{55}$$

where $4g_i^2 \,|\, (4g_i^2r_i^2 + m_i^2 w_i)$. We need to show that under this assumption r_i-s can take at most two different values. The proof proceeds exactly as for $d = 1$ (using Proposition 8.1 with w_i-s instead of A_i-s) and will be omitted. This finishes the proof of Lemma 1.14 for $d > 1$.

□

We now turn to proving that no finiteness statement analogous to Lemma 1.14 could be true for simplices of codimension two or less. As remarked in the introduction, it suffices to construct an infinite sequence Δ_j of simplices with vertices in $\mathbb{Z}^n$ of codimension two, and a sequence S_k of spheres in $\mathbb{R}^n$ centered at the origin of increasing radii such that on S_k there are Δ_j-resonant points for $1 \le j \le k$. The main applications of that are various "existence" statements for quantum limits (those with infinitely many frequencies for $\mathbb{T}^3$, with $\sum_\tau |c_\tau| = \infty$ on $\mathbb{T}^4$ etc; see also Propositions 1.7 and 1.11).

It suffices to prove the existence of such simplices in a single coordinate subspace of codimension two in $\mathbb{R}^n$. It is easy to see that for every n such that $r_d(n) > 0$, there exists a d-dimensional simplex in $\mathbb{R}^d$ whose vertices are lattice points on the sphere of radius $\sqrt{n}$ centered at the origin. Accordingly, it suffices to produce an increasing sequence of a_j-s and a sequence of spheres S_k (of radius $\sqrt{M_k}$, $M_k \in \mathbb{N}$) in $\mathbb{R}^n$, centered at the origin, such that among the $(n-3)$-dimensional spheres obtained by intersecting S_k with $(n-2)$-dimensional planes $\{x : x_n = c_1, x_{n-1} = c_2\}, c_i \in \mathbb{Z}$, there are spheres of radius a_j for $1 \le j \le k$. For the applications to quantum limits on $\mathbb{T}^4$, we also want to require that $r_2(a_1) < r_2(a_2) < \dots$.

The existence of a sequence of spheres S_k is equivalent to the existence of a sequence of numbers M_k such that $M_k - a_j^2$ is a sum of two squares for $1 \le j \le k$. Accordingly, we state

Conjecture 8.2. *There exists an increasing sequence $a_j \in \mathbb{N}, j = 1, 2, \dots$ such that $r_2(a_j) > r_2(a_{j-1})$, and an increasing sequence $M_k \in \mathbb{N}$ such that $M_k - a_j^2$ is a sum of two squares for $1 \le j \le k$.*

We remark that Hooley proved in [Ho] that for arbitrary $b > c > 0$ there exist infinitely many M_k-s such that each of $M_k, M_k + b, M_k + c$ is a sum of two squares. We shall now show that a weak form of a Hardy-Littlewood conjecture about k-tuples of primes (cf. [HL]) implies conjecture 8.2.

It suffices to have a sequence M_k such that $M_k \pm a_j$ is a prime $p \equiv 1 \pmod 4$ for $1 \le j \le k$ (since every such number is a sum of two squares. So, it would suffice to have find (for each k) $2k$ pairs of twin primes $(p_1, p_1+2), (p_2, p_2+2), \dots, (p_{2k}, p_{2k}+2)$ such that

$$p_1 = M_k - a_k,\ p_2 = M_k - a_{k-1}, \dots, p_{2k} = M_k + a_k \tag{56}$$

for some $M_k > a_k$.

Such a $(4k)$-tuple of primes would clearly be impossible if for some k and for some prime p the numbers

$$\pm a_1, \dots, \pm a_k, 2 \pm a_1, \dots, 2 \pm a_k \tag{57}$$

form a complete system of residues mod p. Hardy and Littlewood conjectured that if this is *not* the case for all primes p and for all $k \in \mathbb{N}$, then infinitely many such $(4k)$-tuples exist; moreover, their density is predicted. We do not need the full strength of the above hypothesis. In fact, it suffices to assume that if $\{b_l\}$ is an increasing sequence b_l such that for each $m \in \mathbb{N}$ the numbers $\{b_1, b_2, \dots, b_m\}$ do not form a complete system of residues mod p for any prime p, then for every m there exists an $(m+1)$-tuple of primes $\{p_0, p_1, \dots, p_m\}$ such that $p_m - p_0 = b_m$.

Now, it is easy to construct by induction an increasing sequence $\{a_k\}$ such that the numbers (57) will not form a complete system of residues for any prime p (we need to worry only about the primes which are less than $4k$). Then the weak form of Hardy-Littlewood's conjecture stated above would imply for each k the existence of $2k$ pairs of twin primes such that the smaller primes in each of $2k$ pairs satisfy (56). This implies conjecture 8.2.

Finally, we consider simplices of codimension three. We will show that for $n \ge 4$ there exists a sequence of spheres S_k (of radius $M_k \in \mathbb{N}$) in $\mathbb{R}^n$, centered at the origin, that among the $(n-4)$-dimensional spheres obtained by intersecting S_k with $(n-3)$-dimensional planes $\{x : x_n = c_1, x_{n-1} = c_2, x_{n-2} = c_3\}, c_i \in \mathbb{Z}$, there are spheres of radius j for $1 \le j \le k$. This is equivalent to the system

$$M_k = j^2 + u_j^2 + v_j^2 + w_j^2$$

having integer solutions (u_j, v_j, w_j) for $1 \le j \le k$.This would clearly imply the existence of a sequence Δ_j of simplices of codimension three such that on S_k there are Δ_j-resonant points for $1 \le j \le k$.

Accordingly, it suffices to have a sequence M_k such that $M_k - j^2$ be representable as a sum of three squares for $1 \leq j \leq k$. It is well-known that n is representable as a sum of three squares if $n \not\equiv 0, 4, 7 (\mathrm{mod}\, 8)$. So, it suffices to take $M_k > k^2$ and $M_k \equiv 2, 6 (\mathrm{mod}\, 8)$. This proves unconditionally that there is no analogue of Lemma 1.14 for simplices of codimension three and completes our discussion of the Geometric Lemma.

References

[Bak73] A. Baker. A sharpening of the bounds for linear forms in logarithms II. *Acta Arithmetica*, 24:33–36, 1973.

[Bak75] A. Baker. Transcedental number theory. Cambridge Univ. Press, 1975.

[B-D] A. Baker and H. Davenport. On the equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$. *Quarterly Journal of Mathematics, Oxford Series*, 20:129–137, 1969.

[Bou93] J. Bourgain. Eigenfunction bounds for the Laplacian on the n-torus. *Intern. Math. Res. Notices*, 3:61–66, 1993.

[Bou94] J. Bourgain. Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Shroödinger equations. preprint, 1994.

[CC] J. Cilleruelo and A. Cordoba. Trigonometric polynomials and lattice points. *Proc. AMS*, 115:899–906, 1992.

[Con] B. Connes. Sur les coefficients des séries trigonométriques convergents sphériquement. *C. R. Acad. Sc. Paris*, 283A:159–161, 1975.

[DSP] W. Duke and R. Schulze-Pillot. Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids. *Inv. Math*, 99:49–57, 1990.

[HL] G. Hardy and J. Littlewood. Some Problems of 'Partitio Numeronum': III. On the expression of a number as a sum of primes. *Collected papers of G. H. Hardy*, 1:561–631 Oxford Univ. Press, 1966.

[Ho] C. Hooley On the Intervals between Numbers that are Sums of Two Squares: II. *Jour. Num. Theory*, 5:215–217, 1973.

[Jak1] D. Jakobson. Ph. D. Thesis. Princeton University, 1995.

[Jak2] D. Jakobson. On quantum limits on flat tori. *Electronic research announcements of the AMS*, 1(2):80–86, 1995.

[Mi] M. Mignotte. Intersection des images de certaines suites récurrentes linéaires. *Theor. Computer Science*, 7:117-122, 1978.

[Sa93] P. Sarnak. Arithmetic Quantum Chaos. *Israel Math. Conf. Proc*, 8:183–236, 1995.

[Zyg] A. Zygmund. On Fourier coefficients and Transforms of functions of two variables. *Studia Mathematica*, 50:189–201, 1974.

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