LOWER BOUNDS FOR RESONANCES OF INFINITE AREA RIEMANN SURFACES

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Abstract. For infinite area, geometrically finite surfaces $X = \Gamma\backslash\mathbb{H}^2$, we prove new lower bounds on the local density of resonances $\mathcal{D}(z)$ when z lies in a logarithmic neighborhood of the real axis. These lower bounds involve the dimension δ of the limit set of Γ. The first bound is valid when $\delta > \frac{1}{2}$ and shows logarithmic growth of the number $\mathcal{D}(z)$ of resonances at high energy i.e. when $|\mathrm{Re}(z)| \to +\infty$. The second bound holds for $\delta > \frac{3}{4}$ and if Γ is an infinite index subgroup of certain arithmetic groups. In this case we obtain a polynomial lower bound. Both results are in favor of a conjecture of Guillopé-Zworski on the existence of a fractal Weyl law for resonances.

1. Introduction and results

Resonances arise in spectral theory on non-compact Riemannian manifolds when one tries to figure out what should be the natural replacement data for the missing eigenvalues of the Laplacian. The basic problem of the mathematical theory of resonances is to relate the resonances spectrum (which is a discrete set of complex numbers) to the geometry of the underlying manifold and its geodesic flow. In this paper we will focus on a particular setting where the spectral and scattering theory are already well developed: infinite area surfaces with constant negative curvature. For a detailed account of the spectral theory of infinite area surfaces, we refer the reader to [2]. Let $\mathbb{H}^2$ be the hyperbolic plane endowed with its standard metric of constant gaussian curvature -1. Let Γ be a geometrically finite discrete group of isometries acting on $\mathbb{H}^2$. This means that Γ admits a finite sided polygonal fundamental domain in $\mathbb{H}^2$. We will require that Γ has no *elliptic* elements different from the identity and that the quotient $\Gamma\backslash\mathbb{H}^2$ is of *infinite hyperbolic area.* Under these assumptions, the quotient space $X = \Gamma\backslash\mathbb{H}^2$ is a nice Riemann surface whose geometry can be described as follows. The surface X can be decomposed into a finite area surface with geodesic boundary N, called the Nielsen region, on which $f \geq 1$ non-compact, infinite area ends F_i are glued : the funnels. Each funnel F_i is isometric to a half cylinder

$$F_i = (\mathbb{R}/l_i\mathbb{Z})_\theta \times (\mathbb{R}^+)_t,$$

where $l_i > 0$, with the warped metric

$$ds^2 = dt^2 + \cosh^2(t)d\theta^2.$$

The Nielsen region N is itself decomposed into a compact surface K with geodesic and horocyclic boundary on which c non-compact, finite area ends C_i are glued: the cusps. A cusp C_i is isometric to a half cylinder

$$C_i = (\mathbb{R}/h_i\mathbb{Z})_\theta \times ([1, +\infty))_y,$$

where $h_i > 0$, endowed with the familiar Poincaré metric

$$ds^2 = \frac{d\theta^2 + dy^2}{y^2}.$$

Let Δ_X be the hyperbolic Laplacian on X. Its spectrum on $L^2(X)$ has been described by Lax and Phillips [13] :$[1/4, +\infty)$ is the continuous spectrum, has no embedded eigenvalues. The rest of the spectrum is made of a (possibly empty) finite set of eigenvalues, starting at $\delta(1-\delta)$, where $0 \le \delta < 1$ is the Hausdorff dimension of the limit set of Γ. The fact that the bottom of the spectrum is related to the dimension δ was first pointed out by Patterson [17] for convex co-compact groups (which amounts to say that there are no cusps on X or equivalently, no *parabolic* elements in Γ). This result was later extented for geometrically finite groups by Sullivan [22, 21].

By the preceding description of the spectrum, the resolvent

$$R(\lambda) = \left(\Delta_X - \frac{1}{4} - \lambda^2\right)^{-1} : L^2(X) \to L^2(X),$$

is therefore well defined and analytic on the lower half-plane $\{\mathrm{Im}(\lambda) < 0\}$ except at a possible finite set of poles corresponding to the finite point spectrum. *Resonances* are then defined as poles of the meromorphic continuation of

$$R(\lambda) : C_0^\infty(X) \to C^\infty(X)$$

to the whole complex plane. The set of poles is denoted by $\mathcal{R}_X$. This continuation is usually performed via the analytic Fredholm theorem after the construction of an adequate parametrix. The first result of this kind in the more general setting of asymptotically hyperbolic manifolds is due to Mazzeo and Melrose [15]. A more precise parametrix for surfaces was constructed by Guillopé and Zworski [8, 7] which allowed them to obtain global counting results for resonances of the following type. Let $N(R)$ be the number of resonances (counted with multiplicity) of modulus smaller than R. We have for all $R \ge 0$,

$$C^{-1}R^2 \le N(R) \le C + CR^2,$$

for some $C > 0$. Hence the set of resonances satisfy a quadratic growth law similar to the usual Weyl law for finite area surfaces. These bounds however do not reflect the geometry of X and its trapped set (the closure of the set of periodic geodesics in T_1X, the unit tangent bundle of X), whose Hausdorff dimension is $2\delta+1$. It is therefore necessary to examinate finer properties of $\mathcal{R}_X$ to capture some geometrical informations. The most natural thing to do is to look at resonances that are close to the real axis. From a physical point of view, these are the most relevant resonances, because they correspond to metastable states that live the longest (the imaginary part corresponding to the decay rate). In the case of Schottky groups (equivalently convex co-compact quotients in dimension 2), Zworski [24], and Guillopé-Lin-Zworski [6], have obtained a "fractal" upper bound. Let $N_C(T)$ be defined by

$$N_C(T) = \#\{z \in \mathcal{R}_X \ : \ \mathrm{Im}(z) \le C, |\mathrm{Re}(z)| \le T\},$$

then we have

$$N_C(T) = O(T^{1+\delta}). \tag{1}$$

The first proof of a geometric bound of the above type involving fractal dimension is due to Sjostrand for obstacle and potential scattering [20]. This upper bound, together with numerical experiments, has led Guillopé and Zworski to the following conjecture, known as the "fractal Weyl law".

Conjecture 1.1 (Guillopé-Zworski)**.** *There exist $C > 0$ and $A > 0$ such that for all T large enough,*

$$A^{-1}T^{1+\delta} \leq N_C(T) \leq AT^{1+\delta}.$$

The only existing lower bound can be found in [5], where the authors show that for all $\epsilon > 0$, one can find $C_\epsilon > 0$ such that

$$N_{C_\epsilon}(T) = \Omega(T^{1-\epsilon}),$$

where $\Omega(.)$ means being not a $O(.)$, in other words, one can find a sequence $(T_i)_{i\in\mathbb{N}}$ with $T_i \to \infty$ such that

$$\lim_{i\to\infty} \frac{N_{C_\epsilon}(T_i)}{T_i^{1-\epsilon}} = +\infty.$$

This is a frustrating lower bound: not only it does not involve δ but it is not even optimal in the computable case of elementary groups where $N_C(T)$ grows linearly. In the paper [6], they actually prove a stronger statement than (1). Let $\mathcal{D}(z)$ be the number of resonances in the disc centered at z and radius one:

$$\mathcal{D}(z) := \#\{\lambda \in \mathcal{R}_X \ : \ |\lambda - z| \leq 1\}.$$

Then if $\mathrm{Im}(z) \leq C$, we have $\mathcal{D}(z) = O(|\mathrm{Re}(z)|^\delta)$, the implied constant depending solely on C. Note that if the *Guillopé-Zworski conjecture holds*, then by the box principle, for all $\epsilon > 0$, one can find a sequence (z_i) with $|\mathrm{Re}(z_i)| \to +\infty$ and $\mathrm{Im}(z_i) \leq C$ such that for all $i \in \mathbb{N}$,

$$\mathcal{D}(z_i) \geq |\mathrm{Re}(z_i)|^{\delta-\epsilon}. \tag{2}$$

To state our results, we need one more notation. Let $A > 0$ and set

$$W_A = \{\lambda \in \mathbb{C} \ : \ \mathrm{Im}(\lambda) \leq A\log(1 + |\mathrm{Re}(\lambda)|)\}.$$

In [8], Guillopé and Zworski have shown that in logarithmic regions W_A, the density of resonances grows at least linearly. We shall prove the following thing.

Theorem 1.2. *Let Γ be a geometrically finite group as above. Assume that $\delta > \frac{1}{2}$, and fix arbitrarily small $\epsilon > 0$ and $A > 0$. Then there exists a sequence $(z_i)_{i\in\mathbb{N}}$ with $z_i \in W_A$ and $|\mathrm{Re}(z_i)| \to +\infty$, such that for all $i \geq 0$,*

$$\mathcal{D}(z_i) \geq (\log|\mathrm{Re}(z_i)|)^{\frac{\delta-1/2}{\delta}-\epsilon}.$$

In other words, the local density $\mathcal{D}(z)$ of resonances in logarithmic regions W_A is not bounded, and sensitive to the dimension of the trapped set. This implies in particular that the resonance set $\mathcal{R}_X \cap W_A$ is different from a lattice when $\delta > \frac{1}{2}$, which clearly could not follow from the existing lower bound in strips nor the global counting results. Building groups with $\delta > \frac{1}{2}$ is easy: if there is a parabolic element this is always the case and if one wants to consider only convex-cocompact groups, pinching a pair of pants will do it, see §4. We point out that the proof is based on Dirichlet box arguments, a technique that as proved usefull to obtain lower bounds for the remainder in Weyl's law on compact negatively curved manifolds, see [11, 10].

It is possible to obtain significantly better lower bounds that are closer to (2), by using infinite index subgroups of *arithmetic groups*. Arithmetic groups are algebraically defined discrete groups of isometries of $\mathbb{H}^2$, the most celebrated being the modular group $\mathrm{PSL}_2(\mathbb{Z})$. For more details on definitions and references, see §3. Our result is as follows.

Theorem 1.3. *Let Γ be a geometrically finite group as above, and assume that Γ is an infinite index subgroup of an arithmetic group Γ_0 derived from a quaternion algebra. Suppose $\delta > \frac{3}{4}$, and fix arbitrarily small $\epsilon > 0$ and $A > 0$. Then there exists a sequence $(z_i)_{i\in\mathbb{N}}$ with $z_i \in W_A$ and $|\mathrm{Re}(z_i)| \to +\infty$, such that for all $i \geq 0$,*

$$\mathcal{D}(z_i) \geq |\mathrm{Re}(z_i)|^{2\delta - \frac{3}{2} - \epsilon}.$$

This improvement is based on the very special structure of closed geodesics on arithmetic surfaces: the set of lengths has high multiplicities and good separation (see §3 for more details). This lower bound is clearly in favor of Guillopé-Zworski's conjecture, at least for the class of groups considered above. One may wonder at this point if Theorem 1.3 is not empty: we prove in §4 the existence of convex co-compact subgroups Γ of $\mathrm{PSL}_2(\mathbb{Z})$ with dimension $\delta > \frac{3}{4}$.

The lower bounds obtained above are to our knowledge the first examples in the literature which are related to the dimension of the trapped set, at least for fractal dimensions. Similar results should hold for higher dimensional convex-compact manifolds, by applying a similar strategy of proof based on the trace formula in [4].

The plan of the paper is as follows: in §2 we recall the necessary material for the proofs, including the wave trace formula which is at the basis of our results. We then prove Theorem 1.2 by a Dirichlet box-principle argument. Section §3 is devoted to the case of arithmetically built groups. The heart of the proof is based on a trick of Selberg and Hejhal on mean square estimates. This is where the high multiplicity and the separation play a key role. In §4 we discuss various examples of geometrically finite groups with δ large, and we construct an explicit family of convex co-compact subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ with $\delta > 3/4$.

2. Wave trace and log lower bounds

In this section, we prove Theorem 1.2. Some of the technical estimates below will be of some use in the next section. We use the notations of the introduction. The constant $A > 0$ defining the logarithmic region W_A is set once for all.

The variant of Selberg's trace formula we need here is due to Guillopé and Zworski [5]. We denote by $\mathcal{P}$ the set of *primitive closed geodesics* on the surface $X = \Gamma\backslash\mathbb{H}^2$, and if $\gamma \in \mathcal{P}$, $l(\gamma)$ is the length. In the following, c is the number of cusps, and N is the Nielsen region. Let $\varphi \in C_0^\infty((0,+\infty))$ i.e. a smooth function, compactly supported in $\mathbb{R}_+^*$. We have the identity:

$$\sum_{\lambda\in\mathcal{R}_X} \widehat{\varphi}(-\lambda) = -\frac{\mathrm{Vol}(N)}{4\pi}\int_0^{+\infty} \frac{\cosh(t/2)}{\sinh^2(t/2)}\varphi(t)dt$$

$$+\frac{c}{2}\int_0^{+\infty} \frac{\cosh(t/2)}{\sinh(t/2)}\varphi(t)dt \tag{3}$$

$$+\sum_{\gamma\in\mathcal{P}}\sum_{k\geq 1} \frac{l(\gamma)}{2\sinh(kl(\gamma)/2)}\varphi(kl(\gamma)),$$

where $\widehat{\varphi}$ is the usual Fourier transform

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x)e^{-ix\xi}dx.$$

We recall that $\mathcal{R}_X$ (except a possible finite number of term on the imaginary axis starting at $\lambda = i(\frac{1}{2}-\delta)$) is included in the upper half-plane. Note that we have omitted the main singular terms at $t = 0$ which are not relevant for our problem, see [5] for the formula in full detail. Proofs of Theorem 1.2 and 1.3 are based on the use of test functions of the form

$$\varphi_{t,\alpha}(x) = e^{-itx}\varphi_0(x-\alpha),$$

where $t > 0$, $\alpha > 0$ will be large and $\varphi_0 \in C_0^\infty(\mathbb{R})$ is a positive function, supported on the interval $[-1,+1]$ identical to 1 on $[-\frac{1}{2},+\frac{1}{2}]$. The basic idea is to use the full length spectrum (the set of lengths of closed geodesics) in the contribution from the geometric side instead of one single closed primitive geodesic and its iterates as in the proof of [5]. The price to pay for that is to lose positivity and deal with oscillating contributions. We start with some usefull Lemmas that consist mainly of brute force estimates. They will be used to control sums over resonances in the proof of Theorem 1.2 and 1.3. The reader can skip it for its first reading.

Lemma 2.1. *For all $N \geq 0$, one can find $C_N > 0$ such that for all $\xi \in \mathbb{C}$,*

$$|\widehat{\varphi_{t,\alpha}}(\xi)| \leq C_N \frac{e^{\alpha \mathrm{Im}(\xi)+|\mathrm{Im}\xi|}}{(1+|t+\xi|)^N}.$$

Proof. Write $\widehat{\varphi_{t,\alpha}}(\xi) = e^{-i\alpha(t+\xi)}\widehat{\varphi_0}(t+\xi)$, and integrate by parts N times. □

Lemma 2.2. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be either $f(x) = (\log(1+x))^\beta$ or $f(x) = x^\beta$ with $0 < \beta < 1$. Assume that for all $z \in W_A$ with $|\mathrm{Re}(z)|$ large enough one has*

$$\mathcal{D}(z) = O(f(|\mathrm{Re}(z)|)),$$

then for all α, t large and all $k \geq 0$ one has

$$\left|\sum_{\lambda\in W_A\cap\mathcal{R}_X} \widehat{\varphi_{\alpha,t}}(-\lambda)\right| = O\left(\frac{e^{\alpha(\delta-1/2)}}{t^k}\right) + O(f(t)),$$

where the implied constants do not depend on α, t.

Proof. Let us assume that $\mathcal{D}(z) = O(f(|\mathrm{Re}(z)|))$ whenever $|\mathrm{Re}(z)| \geq p_0 \geq 1$ and $z \in W_A$. Let $t > 0$ be so large that $t > p_0 + 1$, assume that $\alpha > 1$. By absolute convergence one can write

$$\sum_{\lambda \in W_A \cap \mathcal{R}_X} \widehat{\varphi_{\alpha,t}}(-\lambda) = \sum_{p \in \mathbb{Z}} \sum_{\substack{p \leq \mathrm{Re}(\lambda) \leq p+1 \\ \lambda \in W_A \cap \mathcal{R}_X}} \widehat{\varphi_{\alpha,t}}(-\lambda).$$

Let us set

$$S_p(\alpha, t) = \sum_{\substack{p \leq \mathrm{Re}(\lambda) \leq p+1 \\ \lambda \in W_A \cap \mathcal{R}_X}} \widehat{\varphi_{\alpha,t}}(-\lambda).$$

We split the above sum as

$$\sum_{\lambda \in W_A \cap \mathcal{R}_X} \widehat{\varphi_{\alpha,t}}(-\lambda) = \sum_{p < -p_0} S_p(\alpha, t) + \sum_{-p_0 \leq p \leq p_0} S_p(\alpha, t) + \sum_{p > p_0} S_p(\alpha, t).$$

The middle term involves only finitely many resonances $\lambda \in W_A$, and they satisfy $\mathrm{Im}(\lambda) \geq \frac{1}{2} - \delta$. Therefore using Lemma 2.1, we have

$$\left| \sum_{-p_0 \leq p \leq p_0} S_p(\alpha, t) \right| \leq C_k \frac{e^{(-\alpha+1)(1/2-\delta)}}{(1 + |t - p_0 - 1|)^k} \sum_{\substack{\lambda \in \mathcal{R}_X \cap W_A \\ |\mathrm{Re}(\lambda)| \leq p_0}} 1$$

$$= O\left(\frac{e^{\alpha(\delta - 1/2)}}{t^k} \right).$$

The first term can be estimated as

$$\left| \sum_{p < -p_0} S_p(\alpha, t) \right| \leq C_2 \sum_{p < -p_0} \frac{1}{(1 + |p + 1 - t|)^2} \sum_{\substack{p \leq \mathrm{Re}(\lambda) \leq p+1 \\ \lambda \in \mathcal{R}_X \cap W_A}} e^{(-\alpha+1)\mathrm{Im}(\lambda)},$$

while the last term is of size

$$\left| \sum_{p > p_0} S_p(\alpha, t) \right| \leq C_2 \sum_{p > p_0} \frac{\widetilde{S}_p(\alpha)}{(1 + \min\{|p - t|, |p + 1 - t|\})^2},$$

where we have set

$$\widetilde{S}_p(\alpha) = \sum_{\substack{p \leq \mathrm{Re}(\lambda) \leq p+1 \\ \lambda \in W_A \cap \mathcal{R}_X}} e^{(-\alpha+1)\mathrm{Im}(\lambda)}.$$

The following Lemma will be convenient (this is where the hypothesis on $\mathcal{D}(z)$ is used).

Lemma 2.3. *Under the hypothesis of Lemma 2.2, there exists a constant M, independent of α, p and such that for all $|p| \geq p_0$, we have*

$$\widetilde{S}_p(\alpha) \leq M f(|p|).$$

Let us postpone the proof of this result for a moment and show how to end the proof of Lemma 2.2. Clearly, using Lemma 2.3, the sum of the first and last terms is smaller than

$$C \sum_{p \in \mathbb{Z}} \frac{f(|p|)}{(1 + |p - t|)^2},$$

for a constant $C > 0$ large enough. We can now write ($[t]$ is the integer part of t)

$$\sum_{p\in\mathbb{Z}} \frac{f(|p|)}{(1+|p-t|)^2} = \sum_{q\in\mathbb{Z}} \frac{f(|q+[t]|)}{(1+|q+[t]-t|)^2} \leq C' \sum_{q\in\mathbb{Z}} \frac{f(|q|+[t])}{(1+|q|)^2},$$

again for a well chosen $C' > 0$ (we have used the fact that f is increasing). To end the proof, simply write

$$\sum_{q\in\mathbb{Z}} \frac{f(|q|+[t])}{(1+|q|)^2} = \sum_{|q|\leq[t]} \frac{f(|q|+[t])}{(1+|q|)^2} + \sum_{|q|>[t]} \frac{f(|q|+[t])}{(1+|q|)^2},$$

which yields

$$\sum_{q\in\mathbb{Z}} \frac{f(|q|+[t])}{(1+|q|)^2} \leq f(2[t]) \sum_{q\in\mathbb{Z}} \frac{1}{(1+|q|)^2} + \sum_{|q|>[t]} \frac{f(2|q|)}{(1+|q|)^2}.$$

Since $f(2|q|) = O(|q|^{1-\epsilon})$, the second term is clearly bounded in t and we get the upper bound of size $O(f(2t))$. It remains to prove Lemma 2.3. It will follow from a standard covering argument. It is enough to consider just the case $p > p_0$. We recall that for all $\lambda \in \mathcal{R}_X$, then for $\mathrm{Re}(\lambda) \neq 0$ we have actually $\mathrm{Im}(\lambda) \geq 0$ by definition. Let $\mathcal{A}_p$ denote the set

$$\mathcal{A}_p = \{z \in W_A \ : \ p \leq \mathrm{Re}(z) \leq p+1\},$$

let $D(z)$ denote the unit disc centered at $z \in \mathbb{C}$, and set

$$K(p) = \max\{k \geq 0 \ : \ k\sqrt{3} \leq A\log(1+p)\}.$$

For $1 \leq k \leq K(p)$, we define the rectangle $R(k)$ by

$$R(k) = \{z \in \mathcal{A}_p \ : \ (k-1)\sqrt{3} \leq \mathrm{Im}(z) \leq k\sqrt{3}\}.$$

Set $l = A\log(1+p) - K(p)\sqrt{3} < \sqrt{3}$. One can check that we have for p large enough,

$$\mathcal{A}_p \subset \left(\bigcup_{k=1}^{K(p)} R(k)\right) \cup D\left(p+\tfrac{1}{2}+i(K(p)+l/2)\right) \cup D\left(p+\tfrac{1}{2}+i(K(p)+l)\right).$$

Indeed,

$$\mathcal{A}_p \setminus \left(\bigcup_{k=1}^{K(p)} R(k)\right)$$

is exactly the set

$$\{z \in \mathbb{C} \ : \ p \leq \mathrm{Re}(z) \leq p+1 \text{ and } K(p)\sqrt{3} \leq \mathrm{Im}(z) \leq A\log(1+\mathrm{Re}(z))\},$$

which is clearly covered by the union of the two above discs as long as

$$A\log(1+p+1) - A\log(1+p) = A\log\left(1+\frac{1}{p+1}\right) \leq \frac{\sqrt{3}}{2}.$$

Remark that for all $k = 1, \ldots, K(p)$, $R(k) \subset D\left(p+\frac{1}{2}+i(\frac{\sqrt{3}}{2}+(k-1)\sqrt{3})\right)$. We can now conclude by estimating

$$\widetilde{S}_p(\alpha) = \sum_{\lambda\in\mathcal{A}_p\cap\mathcal{R}_X} e^{(-\alpha+1)\mathrm{Im}(\lambda)} \leq \sum_{j=0}^{K(p)-1} \mathcal{D}\left(p+\tfrac{1}{2}+i(\tfrac{\sqrt{3}}{2}+j\sqrt{3})\right) e^{(-\alpha+1)j\sqrt{3}}$$

$$+\mathcal{D}\left(p+\tfrac{1}{2}+i(K(p))+l/2)\right)+\mathcal{D}\left(p+\tfrac{1}{2}+i(K(p))+l/2)\right).$$

Recalling that $\alpha > 1$ and $\mathcal{D}(z) \leq Cf(|\mathrm{Re}(z)|)$ for all $z \in W_A$ with $|\mathrm{Re}(z)| \geq p_0$, we thus obtain

$$\widetilde{S}_p(\alpha) \leq 2Cf(p+\tfrac{1}{2}) + C\frac{f(p+\frac{1}{2})}{1-e^{(-\alpha+1)\sqrt{3}}},$$

and therefore $\widetilde{S}_p(\alpha) = O(f(p))$, uniformly in α. □

Before we start the proof of Theorem 1.2, we need one more Lemma, which is the key observation that motivates the definition of the region W_A.

Lemma 2.4. *There exist some constants $\alpha_0, C_0 > 0$, independent of α, t such that for all $\alpha \geq \alpha_0$,*

$$\left|\sum_{\lambda\in\mathcal{R}_X\setminus W_A}\widehat{\varphi_{\alpha,t}}(-\lambda)\right| \leq C_0.$$

Proof. We assume first that $\alpha > 1$. If $\lambda \notin W_A$, then $\mathrm{Im}(\lambda) \geq 0$ and

$$|\lambda|^2 = (\mathrm{Re}(\lambda))^2 + (\mathrm{Im}(\lambda))^2 \leq e^{\frac{2}{A}\mathrm{Im}(\lambda)} + (\mathrm{Im}(\lambda))^2$$
$$\leq e^{\frac{3}{A}\mathrm{Im}(\lambda)},$$

whenever $\mathrm{Im}(\lambda) \geq C_A$ where C_A is a large enough constant depending on A. We can assume in the sequel that $C_A \geq 1$. Using Lemma 2.1 with $N = 0$, we get

$$\left|\sum_{\lambda\in\mathcal{R}_X\setminus W_A}\widehat{\varphi_{\alpha,t}}(-\lambda)\right| \leq C_0\#\{\lambda\in\mathcal{R}_X\setminus W_A\ :\ \mathrm{Im}(\lambda)\leq C_A\}$$
$$+\sum_{\substack{\lambda\in\mathcal{R}_X\setminus W_A\\ \mathrm{Im}(\lambda)\geq C_A}}\frac{1}{|\lambda|^{(\alpha-1)2A/3}}.$$

The first term is clearly independent of α while the second can be bounded by the Stieltjes integral

$$\sum_{\substack{\lambda\in\mathcal{R}_X\setminus W_A\\ \mathrm{Im}(\lambda)\geq C_A}}\frac{1}{|\lambda|^{(\alpha-1)2A/3}} \leq \int_1^{+\infty} u^{-(\alpha-1)2A/3}dN(u),$$

where $N(u) = O(u^2)$ is the counting function for resonances in discs defined in §1. By integration by parts, the above integral is clearly convergent and bounded in α as long as

$$A(\alpha-1) > 3.$$

The proof is complete. □

We can now start the proof of Theorem 1.2. Let's test the trace formula (3) with the family $\varphi_{\alpha,t}$ where α is a large positive number:

$$\sum_{\lambda\in\mathcal{R}_X}\widehat{\varphi_{\alpha,t}}(-\lambda) = -\frac{\mathrm{Vol}(N)}{4\pi}\int_{\alpha-1}^{\alpha+1}\frac{\cosh(t/2)}{\sinh^2(t/2)}\varphi_{\alpha,t}(t)dt$$

$$+\frac{c}{2}\int_{\alpha-1}^{\alpha+1}\frac{\cosh(t/2)}{\sinh(t/2)}\varphi_{\alpha,t}(t)dt$$
$$+\sum_{\alpha-1\leq kl(\gamma)\leq\alpha+1}\frac{l(\gamma)}{2\sinh(kl(\gamma)/2)}e^{-itkl(\gamma)}\varphi_0(kl(\gamma)-\alpha).$$

The first two terms on the right side are clearly bounded with respect to α and t. To get an appropriate control on the sum

$$\mathcal{S}_{\alpha,t}:=\sum_{\alpha-1\leq kl(\gamma)\leq\alpha+1}\frac{l(\gamma)}{2\sinh(kl(\gamma)/2)}e^{-itkl(\gamma)}\varphi_0(kl(\gamma)-\alpha),$$

we will use the following Lemma, also known as the Dirichlet box theorem.

Lemma 2.5. *Let $\alpha_1,\ldots,\alpha_N\in\mathbb{R}$, and $D\in\mathbb{N}^*$. For all $Q\geq 2$ one can find an integer $q\in\{D,\ldots,DQ^N\}$ such that*

$$\max_{1\leq j\leq N}\|q\alpha_j\|\leq\frac{1}{Q},$$

where $\|x\|=\min_{n\in\mathbb{Z}}|x-n|$.

Proof. Use the box principle. □

By N_α we denote

$$N_\alpha:=\#\{(k,l(\gamma))\in\mathbb{N}^*\times\mathcal{P}\ :\ kl(\gamma)\in[\alpha-1,\alpha+1]\}.$$

Fix a constant $\varepsilon_0>0$ and set $D_\alpha=\left[(4\pi)^{\varepsilon_0 N_\alpha}\right]$. By Lemma (2.5) with $Q=[4\pi]$, for all $\alpha>>1$, one can find $q_\alpha\in\{D_\alpha,\ldots,D_\alpha Q^{N_\alpha}\}$ such that

$$\max_{\alpha-1\leq kl(\gamma)\leq\alpha+1}\|q_\alpha kl(\gamma)\|\leq\frac{1}{Q}.$$

Set $t_\alpha:=2\pi q_\alpha$, we have for all $\alpha-1\leq kl(\gamma)\leq\alpha+1$,

$$\left|e^{it_\alpha kl(\gamma)}-1\right|\leq\frac{2\pi}{Q}<\frac{2}{3}.$$

Hence we get

$$|\mathcal{S}_{\alpha,t_\alpha}|\geq\frac{1}{3}\left(\sum_{\alpha-1\leq kl(\gamma)\leq\alpha+1}\frac{l(\gamma)}{2\sinh(kl(\gamma)/2)}\varphi_0(kl(\gamma)-\alpha)\right)$$
$$\geq C_0e^{-\alpha/2}\left(\sum_{\alpha-\frac{1}{2}\leq kl(\gamma)\leq\alpha+\frac{1}{2}}1\right),$$

for a well chosen constant $C_0>0$. We now recall that by the prime geodesic theorem (see [16] for a proof and references in the case of infinite area surfaces), one has (as $T\to+\infty$),

$$\#\{(k,l(\gamma))\in\mathbb{N}^*\times\mathcal{P}\ :\ kl(\gamma)\leq T\}=\frac{e^{\delta T}}{\delta T}\left(1+o(1)\right).$$

This yields for α large,

$$|\mathcal{S}_{\alpha,t_\alpha}|\geq C_1\frac{e^{(\delta-\frac{1}{2})\alpha}}{\alpha},$$

where C_1 is again a suitable constant. Using the prime geodesic theorem, one shows also that

$$C_2^{-1}\frac{e^{\delta\alpha}}{\alpha} \leq N_\alpha \leq C_2 e^{\delta\alpha},$$

with $C_2 > 0$ and α large. We have therefore

$$\log\log t_\alpha \leq \delta\alpha + \text{constants},$$

which can be more conveniently restated as : for all $\epsilon > 0$ and α large,

$$\log\log t_\alpha \leq (\delta+\epsilon)\alpha.$$

Similarly we get the lower bound

$$\log\log t_\alpha \geq (\delta-\epsilon)\alpha.$$

We can now conclude the proof. Assume that $\delta > \frac{1}{2}$. Suppose that for all $z \in W_A$ with $|\mathrm{Re}(z)| \geq R_0$, one has $\mathcal{D}(z) \leq (\log|\mathrm{Re}(z)|)^\beta$, where $\beta > 0$ will be determined later on. Then by Lemma (2.2) with $k = 1$, and Lemma (2.4), one gets as $\alpha \to +\infty$,

$$C_1\frac{e^{(\delta-\frac{1}{2})\alpha}}{\alpha} \leq |\mathcal{S}_{\alpha,t_\alpha}| \leq O(1) + O\left(\frac{e^{\alpha(\delta-1/2)}}{t_\alpha}\right) + O((\log t_\alpha)^\beta).$$

Now recall that

$$\frac{\log\log t_\alpha}{\delta+\epsilon} \leq \alpha \leq \frac{\log\log t_\alpha}{\delta-\epsilon},$$

so that we have

$$\frac{C_1(\delta+\epsilon)}{\log\log t_\alpha}(\log t_\alpha)^{\frac{\delta-1/2}{\delta+\epsilon}} \leq O(1) + O\left(\frac{(\log t_\alpha)^{\frac{\delta-1/2}{\delta-\epsilon}}}{t_\alpha}\right) + O((\log t_\alpha)^\beta).$$

We have a contradiction whenever $\beta < \frac{\delta-1/2}{\delta+\epsilon}$. As a conclusion, for all $\epsilon > 0$ and all $R_0 \geq 0$ one can find $z \in W_A$ with $|\mathrm{Re}(z)| \geq R_0$ and $\mathcal{D}(z) > (\log|\mathrm{Re}(z)|)^{\frac{\delta-1/2}{\delta}-\epsilon}$. This ends the proof of Theorem 1.2. □

3. Mean square lower bounds and arithmetic length spectrum

The goal of this section is to prove Theorem 1.3. First we need to a recall a few basic facts about arithmetic group. Instead of detailing the construction of such groups, we refer the reader to the introductory book [12], and will use a characterization of arithmetic groups derived from quaternion algebra due to Takeuchi [23], which is all we need for this section.

We recall that a discrete group of isometries of the hyperbolic plane $\mathbb{H}^2$ can be viewed as a discrete subgroup of $PSL_2(\mathbb{R})$. If $M \in PSL_2(\mathbb{R})$ correspond to a hyperbolic isometry, then $\mathrm{Tr}(M)$ is related to the *translation length* l of M by the formula $2\cosh(l/2) = |\mathrm{Tr}(M)|$. Takeuchi's result is as follows.

Theorem 3.1 (Takeuchi)**.** *Let* Γ *be a discrete, cofinite subgroup of* $PSL_2(\mathbb{R})$*. Set* $\mathrm{Tr}(\Gamma) := \{\mathrm{Tr}(T)\ :\ T \in \Gamma\}$*. Then* Γ *is derived from a quaternion algebra if and only if:*

(1) The field $K = \mathbb{Q}(\mathrm{Tr}(\Gamma))$ *is an algebraic field of finite degree and* $\mathrm{Tr}(\Gamma)$ *is a subset of the ring of integers of* K*.*

(2) *For all embedding* $\varphi : K \to \mathbb{C}$, $\varphi \neq Id$, *the set* $\varphi(\mathrm{Tr}(\gamma))$ *is bounded in* $\mathbb{C}$.

For a proof of the above characterization, see [12, 23]. Condition (2) has some strong implications on the structure of the trace set $\mathrm{Tr}(\Gamma)$, as the next result shows. A similar statement can be found in [14].

Lemma 3.2. *Let* Γ_0 *be an arithmetic group derived from a quaternion algebra.*

(1) *There exists a constant* $C_0 > 0$ *depending solely on* Γ_0 *such that for all* $x, x' \in \mathrm{Tr}(\Gamma_0)$ *with* $x \neq x'$, $|x - x'| \geq C_0$.
(2) *There exists a constant* M_0 *depending only on* Γ_0 *such that for all* R *large,*

$$\Pi_0(x) := \#\{x \in \mathrm{Tr}(\Gamma_0) \ : \ |x| \leq R\} \leq M_0 R.$$

Proof. Clearly (1) implies (2) by a box argument. Let us prove (1). The field $K = \mathbb{Q}(\mathrm{Tr}(\Gamma_0))$ is a totally real number field of degree say $n = [K : \mathbb{Q}]$. Let $\varphi_1 = id, \varphi_2, \ldots, \varphi_n$ be the n distinct embeddings of K into $\mathbb{C}$. The set $\mathrm{Tr}(\Gamma_0)$ is a subset of the ring of integers O_K of K. We denote by $N_{\mathbb{Q}}^K(.)$ the norm on K. We recall that if $x \in O_K$ then $N_{\mathbb{Q}}^K(x) \in \mathbb{Z}$. Let $x \neq x'$ belong to $\mathrm{Tr}(\Gamma_0)$, we have

$$1 \leq |N_{\mathbb{Q}}^K(x - x')| = \prod_{i=1}^{n} |\varphi_i(x - x')| \leq |x - x'| M^{n-1},$$

where $M > 0$ is given by property (2) of Takeuchi's characterization. □

This important feature of the trace set was noticed by physicists working on quantum chaos [1] and was clearly emphasized by Luo and Sarnak [14] in their work on the number variance of arithmetic surfaces. Selberg and Hejhal [9], when trying to obtain sharp lower bounds for the error term in Weyl's law, had already noticed similar properties for some examples of co-compact arithmetic groups.

In the rest of this section we will work with a geometrically finite group Γ as defined in §1, and we assume in addition that Γ is an (infinite index) subgroup of an arithmetic group Γ_0, derived from a quaternion algebra. The simplest examples of such groups Γ that one can think of are finitely generated Schottky subgroups of $PSL_2(\mathbb{Z})$, but there are definitely many other examples, see the next section.
Given such a group Γ, one can define the length spectrum of $X = \Gamma \backslash \mathbb{H}^2$ by

$$\mathcal{L}_\Gamma := \{kl(\gamma) \ : \ (k, \gamma) \in \mathbb{N}^* \times \mathcal{P}\},$$

where as in the preceding section, $\mathcal{P}$ is the set of primitive closed geodesics. We have the following key properties.

Proposition 3.3. *Let* Γ *be a Fuchsian group as above, then we have:*

(1) *Let* $l_1, l_2 \in \mathcal{L}_\Gamma$ *with* $2\cosh(l_i/2) = \mathrm{Tr}(M_i)$, $i \in \{1, 2\}$, *then*

$$|l_1 - l_2| \geq e^{-\frac{\max(l_1, l_2)}{2}} |\mathrm{Tr}(M_1) - \mathrm{Tr}(M_2)|.$$

(2) There exists a constant $C_1 > 0$ depending only on Γ_0 such that for all α large,

$$\#\{l \in \mathcal{L}_\Gamma \ : \ \alpha - 1 \leq l \leq \alpha + 1\} \leq C_1 e^{\frac{\alpha}{2}}.$$

Proof. The set of closed geodesics on $X = \Gamma \backslash \mathbb{H}^2$ is in one-to-one correspondence with the set of conjugacy classes of hyperbolic elements in the fundamental group Γ, each closed geodesic γ having its length $l(\gamma)$ given by the formula

$$2\cosh(l(\gamma)/2) = |\mathrm{Tr}(T_\gamma)|,$$

where $T_\gamma \in \Gamma$ is an hyperbolic isometry. The length spectrum $\mathcal{L}_\Gamma$ is therefore in one-to-one correspondence with the trace set $\mathrm{Tr}(\Gamma)$ via the above formula (except for the conjugacy classes of parabolic elements with trace 2). Since we have $\mathrm{Tr}(\Gamma) \subset \mathrm{Tr}(\Gamma_0)$, we can use the preceding Lemma and crude bounds give estimate (2). To obtain the first lower bound (1), one simply writes (assuming $l_2 > l_1$),

$$l_2 - l_1 = 2\int_{x_1}^{x_2} \frac{dt}{t} \geq 2\frac{x_2 - x_1}{x_2},$$

where we have

$$x_i = e^{l_i/2} = \tfrac{1}{2}\left(\mathrm{Tr}M_i + \sqrt{(\mathrm{Tr}M_i)^2 - 4}\right).$$

Clearly one gets

$$x_2 - x_1 = \tfrac{1}{2}\int_{\mathrm{Tr}M_1}^{\mathrm{Tr}M_2} \left(1 + \frac{u}{\sqrt{u^2-4}}\right) du \geq \tfrac{1}{2}(\mathrm{Tr}(M_2) - \mathrm{Tr}(M_1)),$$

and the proof is done. □

When compared with the prime geodesic theorem, see §2, estimate (2) shows that whenever $\delta > \frac{1}{2}$ there must be some exponentially large multiplicities in the length spectrum. This is the key observation of Selberg and Hejhal ([9] chapter 2, section 18) that will allow us to produce a better lower bound than in §2. More precisely, we prove the following.

Proposition 3.4. *Let Γ be a group as above, δ being the dimension of its limit set. Let $\mathcal{S}_{\alpha,t}$ be the sum defined by*

$$\mathcal{S}_{\alpha,t} := \sum_{\alpha-1 \leq kl(\gamma) \leq \alpha+1} \frac{l(\gamma)}{2\sinh(kl(\gamma)/2)} e^{-itkl(\gamma)} \varphi_0(kl(\gamma) - \alpha).$$

There exists a constant $A > 0$ such that for all T large, if one sets $\alpha = 2\log T - A$ then the integral $\mathcal{I}(T)$ defined by

$$\mathcal{I}(T) = \int_T^{3T} \left(1 - \frac{|t - 2T|}{T}\right) |\mathcal{S}_{\alpha,t}|^2 dt,$$

enjoys the lower bound

$$\mathcal{I}(T) \geq C_2 \frac{T^{1+4\delta-3}}{(\log T)^2},$$

for some constant $C_2 > 0$ independent of T.

Let us show how Theorem 1.3 follows from this lower bound. First we assume that for all $z \in W_A$ with $|\mathrm{Re}(z)| \geq R_0$, we have

$$\mathcal{D}(z) \leq |\mathrm{Re}(z)|^{\beta},$$

for some $0 < \beta < 1$. Set $\alpha = 2\log T - A$, where A is given by the above proposition. We have

$$C_2 \frac{T^{1+4\delta-3}}{(\log T)^2} \leq \mathcal{I}(T) \leq \int_T^{3T} |\mathcal{S}_{\alpha,t}|^2 dt.$$

By the trace formula (3) applied to $\varphi_{\alpha,t}$, and Lemma 2.2 with $k = 2$, Lemma 2.4, we have

$$|\mathcal{S}_{\alpha,t}| \leq O(1) + O\left(\frac{T^{2\delta-1}}{T^2}\right) + O(t^{\beta}),$$

therefore we get

$$\int_T^{3T} |\mathcal{S}_{\alpha,t}|^2 dt = O\left(T^{2\beta+1}\right),$$

which produces a contradiction whenever $\beta < 2\delta - 3/2$. Theorem 1.3 is proved. □

We now devote the end of this section to the proof of Proposition 3.4. We start with an elementary observation. For all $\lambda \in \mathbb{R}$ and $T > 0$ set

$$J(T, \lambda) = \int_T^{3T} \left(1 - \frac{|t - 2T|}{T}\right) e^{-i\lambda t} dt.$$

Lemma 3.5. *With the above notations, we have for all $\lambda \neq 0$,*

$$|J(T, \lambda)| \leq \frac{4}{\lambda^2 T},$$

while $J(T, 0) = T$.

Proof. It follows by direct computation. □

At this point we need some more notations. If $\ell \in \mathcal{L}_\Gamma$, we denote by $\mu(\ell)$ the *multiplicity* of ℓ as the length of a closed geodesic. If $\ell \in \mathcal{L}_\Gamma$, then let $\widetilde{\ell}$ denote the *primitive length* of ℓ, i.e. if $\ell = kl(\gamma)$ with γ a primitive closed geodesic, then $\widetilde{\ell} = l(\gamma)$. Using these notations, we have

$$\mathcal{I}(T) = \sum_{\ell, \ell' \in \mathcal{L}_\Gamma} \frac{\widetilde{\ell}\widetilde{\ell'}\mu(\ell)\mu(\ell')}{4\sinh(\ell/2)\sinh(\ell'/2)} J(T, \ell - \ell')\varphi_0(\ell - \alpha)\varphi_0(\ell' - \alpha).$$

We now set $\mathcal{I}(T) = \mathcal{I}_1(T) + \mathcal{I}_2(T)$ where

$$\mathcal{I}_1(T) = T \sum_{\ell \in \mathcal{L}_\Gamma} \frac{(\widetilde{\ell}\mu(\ell))^2}{4\sinh^2(\ell/2)} \varphi_0^2(\ell - \alpha),$$

and

$$\mathcal{I}_2(T) = \sum_{\substack{\ell, \ell' \in \mathcal{L}_\Gamma \\ \ell \neq \ell'}} \frac{\widetilde{\ell}\widetilde{\ell'}\mu(\ell)\mu(\ell')}{4\sinh(\ell/2)\sinh(\ell'/2)} J(T, \ell - \ell')\varphi_0(\ell - \alpha)\varphi_0(\ell' - \alpha).$$

By Lemma 3.5, we have

$$|\mathcal{I}_2(T)| \leq \frac{4}{T} \sum_{\substack{\ell,\ell' \in \mathcal{L}_\Gamma \\ \ell \neq \ell'}} \frac{\widetilde{\ell\ell'}\mu(\ell)\mu(\ell')\varphi_0(\ell-\alpha)\varphi_0(\ell'-\alpha)}{4\sinh(\ell/2)\sinh(\ell'/2)(\ell-\ell')^2}.$$

Using the inequality $ab \leq \frac{1}{2}(a^2+b^2)$ for all $a,b \in \mathbb{R}$, we get by symmetry of the summation

$$|\mathcal{I}_2(T)| \leq \frac{4}{T} \sum_{\substack{\ell,\ell' \in \mathcal{L}_\Gamma \\ \ell \neq \ell'}} \frac{(\widetilde{\ell}\mu(\ell))^2\varphi_0^2(\ell-\alpha)}{4\sinh(\ell/2)\sinh(\ell'/2)(\ell-\ell')^2}.$$

Therefore, one can find a constant $C>0$ such that for all α and T large one has

$$|\mathcal{I}_2(T)| \leq C\frac{e^{-\alpha}}{T} \sum_{\ell \in \mathcal{L}_\Gamma} (\widetilde{\ell}\mu(\ell))^2\varphi_0^2(\ell-\alpha) \sum_{\substack{\ell' \in \mathcal{L}_\Gamma \cap [\alpha-1,\alpha+1] \\ \ell' \neq \ell}} \frac{1}{(\ell-\ell')^2}.$$

By Proposition 3.3 (1), we can write $x = 2\cosh(\ell/2)$, where $x \in \mathrm{Tr}(\Gamma)$, and thus

$$\sum_{\substack{\ell' \in \mathcal{L}_\Gamma \cap [\alpha-1,\alpha+1] \\ \ell' \neq \ell}} \frac{1}{(\ell-\ell')^2} \leq e^{\alpha+1} \sum_{x' \in \mathrm{Tr}(\Gamma)} \frac{1}{(x-x')^2}.$$

We can now bound

$$\sum_{x' \in \mathrm{Tr}(\Gamma)} \frac{1}{(x-x')^2} \leq \int_2^{x-C_0} \frac{d\Pi_0(u)}{(x-u)^2} + \int_{x+C_0}^{+\infty} \frac{d\Pi_0(u)}{(x-u)^2},$$

where Π_0 is the counting function for the trace set of the arithmetic group Γ_0 and the constant C_0 is given by Lemma 3.2. Using the fact that the growth $\Pi_0(u) = O(u)$, two Stieltjes integration by parts show that there exists a constant $\widetilde{C_0}$ depending only on Γ_0 such that for all $x \in \mathrm{Tr}(\Gamma)$,

$$\sum_{x' \in \mathrm{Tr}(\Gamma)} \frac{1}{(x-x')^2} \leq \widetilde{C_0}.$$

Going back to $\mathcal{I}_2(T)$, we have obtained for T and α large,

$$|\mathcal{I}_2(T)| \leq \frac{C'}{T} \sum_{\ell \in \mathcal{L}_\Gamma} (\widetilde{\ell}\mu(\ell))^2\varphi_0^2(\ell-\alpha).$$

Recall that

$$\mathcal{I}_1(T) = T\sum_{\ell \in \mathcal{L}_\Gamma} \frac{(\widetilde{\ell}\mu(\ell))^2}{4\sinh^2(\ell/2)}\varphi_0^2(\ell-\alpha) \geq C''e^{-\alpha}T \sum_{\ell \in \mathcal{L}_\Gamma} (\widetilde{\ell}\mu(\ell))^2\varphi_0^2(\ell-\alpha),$$

again for α large and some $C''>0$. Therefore $|\mathcal{I}_2| \leq \frac{1}{2}\mathcal{I}_1$ as long as

$$e^\alpha \leq \tfrac{1}{2}\frac{C''}{C'}T^2,$$

which is definitely achieved if one sets $\alpha = 2\log T - A$, where $A >> 1$. We have thus

$$|\mathcal{I}(T)| \geq \tfrac{1}{2}|\mathcal{I}_1(T)| \geq C''e^{-\alpha}T \sum_{\ell \in \mathcal{L}_\Gamma \cap [\alpha-1,\alpha+1]} (\widetilde{\ell}\mu(\ell))^2\varphi_0^2(\ell-\alpha)$$

$$\geq \widetilde{C''} e^{-\alpha} T \sum_{\ell \in \mathcal{L}_\Gamma \cap [\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]} (\mu(\ell))^2,$$

for some $\widetilde{C''} > 0$. By Schwarz inequality we get

$$\left(\sum_{\alpha - \frac{1}{2} \leq kl(\gamma) \leq \alpha + \frac{1}{2}} 1 \right)^2 = \left(\sum_{\ell \in \mathcal{L}_\Gamma \cap [\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]} \mu(\ell) \right)^2$$

$$\leq \left(\sum_{\ell \in \mathcal{L}_\Gamma \cap [\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]} (\mu(\ell))^2 \right) \left(\sum_{\ell \in \mathcal{L}_\Gamma \cap [\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]} 1 \right).$$

By Proposition 3.3 (2),

$$\sum_{\ell \in \mathcal{L}_\Gamma \cap [\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]} 1 = O(e^{\alpha/2}),$$

while the prime geodesic theorem yields

$$\sum_{\alpha - \frac{1}{2} \leq kl(\gamma) \leq \alpha + \frac{1}{2}} 1 \geq B \frac{e^{\delta \alpha}}{\alpha},$$

where $B > 0$. Hence we have obtained

$$\sum_{\ell \in \mathcal{L}_\Gamma \cap [\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]} (\mu(\ell))^2 \geq B^2 \frac{e^{(2\delta - 1/2)\alpha}}{\alpha^2}.$$

Going back to $\mathcal{I}(T)$ and recalling that $\alpha = 2 \log T - A$ we get

$$|\mathcal{I}(T)| \geq B' \frac{T^{1+4\delta-3}}{\log T^2}.$$

The proof is now complete. □

4. Examples

In this section with discuss examples of geometrically finite surfaces $X = \Gamma \backslash \mathbb{H}^2$ satisfying the assumptions of Theorem 1.2 and 1.3. By the work of Patterson [17], we know that if X has at least one cusp, i.e. if Γ has at least one non trivial parabolic element, then the dimension $\delta > \frac{1}{2}$. If one wants examples without cusps, then δ can be made arbitrarily close to 1 by "pinching" the geodesic boundary of Nielsen's region. Let us explain what we mean. By Patterson [17] and the spectral analysis of Lax-Phillips [13], we have $\delta > 1/2$ if and only if $\lambda_0(X) < 1/4$, where $\lambda_0(X)$ is the bottom of the spectrum of the Laplacian Δ_X. In that case $\lambda_0(X) = \delta(1-\delta)$. Hence to get $\delta > 1/2$, it is enough to show that the Rayleigh quotient

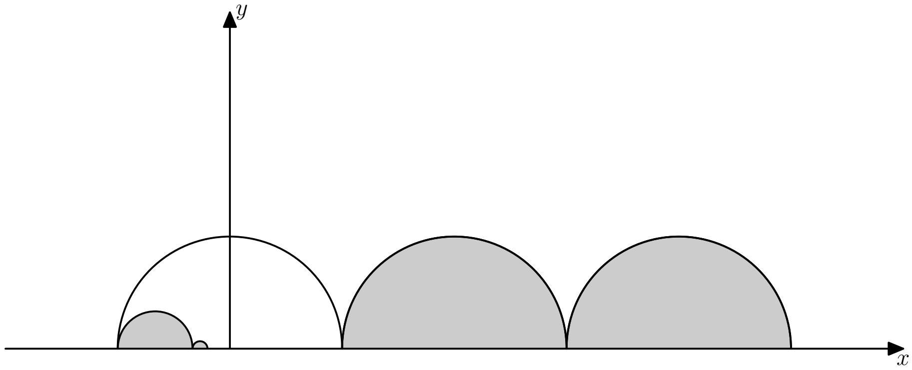
$$\lambda_0(X) = \inf_{f \neq 0} \frac{\int_X |\nabla f|^2 d\mathrm{Vol}}{\int_X f^2 d\mathrm{Vol}} < \frac{1}{4},$$

where f is an L^2 function on X with an L^2 gradient ∇f. Based on the above formula, a result of Pignataro and Sullivan [19] says the following thing. Let $\ell(X)$ denote the maximum length of the closed geodesics which are the boundary of the Nielsen region of X (the convex core), we have

Proposition 4.1 (Pignatoro, Sullivan)**.** *There exists a constant $C(X) > 0$ depending only the topology of X such that*

$$\lambda_0(X) \leq C(X)\ell(X).$$

Therefore if $\ell(X)$ is small enough, one definitely has $\delta > 1/2$. Applying the same strategy to find examples satisfying the hypotheses of Theorem 1.3 is harder. Indeed, the discreteness of arithmetic groups makes it difficult to perform deformations. What we are looking for are geometrically finite, infinite index subroups Γ of arithmetic groups derived from quaternion algebras with $\delta(\Gamma) > 3/4$. The easiest thing to do is to consider first $\mathrm{PSL}_2(\mathbb{Z})$ and look at some of its subgroups.



Fundamental domains for Λ_2 and Γ_2.

Let us fist consider the group Λ_N obtained as

$$\Lambda_N := \langle g_0, g_1, \dots, g_N \rangle,$$

where

$$g_0(z) = \frac{-1}{z},\ g_k = \tau^{2k} g_0 \tau^{-2k},\ \tau(z) = z + 1.$$

Let D_j, $j = 0, \dots, N$ be the unit closed disc centered at $2j$. A fundamental domain for the action of Λ_N on $\mathbb{H}^2$ is given by

$$\mathcal{F} = \overline{\mathbb{H}^2 \setminus (D_0 \cup \ldots \cup D_N)},$$

see the above picture for $N = 2$. We will need the following remark.

Lemma 4.2. *The group Λ_N is geometrically finite and has no parabolic elements. The elliptic elements are the conjugacy classes of $g_0, \dots, g_N$.*

Proof. Λ_N is geometrically finite because it is finitely generated. Assume that h is a parabolic element in Λ_N. Then because $\mathcal{F}$ is a locally finite convex fundamental domain, h is conjugated in Λ_N to some element g fixing a cusp point of $\partial\mathcal{F}$, say $v = 2i_0 + 1$, $i_0 \in \{0, \dots, N-1\}$ Write

$$g = g_{i_p} \cdots g_{i_2} g_{i_1},$$

where $g_{i_{j+1}} \neq g_{i_j}$. Remark that if $z \in \mathbb{C}$ is such that $z \notin D_j$ then $g_j(z)$ is mapped strictly inside D_j i.e. $g_j(z) \in \mathrm{Int}(D_j)$. This implies either $g_{i_1} = g_{i_0}$ or $g_{i_1} = g_{i_0+1}$. By induction on p, one gets that the sequence $i_1, i_2, \dots, i_p$ is monotonic. The only possibility left for v to be a fixed point of g is

then to have $p = 1$ but none of the g_i will fix a cusp point. Hence despite the shape of the fundamental domain, there are no parabolic elements in Λ_N. An alternative way to see that is to observe that the limit set of Λ_N is separated from the cusp points of $\partial\mathcal{F}$, see the next figure. Any elliptic element in Λ_N has to be conjugated to an elliptic isometry in Λ_N fixing a point of $\partial\mathcal{F} \cap \mathbb{H}^2$. By a similar procedure as above, one then shows that it is actually conjugated to one of the g_i. □
For $k = 1, \ldots, N$, set $h_k = g_0 g_k$, and consider the subgroup

$$\Gamma_N = \langle h_1, \ldots, h_N; h_1^{-1}, \ldots, h_N^{-1} \rangle.$$

We have the following fact.

Lemma 4.3. *The group Γ_N is of index 2 in Λ_N and is a convex co-compact group.*

Proof. Let $w \in \Lambda_N$, we have

$$w = g_{i_1} g_{i_2} \ldots g_{i_p},$$

with $i_j \neq i_{j+1}$. If p is even, then one writes

$$w = (g_{i_1} g_0)(g_0 g_{i_2}) \ldots (g_{i_{p-1}} g_0)(g_0 g_{i_p}) = h_{i_1}^{-1} h_{i_2} \ldots h_{i_{p-1}}^{-1} h_{i_p} \in \Gamma_N,$$

while if $p = 2k + 1$, one has

$$w = g_{i_1} g_{i_2} \ldots g_{i_{2k}} g_{i_{2k+1}} = h_{i_1}^{-1} h_{i_2} \ldots h_{i_{2k-1}}^{-1} h_{i_{2k}} h_{i_{2k+1}}^{-1} g_0 \in \Gamma_N g_0.$$

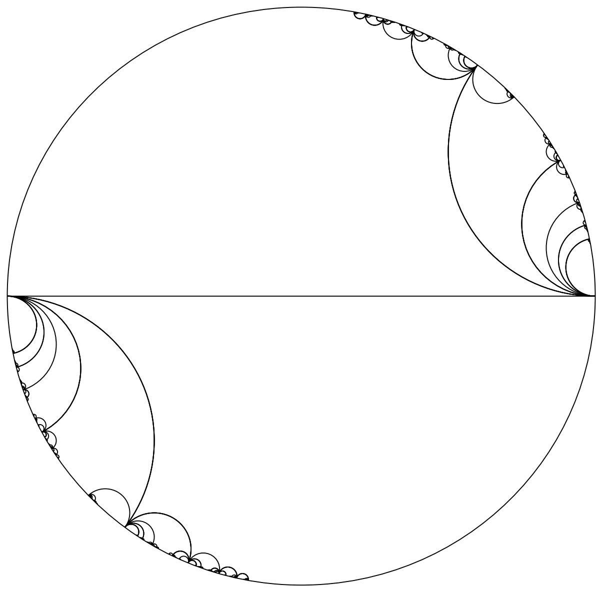
Clearly g_0 does not belong to Γ_N: none of the non trivial elements of Γ_N fixes i while g_0 does. Hence Γ_N is of index 2. If Γ_N contained a non-trivial elliptic element h, then h would be conjugated (in Λ_N) to some g_i, i.e. $h = \gamma g_i \gamma^{-1}$. Either $\gamma \in \Gamma_N$ or $\gamma \in \Gamma_N g_0$, which would force g_0 to belong to Γ_N. The group Γ_N has no parabolic nor elliptic elements, is finitely generated, therefore convex co-compact. □

A fundamental domain for the action of Γ_N is given by $\mathcal{F} \cup g_0(\mathcal{F})$, the region outside the grey discs in the preceding picture. Because Γ_N is of finite index the critical exponents $\delta(\Gamma_N)$ and $\delta(\Lambda_N)$ are the same: the critical exponent is defined as the infimum of positive real numbers σ such that the Poincaré series

$$P(\sigma) := \sum_{\gamma \in \Gamma} e^{-\sigma d(i, \gamma i)},$$

are convergent. Here d is the hyperbolic distance in the half-plane model. A classical result of Sullivan [22] shows that for geometrically finite groups, the critical exponent is also the Hausdorff dimension of the limit set, hence Λ_N and Γ_N have same dimension for their limit set.

The group Λ_N is also considered in the paper of Gamburd [3], where he shows using the min-max argument that $\delta(\Lambda_N)$ can be made as close to 1 as we want, provided N is large enough (estimates are effective). The next figure shows a view of the limit set of Λ_N for $N = 5$ in the disc model. The Poincaré half plane is mapped onto the disc via $z \mapsto \frac{iz+1}{z+i}$.



The limit set of Λ_N for $N = 5$.

As a conclusion, we have found examples of convex co-compact subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ with $\delta > \frac{3}{4}$. By a similar technique, one should be able to produce several examples with cusps. In that direction, let us point out that the Hecke group Γ_3 generated by $g : z \mapsto \frac{-1}{z}$ and $h : z \mapsto z + 3$ is a good candidate: its Hausdorff dimension was estimated by Phillips and Sarnak in [18] to be $\delta = 0.753 \pm 0.003$. Can one prove (or disprove) rigourously that $\delta > 0.75$?

It would be interesting in itself to find similar constructions for arithmetic groups that were not considered in this paper. In a sequel, the authors plan to address the case of arithmetic groups derived from quaternion division algebras (which are co-compact surface groups). It would also be interesting to consider groups acting on higher-dimensonal hyperbolic spaces.

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References

[1] E. B. Bogomolny, B. Georgeot, M.-J. Giannoni, and C. Schmit. Arithmetical chaos. *Phys. Rep.*, 291(5-6):219–324, 1997.

[2] David Borthwick. *Spectral theory of infinite-area hyperbolic surfaces*, volume 256 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2007.
[3] Alex Gamburd. On the spectral gap for infinite index "congruence" subgroups of $\mathrm{SL}_2(\mathbf{Z})$. *Israel J. Math.*, 127:157–200, 2002.
[4] Colin Guillarmou and Frédéric Naud. Wave 0-trace and length spectrum on convex co-compact hyperbolic manifolds. *Comm. Anal. Geom.*, 14(5):945–967, 2006.
[5] L. Guillopé and M. Zworski. The wave trace for Riemann surfaces. *Geom. Funct. Anal.*, **9**(6):1156–1168, 1999.
[6] Laurent Guillopé, Kevin K. Lin, and Maciej Zworski. The Selberg zeta function for convex co-compact Schottky groups. *Comm. Math. Phys.*, 245(1):149–176, 2004.
[7] Laurent Guillopé and Maciej Zworski. Upper bounds on the number of resonances for non-compact Riemann surfaces. *J. Funct. Anal.*, **129**(2):364–389, 1995.
[8] Laurent Guillopé and Maciej Zworski. Scattering asymptotics for Riemann surfaces. *Ann. of Math. (2)*, **145**(3):597–660, 1997.
[9] Dennis A. Hejhal. *The Selberg trace formula for* $\mathrm{PSL}(2,R)$. *Vol. I.* Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 548.
[10] Dmitry Jakobson and Iosif Polterovich. Estimates from below for the spectral function and for the remainder in local Weyl's law. *Geom. Funct. Anal.*, 17(3):806–838, 2007.
[11] Dmitry Jakobson, Iosif Polterovich, and John A. Toth. A lower bound for the remainder in Weyl's law on negatively curved surfaces. *Int. Math. Res. Not. IMRN*, (2):Art. ID rnm142, 38, 2008.
[12] Svetlana Katok. *Fuchsian groups*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.
[13] Peter D. Lax and Ralph S. Phillips. Translation representation for automorphic solutions of the non-Euclidean wave equation I, II, III. *Comm. Pure. Appl. Math.*, **37,38**:303–328, 779–813, 179–208, 1984, 1985.
[14] W. Luo and P. Sarnak. Number variance for arithmetic hyperbolic surfaces. *Comm. Math. Phys.*, 161(2):419–432, 1994.
[15] Rafe R. Mazzeo and Richard B. Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Funct. Anal.*, **75**(2):260–310, 1987.
[16] Frédéric Naud. Precise asymptotics of the length spectrum for finite-geometry Riemann surfaces. *Int. Math. Res. Not.*, (5):299–310, 2005.
[17] S. J. Patterson. The limit set of a Fuchsian group. *Acta Math.*, **136**(3-4):241–273, 1976.
[18] R. S. Phillips and P. Sarnak. On the spectrum of the Hecke groups. *Duke Math. J.*, 52(1):211–221, 1985.
[19] Thea Pignataro and Dennis Sullivan. Ground state and lowest eigenvalue of the Laplacian for noncompact hyperbolic surfaces. *Comm. Math. Phys.*, 104(4):529–535, 1986.
[20] Johannes Sjöstrand. Geometric bounds on the density of resonances for semiclassical problems. *Duke Math. J.*, 60(1):1–57, 1990.
[21] Dennis Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.*, (50):171–202, 1979.
[22] Dennis Sullivan. Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. *Acta Math.*, **153**(3-4):259–277, 1984.
[23] Kisao Takeuchi. A characterization of arithmetic Fuchsian groups. *J. Math. Soc. Japan*, 27(4):600–612, 1975.
[24] Maciej Zworski. Dimension of the limit set and the density of resonances for convex co-compact hyperbolic surfaces. *Invent. Math.*, **136**(2):353–409, 1999.

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