CLASSICAL LIMITS OF EIGENFUNCTIONS FOR SOME COMPLETELY INTEGRABLE SYSTEMS

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Abstract. We give an overview of some old results on weak* limits of eigenfunctions and prove some new ones. We first show that on $M = (S^n, can)$ every probability measure on S^*M which is invariant under the geodesic flow and time reversal is a weak* limit of a sequence of Wigner measures corresponding to eigenfunctions of Δ. We next show that joint eigenfunctions of Δ and a single Hecke operator on S^n cannot scar on a single closed geodesic. We finally use the estimates of [Z3] on the rate of quantum ergodicity to prove that adding a ΨDO of order $-n+2$ doesn't change the level spacings distribution of Δ (if the former is well defined) on a compact negatively curved manifold of dimension n. In dimension two this shows that the level spacings distributions of quantizations of certain Hamiltonians do not depend on the quantization.

1. Introduction

A general theme of semi-classical analysis is to find relations between the asymptotic properties of the eigenfunctions of a quantum system and the dynamics of the classical limit system. In this paper, the quantum system will consist of the wave group $U_t = e^{it\sqrt{\Delta}}$ of a compact Riemannian manifold (M, g), whose eigenfunctions $U_t\phi_j = e^{it\lambda_j}\phi_j$ represent standing waves, and the classical limit system will consist of the geodesic flow G^t on the unit sphere bundle S^*M. Our purpose is to describe some results on the asymptotic behavior of matrix elements $(A\phi_j, \phi_j)$ of observables $A \in \Psi^o(M)$ relative to eigenfunctions of certain quantum completely integrable systems. In part we will be reviewing known results, but our primary goal is to present some new results on asymptotics of matrix elements, especially in the case of the standard spheres (S^n, can).

Before describing the new results, let us mention a few of the issues of current interest in the semiclassical analysis of matrix elements (for more details, see §1). The basic problem is to determine the possible classical limits of the diagonal and off-diagonal matrix elements $(A\phi_i, \phi_j)$. This problem is raised in many standard texts of quantum mechanics, such as the classic Landau-Lifshitz text [L-L], but the discussions are non-rigorous and often contain implicit assumptions on the behavior of the classical limit systems (such as its complete integrability). One reason to be interested in the diagonal matrix elements $(A\phi_j, \phi_j)$ is that they are the eigenvalues of the quantum time average

$$\langle A\rangle := w - \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} U_t^* A U_t \, dt$$

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of the observable A (at least when the spectrum $Sp(\Delta)$ is simple). The limit points of $\{(A\phi_j, \phi_j)\}$ therefore fill out the essential spectrum $Sp_{ess}(\langle A\rangle)$. Since the high eigenvalue limit is a semiclassical limit, one may hope or suspect that $Sp_{ess}(\langle A\rangle)$ can be described in terms of the classical limit. Were $\langle A\rangle$ a nice observable (i.e. pseudodifferential operator) then $Sp_{ess}(\langle A\rangle)$ would be the essential range of $\langle a\rangle$, the time-averaged symbol. It is an open question to decide how generally this relation actually holds.

Intuitively, the limits of the matrix elements describe the concentration and oscillation properties of the eigenfunctions. In the case of completely integrable systems, one imagines that subsequences of the eigenfunctions correspond to certain invariant tori for the geodesic flow (specifically, the *quantizable* ones), and concentrate on them in the classical limit. Although there are rigorous results in this direction (cf. §2), it is quite an open problem to establish such results for general completely integrable systems. The difficulty is that the tori actually correspond to quasi-modes rather than actual modes (eigenfunctions), and the relation between quasi-modes and modes is notoriously unclear (see [Ar]). For KAM systems, in which a positive measure of the tori break up, essentially nothing rigorous is known. In the case of ergodic systems, it is known that the only point of density in $Sp(\langle A\rangle)$ is the constant $\langle a\rangle$ (the space average of the symbol); but there may exist other limit points. This is often viewed as the 'scarring problem': can sequences of eigenfunctions of quantum ergodic systems singularly concentrate on closed geodesics? The answer is known to be no in the case of some arithmetic hyperbolic manifolds [R-S], but is generally not known. There are also many computer studies of eigenfunctions, devoted to unraveling the patterns of critical points and nodal surfaces. They suggest that eigenfunctions of quantum chaotic systems might have local gaussian limit distributions, a statement going far beyond the analysis of matrix elements (which involve only the first two moments of the distribution). Another much-studied topic in the physics literature is the comparison of matrix elements $(A\phi_i, \phi_j)$ of chaotic systems with those of various kinds of random matrix ensembles.

Let us now state the specific problems and results of this article more precisely. To study matrix elements, we consider the distributions $\Phi_j \in \mathcal{D}'(S^*M)$ defined for $a \in C^\infty(S^*M)$ by:

$$\int_{S^*M} a d\Phi_j := \langle Op(a)\phi_j, \phi_j\rangle$$

Here, Op is a choice of quantization from symbols to pseudodifferential operators; the Φ_j's of course depend on the choice of Op, but the semi-classical limit results will not. The general problem on any manifold (M, g) is to determine the set

$$\mathcal{Q} = w * -lim\{\Phi_j\} \subset \mathcal{M}_I$$

of weak* limit points of the sequence $\{\Phi_j\}$. It is well-known, and easy to prove, that $\mathcal{Q} \subset \mathcal{M}_I$ where $\mathcal{M}_I$ is the convex set of probability measures invariant under G^t. When the eigenfunctions ϕ_j are real, then the functionals Φ_j and their limits are also invariant under the time-reversal involution $\tau(x,\xi) = (x, -\xi)$. But frequently we are interested in complex eigenfunctions such as exponentials.

Generally speaking, it is *very* difficult to determine the set $\mathcal{Q}$. There are essentially no general tools available besides trace formulae and a number of symmetry and convexity principles. But our first result gives a complete solution of this problem on the standard n-spheres $(M, g) = (S^n, can)$ (the sphere with the standard metric).

Theorem 1.1. *Suppose $(M, g) = (S^n, can)$. Then $\mathcal{Q} = \mathcal{M}_I$.*

Our second topic concerns the singular concentration of eigenfunctions on closed geodesics. As mentioned above, one of the well-known problems of quantum chaos is whether eigenfunctions ϕ_{j_k} of quantizations of chaotic systems can singularly concentrate on a closed geodesic, or more generally whether their microlocal lifts Φ_{j_k} can tend to any limit measure besides Liouville measure. A related problem is whether eigenfunctions of quantum chaotic systems can 'scar' on a closed geodesic. As became clear from discussions during the IMA meeting, these two types of concentration are not the same. In this article, we will only be considering the singular concentration of eigenfunctions. Our goal here is to develop the proof of Rudnick-Sarnak that joint eigenfunctions of the Laplacian and of Hecke operators on certain arithmetic quotients cannot concentrate on finite unions of closed geodesics. Specifically, we lift their argument from the configuration space to phase space and adapt it to simple Hecke operators on the spheres. Our results here are only the simplest of the kinds that could be envisioned in this context.

To state the result, we will need to recall the definition of a Hecke operator on a Riemannian manifold (M, g): it is is defined to be a self-adjoint operator on $L^2(M, dvol)$ of the form

$$T_C f(x) = \frac{1}{2N} \sum_{j=1}^{N} f(C_j x) + f(C_j^{-1} x) \tag{1}$$

where the C_j's are isometries of the universal Riemannian cover $(\tilde{M}, \tilde{g})$ and where T_C is assumed to commute with the deck transformation group Γ of $\tilde{M} \to M$ and hence to preserve $C^\infty(M)$. The classical examples are the Hecke operators associated to discrete arithmetic subgroups of $SL_2(\mathbf{R})$ [Shi]. Since the C_j's are isometries, $[\Delta, T_C] = 0$ and hence there exists an orthonormal basis of joint eigenfunctions

$$\begin{cases} \Delta\phi_j = \lambda_j \phi_j \\ T_C \phi_j = \mu_j(C) \phi_j \end{cases}$$

In the notation and terminology above, we are interested in the weak* limits of the linear functionals $\Phi_j(A) = (A\phi_j, \phi_j)$ corresponding to these eigenfunctions on the algebra Ψ^o of bounded pseudodifferential operators. As mentioned above, from the fact that ϕ_j is a Laplace eigenfunction it follows immediately that Φ_j is invariant under the automorphism $\alpha_t(A) = U_t^* A U_t$, where $U_t = e^{it\sqrt{\Delta}}$ is the wave group, and hence any weak limit of the Φ_j's is an invariant probability measure for the geodesic flow G^t. On the other hand the Hecke operator does not define an automorphism of Ψ^o. In fact it does not even define an endomorphism since $T_C^* A T_C$ is not usually a pseudodifferential operator. Hence we have the following questions ([CV4], [R-S], [Z4]):

Question 1.2. *What invariance property of the classical limit measures of $\{\Phi_j\}$ follows from the fact that the ϕ_j's are Hecke eigenfunctions?*

Question 1.3. *Which invariant probability measures for G^t have this additional Hecke-invariance property? Can a periodic orbit measure arise as the classical limit of Hecke eigenfunctions?*

We will give a rather complete answer to these questions in the case of the standard spheres. In particular, we will prove:

Theorem 1.4. *Suppose T_C is a Hecke operator on S^n defined by (1) such that*

(i) *Any closed geodesic γ on S^n is fixed by at most two isometries C_j, C_j^{-1} in (1); and by at most two words of length two in the free group $\mathcal{F}$ generated by the symbols C_j.*

(ii) *Let γ be a closed geodesic fixed by some word $W \in \mathcal{F}$ of length at most two, and assume that W is not a power of another element in $\mathcal{F}$. Then the only words in $\mathcal{F}$ of length at most four fixing γ are those which reduce to the powers of W.*

Then there is no sequence φ_j of joint $T_C - \Delta$-eigenfunctions such that the corresponding Φ_j-s converge to δ_γ where δ_γ is the delta-measure on a single closed geodesic.

Our final result is a small observation related to a very difficult problem, namely that of determining the level spacings distribution μ of a Laplacian. It is defined as follows: Suppose (M, g) is a compact Riemannian manifold of dimension n, and denote the eigenvalues of its Laplacian Δ by $0 = \lambda_0 < \lambda_1 \leq \ldots$. The growth rate of the spectrum is given by Weyl's law:

$$N(\lambda) = \#\{\lambda_j \leq \lambda\} \ \sim \ C_n vol(M)\lambda^{\frac{n}{2}} \tag{2}$$

where C_n is a constant depending only on the dimension. To eliminate the constants, let us rescale the metric so that $C_n vol(M) = 1$. To make the consecutive eigenvalue spacings $\lambda^*_{j+1} - \lambda^*_j$ equal to 1 on average, we also renormalize $\Delta \to \Delta^*$. By the Weyl law, the spectrum of $\Delta^* = \Delta^{\frac{n}{2}}$ has growth rate

$$N^*(\lambda) = \#\{\lambda_j^{\frac{n}{2}} \leq \lambda\} \ \sim \ \lambda \tag{3}$$

so its eigenvalues do have unit mean level spacing.

To detect statistical regularities in the spacings, one then forms the local level spacings distribution μ on $[0, \infty]$ by

$$\mu_N([a,b]) := \frac{\#\{j < N : x_j \in [a,b]\}}{N} \tag{4}$$

for any $0 \leq a \leq b < \infty$. The problem is then to determine whether there exists a unique weak limit $\mu_N \to \mu$ as $N \to \infty$ and if so, to compute it. The physicists conjecture (and sometimes claim to have proved) that there exists a unique weak limit in case the classical limit dynamics is chaotic, namely the GOE level spacings distribution μ_{GOE}. However, from the mathematical standpoint there is no proof that even a uniquely defined limit measure exists. Moreover, it is invisible why the limit measure should be so universal and in particular depend only the principal symbol of Δ. Indeed, addition of a perturbation term $\Delta \to \Delta + V$ with V a pseudodifferential operator of order $\geq -n+2$ will move the eigenvalues by amounts at or above the mean level spacing. We may assume the mean value of V with respect to Liouville measure $d\omega$ on S^*M equals zero since addition of a constant will not change the level spacings. Our observation is that in the borderline case a perturbation $V \in \Psi^{-n+2}$ will not change the level spacings distribution if it exists.

Theorem 1.5. *Suppose the Laplacian Δ of a compact Riemannian n-manifold has a well-defined level spacings distribution ν. Then if the curvature is negative, ν is also the level spacings distribution of $\Delta + V$ where V is any pseudodifferential operator of order $-n+2$.*

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2. Background

The matrix element $(A\phi_j, \phi_j)$ represents the expected value of the observable A in the energy state ϕ_j. The off-diagonal elements $(A\phi_i, \phi_j)$ represent amplitudes for making transitions between energy states. The problem of determining their semi-classical asymptotics can be found in many classical books on quantum mechanics such as the classic text of Landau-Lifshits [L-L, §48, 51]. In this text, the system is assumed to have one degree of freedom, and the discussion is implicitly based on the fact that the system to be completely integrable.

2.1. Completely integrable systems: quantum torus action. A geodesic flow on a compact n-dimensional compact M is completely integrable if there exist n integrals of motion in involution, i.e. n functions $(p_1, \dots p_n)$ on T^*M (homogeneous of degree one) commuting with each other (and with the generator $\sqrt{|\xi|^2}$ of G^t) and forming an elliptic system (i.e. $\sum_i p_i(x,\xi)^2 > 0$ on $T^*M \backslash \{0\}$; it is often the case that $\sum_i p_i^2 = |\xi|^2$). There are stronger and weaker versions of complete integrability; the strongest is called *Hamiltonian torus action.* It assumes the existence of (globally defined) action-angle variables on T^*M. The examples include flat tori and convex surfaces of revolution. For such systems the level sets of $(p_1, \dots p_n)$-s are Lagrangian tori, and one can define the global *action* variables $(q_1, \dots q_n)$ generating 2π-periodic Hamiltonian flows on these tori.

A *quantization* of a completely integrable system ([CV2], [Z2]) is a choice of n commuting ΨDO-s $(P_1, \dots, P_n)$ of order 1 whose principal symbols are p_j-s. For systems with Hamiltonian torus action one can define *action* operators $(Q_1, \dots, Q_n)$ with $\sigma_{Q_j} = q_j$ and $\exp(2\pi i Q_j) = c_j Id$. The joint spectrum of Q_j-s is $C \cap \mathbf{Z}^n + \mu$ where C is a conic set in $\mathbf{R}^n$ and $\mu \in \mathbf{Z}^n/4$ is a Maslov index. The joint spectrum is simple (perhaps with finitely many exceptions). The preimages under $q = (q_1, \dots q_n)$ of the points in the spectrum are Lagrangean tori satisfying the quantization conditions of Bohr-Sommerfeld-Maslov.

In the completely integrable case one can associate sequences of approximate WKB eigenfunctions (quasi-modes) to the tori satisfying the quantization condition ([CV1]). For quantum torus action systems it has been shown (cf. [G-S], [CV2] and [Z2]) that quasimodes correspond to actual eigenfunctions. Accordingly, given a ray $l \in C$ and a sequence of points $\{\xi_j\}$ in the spectrum approaching l[1] there is an associated sequence of eigenfunctions $\varphi_j = \varphi_{\xi_j}$ such that $d\Phi_j$-s converge to the Lebesgue measure on the torus $q^{-1}(l)$.

2.2. General completely integrable systems; special cases. It is generally believed that the eigenfunctions of a completely integrable system split up into subsequences concentrating on invariant tori for the system. However, to the authors' knowledge, this has only been proved for the quantum torus action systems, and then only for the special basis of joint eigenfunctions of the action operators. This

[1] The points cannot lie on a ray pointing in an irrational direction; in that case one can take a sequence of lattice points corresponding to Diophantine approximants of the slope.

is not generally appreciated because the distinction between modes (eigenfunctions) and quasi-modes is often blurred or lost in the physics literature.

For a general completely integrable system, there do not exist global action-angle charts (see [D]). Rather, one needs to cover the phase space with a finite number of charts. Moreover, the moment map $(p_1, \dots, p_n)$ is generally many-to-one. On the quantum level, the joint spectrum of $(P_1, \dots, P_n)$ is multiple, and one cannot associate a unique eigenfunction to each torus satisfying the Bohr-Sommerfeld-Maslov quantization conditions. In fact, the correspondence between actual eigenfunctions and the lattice of BSM tori has not been worked out in very many examples. For some results in the case of separable systems, see J. Toth ([T1]) for geodesic flow on an ellipsoid and motion of a rigid body in vacuum; and Bleher, Kosygin, Minasov, Sinai and others ([Bl], [BKS], [K-M-S]) for geodesic flow on Liouville surfaces.[2]

2.3. **Spheres.** Our results in this paper show that on (S^n, can) the set $\mathcal{Q}$ of weak* limit points of the sequence Φ_j-s is the largest possible. However, the second author showed in [Z5] that for a random choice of an orthonormal basis of eigenfunctions (in the sense made precise in [Z5]), Theorem 2.4 holds, as in the ergodic case. Moreover, J. VanderKam showed in [Va] that a random orthonormal basis on (S^2, can) is *uniquely ergodic*, that is there are *no* exceptional subsequences in the statement of Theorem 2.4.

2.4. **Flat tori.** We next discuss the classical limits on flat tori. Let $\mathbf{T}^n = \mathbf{R}^n/(2\pi \cdot L)$ and let $\varphi_\xi = e^{i\langle \xi, \cdot\rangle}$ be an eigenfunction on $\mathbf{T}^n$ (here $\xi \in L^*$), and let $d\Phi_{n\xi}$ be the distribution on $S^*\mathbf{T}^n$ corresponding to $\varphi_{n\xi}$. Then it is easy to see that $d\Phi_{n\xi}$-s converge to the delta-measure on the invariant torus $T_x i$ for G^t consisting of unit vectors pointing in the direction $\xi/|\xi|$. The projection of any T_ξ onto the base $\mathbf{T}^n$ is an isometry, so delta-measure on any T_ξ projects to the Lebesgue measure on $\mathbf{T}^n$.

In [J1] the first author studied the case $L = \mathbf{Z}^n$, the standard integer lattice. The multiplicity of the eigenvalue $\lambda \in \mathbf{N}$ is then equal to the number of ways of representing λ as a sum of n squares (or, equivalently, to the number of lattice points on the sphere S_λ of radius $\sqrt{\lambda}$ centered at the origin), and it is well known that multiplicities can become arbitrarily large for $n \geq 2$. Since an eigenfunction φ_λ satisfying $\Delta\varphi + \lambda\varphi = 0$ can be arbitrary linear combinations of φ_ξ-s for $|\xi|^2 = \lambda$, unbounded multiplicities may give rise to more complicated limits than delta-measure on T_ξ-s, and in particular their projections may be more complicated than the Lebesgue measure. Most of the results in [J1] deal with those projections, which are nothing but weak* limits of all possible sequences of $|\varphi_j|^2$-s for various eigenfunctions φ_j of Δ with eigenvalues $\lambda_j \to \infty$.

We first state the result of Bourgain (cf. [J1]):

Theorem 2.1. *On $\mathbf{T}^n$, weak* limits of $|\varphi_j|^2$-s are absolutely continuous with respect to the Lebesgue measure for all n.*

Accordingly, we will speak of possible densities (with respect to the Lebesgue measure) of weak* limits. It is easy to see that the number of frequencies in $|\varphi_j|^2$ can be arbitrarily large for $n \geq 2$ (those are just chords connecting lattice points

[2] There are several examples in the literature (cf. [C-P] and [T2]) of eigenfunctions of completely integrable separable systems concentrating on unstable closed geodesics.

on S_λ); accordingly, the following result (proved by the first author in [J1]) seems rather surprising:

Theorem 2.2. *The density of any weak* limit of* $|\varphi_j|^2$*-s on* $\mathbf{T}^2$ *is a trigonometric polynomial all of whose non-zero frequencies lie on the union of (at most) two circles centered at the origin.*

The proof follows from the finiteness of the number of solutions of a system of two Pell equations. It is also shown that the densities of weak* limits on $\mathbf{T}^3$ have absolutely convergent Fourier series, while on $\mathbf{T}^4$ they are in $L^2(\mathbf{T}^4)$. The L^p properties of the densities of weak* limits on $\mathbf{T}^n$ for $n \geq 5$ turn out to be related to possible generalizations (or lack thereof) to higher dimensions ($n-2$ and $n-3$) of the result of Zygmund ([Zyg]) on the uniform bound on $||\varphi||_4/||\varphi||_2$ for the eigenfunctions φ on $\mathbf{T}^2$.

In the present paper we wish to prove a result complementing Theorem 2.1. Namely, we show that

Proposition 2.3. *The Lebesgue measure on* $\mathbf{T}^n$ *is absolutely continuous with respect to any weak* limit of* $|\varphi_j|^2$*-s on* $\mathbf{T}^n$ *for all* n*.*

Proof. The statement follows easily from [Con, Lemma 2]. $\square$

Having studied possible projections of limits of $d\Phi_j$-s on $\mathbf{T}^n$, one wishes to determine the limits dω which project to a given measure $d\nu = f(x)\,dx$ with

$$f(x) \;=\; 1 + \sum_{0 \neq \tau \in \mathbf{Z}^n} c_\tau e^{i\langle \tau, x\rangle}$$

Writing dω as

$$\sum_{\tau \in \mathbf{Z}^n} g_\tau(\theta)\; e^{i\langle \tau, x\rangle}\; dx d\theta$$

(where θ is the variable on S^{n-1} and where $g_\tau(\theta)$ projects to c_τ), one can show easily that for $\tau \neq 0$ the support of g_τ lies on the $(n-2)$-dimensional sphere $S(\tau)$ of the unit vectors on S^{n-1} orthogonal to τ (guaranteeing the invariance of dω under G^t). There are no such restrictions on the support of g_0.

The more detailed information about g_τ is given by the asymptotic distribution of lattice points on certain "parallels" S_j^τ on $S_j := S_{\lambda_j}$-s (namely, the locus of the endpoints of chords on S_j that are translates of τ); the information about g_0 is given by the distribution of lattice points on the whole $S_j^0 := S_j$. If the lattice points on S_j^τ become equidistributed then g_τ can be any nonnegative measure on $S(\tau)$ with certain upper bounds on its mass (depending on c_τ). However, studying the asymptotic distribution of lattice points on S_j^τ for all τ-s is a difficult problem which the authors cannot solve.

2.5. **Ergodic systems.** We next turn to the discussion of weak* limits of Φ_j-s for manifolds with ergodic geodesic flows. The following result is due to Shnirelman ([Shn1], [Shn2]), the second author ([Z6]) and Colin de Verdiere ([CV3]):

Theorem 2.4. *Let* M *be a compact manifold with ergodic geodesic flow. Then for any orthonormal basis* φ_j *of eigenfunctions of* Δ *on* M *there exists a subsequence* φ_{j_k} *of density one such that the weak* limit of* Φ_{j_k}*-s is the Liouville measure on* S^*M*.*

In other words, almost all eigenfunctions become equidistributed. Analogues of this result have been established for non-compact hyperbolic surfaces of finite area in [Z7]; there one also considers the Eisenstein series which comprise the continuous spectrum of the hyperbolic Laplacian on these surfaces.

Theorem 2.4 follows from the estimate

$$\lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |(A\varphi_j, \varphi_j)|^2 \ = \ 0 \tag{5}$$

for every pseudo-differential operator A of order 0 with $\int_{S^*M} \sigma_A \ = 0$. The rate of convergence in (5) was studied in [Z3]. Better bounds in the case of arithmetic hyperbolic surfaces were proved in [L-S] and [J3].

A natural question related to Theorem 2.4 is the existence of "exceptional" subsequences of φ_j-s (on manifolds with ergodic geodesic flows) such that Φ_j-s do not converge to the Liouville measure. Such subsequences are sometimes called *strong scars*. Natural candidates for limits of such subsequences are δ-measures on unstable closed geodesics on manifolds with ergodic geodesic flow. Rudnick and Sarnak conjecture in [R-S] that such subsequences don't exist for compact manifolds of negative curvature (this conjecture is sometimes called *quantum unique ergodicity*). A theorem of Rudnick and Sarnak quoted below supports this conjecture for arithmetic hyperbolic surfaces.[3] Also, quantum unique ergodicity has been established for Eisenstein series for arithmetic subgroups of $SL_2(\mathbf{R})$ by Luo, Sarnak and the first author ([L-S], [J2]).

To end our discussion of weak* limits in the ergodic case we mention a converse result to Theorem 2.4 proved by Sunada and the second author (cf. [Z1]). They prove that if (5) holds (together with another condition on the "off-diagonal" terms $(A\varphi_i, \varphi_j)$) for eigenfunctions of Δ on a compact manifold M then the geodesic flow on M is ergodic; the second condition holds for manifolds with ergodic geodesic flows.

We conclude this section by stating the only rigorous result (to our knowledge) giving answers to questions 1.2 and 1.3 for arithmetic hyperbolic surfaces. This result is due to Rudnick and Sarnak ([R-S]):

Theorem 2.5. *Joint Hecke-Laplace eigenfunctions on arithmetic hyperbolic surfaces cannot singularly concentrate on a finite union of closed geodesics.*

3. Proof of Theorem 1.1.

Our purpose is to prove that, in the case $(M, g) = (S^n, can)$, any invariant probability measure $\nu \in \mathcal{M}_I$ arises as a weak* limit of a sequence of Φ_j's. To do so, it will be convenient to express all the relevant objects, e.g. invariant measures, eigenfunctions, and limits of their microlocal lifts, as integrals over the space $G(S^n, can)$ of unoriented geodesics of the standard metric.

Let us first recall the properties of this space. Since the geodesic flow of can is periodic it defines a free S^1-action on the unit sphere bundle S^*S^n. The quotient space, $G_+(S^n, can)$, is the symplectic manifold formed by the set of all oriented great circles of (S^n, can). Since each oriented great circle determines an oriented plane, $G(S^n, can)$ can be identified with the Grassmann manifold $Gr(2, n)$ of oriented planes thru the origin in $\mathbf{R}^n$. In particular, when n=2, $G(S^2, can) = S^2$. In

[3] See [SQC] for an overview of results about arithmetic hyperbolic manifolds.

general, $SO(n+1)$ acts on $G(S^n, can)$ and the space of oriented geodesics is the symmetric space

$$G(S^n, can) = SO(n+1)/SO(2) \times SO(n-1).$$

When the eigenfunctions are real, the Φ_j's are invariant under the the canonical involution $\tau : (x, \xi) \to (x, -\xi)$ on T^*S^n, which we will refer to as the 'time reversal.' It is obvious that τ takes an oriented closed geodesic γ to the same closed geodesic with the opposite orientation. Hence τ acts on S^*S^n and on $G(S^n, can)$, and the quotient $G(S^n, can)/\tau = G_-(S^n, can)$, the space of unoriented closed geodesics. However, we will be interested in complex eigenfunctions, so will work on the space of oriented closed geodesics.

Corresponding to each oriented closed geodesic γ we now associate two objects: first, the periodic orbit measure μ_γ with $\int_{S^*M} f d\mu = \int_\gamma f ds$, and second the sequence ϕ_k^γ of highest weight spherical harmonics associated to γ. Let us recall the definition of the latter. The spectral decomposition of Δ on (S^n, can) takes the form

$$L^2(S^n, can) = \sum_{k=0}^{\infty} \mathcal{H}_k, \qquad \Delta|_{\mathcal{H}_k} = k(n+k-1), \qquad dim \mathcal{H}_k = p_n(k)$$

where p_n is a certain polynomial of degree n-1. The eigenspaces $\mathcal{H}_k$ are invariant and irreducible under the action of $SO(n+1)$. Henceforth we fix one geodesic γ_o and express any other geodesic in the form $g \cdot \gamma_o$ with $g \in SO(n+1)$. We then put: $\phi_k^\gamma = g \cdot \phi_k^{\gamma_o}$. Hence it suffices to define $\phi_k^{\gamma_o}$.

To do so we recall that the Lie algebra $\mathbf{o}(n+1)$ has the root space decomposition

$$\mathbf{o}(n+1) = \mathbf{t} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$$

where $\mathbf{t}$ is a choice of Cartan subalgebra and where $\{\alpha\}$ runs over the set of positive roots of $\mathbf{o}(n+1)$. As is well-known, a presentation of $\mathbf{o}(n+1)$ is then given by a (Chevalley) basis $\{H_\alpha, X_\alpha^+, X_\alpha^-\}$ where each triple (for fixed α) forms an $\mathbf{sl}_2$. By a highest weight vector in $\mathcal{H}_k$ relative to this basis (i.e. choice of $\mathbf{t}$) we mean a vector satisfying $X_\alpha^+ v = 0$ for all α. Such a vector is unique by the theorem of the highest weight [B-D]. In the case $n = 2$ the Cartan subalgebra is one-dimensional and corresponds to a choice of an axis of revolution. The $\mathbf{o}(3)$ basis is then given by L_z, L^+, L^- in the usual physics notation; and the highest weight vector in $\mathcal{H}_k$ is the spherical harmonic usually denoted by $Y_k^k := C_k(x+iy)^k$. For $n > 2$, one can choose the maximal torus in $SO(n+1)$ and a basis $e_1, \ldots, e_\mu$ with $\mu = [(n+1)/2]$ for its Lie algebra $\mathbf{t}$ such that e_1 generates rotations in the x_1x_2-plane. The highest weight vector is then $\phi_k^{\gamma_o} = C_k(x_1 + ix_2)^k$, where γ_o is the great circle $x_1^2 + x_2^2 = 1, x_3 = \ldots = 0$ ([Ta, (3.34)]).

Since $\mathcal{H}_k$ is irreducible, every vector in it is cyclic. In particular, any vector $v \in \mathcal{H}_k$ may be expressed as a linear combination of the elements ϕ_k^γ. That is, we may express each eigenfunction in the form

$$\phi_k^\mu = \int_{G(S^n, can)} \phi_k^\gamma d\mu(\gamma) = \int_{SO(n+1)} T_g \phi_k^{\gamma_o} d\mu(g)$$

with $d\mu(g)$ a probability measure on $SO(n+1)$ and with T_g the translation operator by g on $L^2(S^n)$, i.e. $T_g f(x) = f(g^{-1}x)$. In particular, matrix elements may be written in the form

$$\langle A\phi_k^\mu, \phi_k^\mu \rangle = \langle T_\mu^* A T_\mu \phi_k^{\gamma_o}, \phi_k^{\gamma_o} \rangle$$

with

$$T_\mu := \int_{SO(n+1)} T_g d\mu(g).$$

As above, let us denote by Φ_k^μ the linear functional

$$\langle \Phi_k^\mu, a \rangle := \langle Op(a)\phi_k^\mu, \phi_k^\mu \rangle$$

where Op is a fixed choice of quantization. We then make the standard observation that $\Phi_k^\mu(a)$ depends only on the diagonal part $\Pi_k Op(a)\Pi_k$ of $Op(a)$ where $\Pi_k : L^2(S^n) \to \mathcal{H}_k$ is the orthogonal projection. As is also well-known, the direct sum of the diagonal parts is just the time average of $Op(a)$, that is,

$$\bigoplus_{k=0}^{\infty} \Pi_k Op(a)\Pi_k =: Op^{ave}(a) := \frac{1}{2\pi}\int_0^{2\pi} U_t^* Op(a) U_t dt$$

with $U_t = exp(itD)$, $D = \sqrt{\Delta + (\frac{n-1}{4})^2} - \frac{n-1}{4}$. As may be easily verified, $D|_{\mathcal{H}_k} = k$ so that U_t is a periodic group and $Op^{ave}(a)$ commutes with Δ. It is also a standard observation (by the Egorov theorem) that the principal symbol of $Op^{ave}(a)$ is the classical time average

$$a^{ave} := \frac{1}{2\pi}\int_0^{2\pi} a \circ G^t dt$$

where G^t denotes the geodesic flow. Hence the weak limits of Φ_k^μ depend only on a^{ave} and it suffices to consider flow-invariant symbols. Thus, we may assume henceforth that $a \in C^\infty(G(S^n, can))$.

With these identifications, our aim is to construct a sequence of $\phi_k^{\mu_k}$'s so that $\Phi_k^{\mu_k}(a) \to \int_{S^*S^n} a d\nu$ for all flow-invariant $a \in C(S^*S^n)$, where $\nu \in \mathcal{M}_I$ is a given invariant measure. Since ν corresponds to a measure $\bar{\nu}$ on $C(G(S^n, can))$ by

$$\int_{S^*S^n} a d\nu \ = \ \int_{G(S^n,can)} R(a)(\gamma) d\bar{\nu}(\gamma)$$

where $a \in C(S^*S^n)$, and where

$$R(a)(\gamma) = \int_\gamma a ds \in C(G(S^n, can))$$

is the Radon transform, it is equivalent to ask that the limit be $\int_{G(S^n,can)} a d\bar{\nu}$. That is, our aim is to prove that the set $\bar{\mathcal{Q}}$ of weak* limit points, viewed as a class of measures on $C(G(S^n, can))$, equals the entire class $\mathcal{M}_{G(S^n,can)}$ of regular Borel probability measures on this space.

To this end, it suffices to show that any discrete probability measure

$$d\rho = \sum_{j=1}^{N} c_j \delta_{\gamma_j}, \qquad c_j > 0, \sum c_j = 1$$

arises as a weak* limit point of some sequence $\{d\Phi_k^\mu\}$. Indeed, it is well known (by the Krein-Milman theorem) that convex combinations of point masses are weak* dense in the compact convex set $\mathcal{M}_{G(S^n,can)}$ (equipped with the weak* topology). Moreover, this space is Hausdorff so that the set of weak* limit points of the class of the measures $\{d\Phi_k^\mu\}$ is necessarily closed in $\mathcal{M}_{G(S^n,can)}$. The theorem then follows from:

Lemma 3.1. *Let $d\rho$ be as above, and let $d\mu = \sum_{j=1}^{N} \sqrt{c_j}\delta_{g_j}$ where $g_j\gamma_o = \gamma_j$. Also, let Φ_k^μ be the microlocal lift of ϕ_k^μ. Then $\Phi_k^\mu \to d\rho$.*

Proof: What we want to prove is that

$$\langle T_\mu^* Op^{ave}(a) T_\mu \phi_k^{\gamma_o}, \phi_k^{\gamma_o}\rangle \to \int_{G(S^n, can)} a d\rho, \qquad \forall a \in C(G(S^n, can)).$$

To do so, we form the generating function

$$\Upsilon_\mu(\theta) := \sum_{k=0}^{\infty} \langle T_\mu^* Op^{ave}(a) T_\mu \phi_k^{\gamma_o}, \phi_k^{\gamma_o}\rangle e^{ik\theta}.$$

In a well-known way (cf. [B-G]), the singularities of $\Upsilon_\mu(\theta)$ will determine the asymptotics of the Fourier coefficients. The main point is to prove that $\Upsilon_\mu(\theta)$ is a Lagrangean distribution and to determine its principal symbol. Since it is also a Hardy function (i.e. has only positive frequencies), the Lagrangean property will automatically imply a complete asymptotic expansion as $k \to \infty$ of $\langle T_\mu^* Op^{ave}(a) T_\mu \phi_k^{\gamma_o}, \phi_k^{\gamma_o}\rangle$ with principal asymptote given by the principal symbol of Υ_μ.

To show that Υ_μ is Lagrangean and to determine its symbol, we express it as the trace:

$$\Upsilon_\mu(\theta) = Tr\ e^{i\theta D} \Pi_{\gamma_o} T_\mu^* Op^{ave}(a) T_\mu$$

where D is (as above) the positive elliptic first order pseudodifferential operator equaling k on $\mathcal{H}_k$ and where

$$\Pi_{\gamma_o} = \sum_{k=0}^{\infty} \phi_k^{\gamma_o} \otimes \phi_k^{\gamma_o *}$$

is the canonical Toeplitz projection associated to γ_o. That this projection *is* a Toeplitz projector follows from a theorem due to V.Guillemin [G]. We now claim that the trace above is a clean composition of Fourier Integral and Toeplitz operators (see [B-G] for the terminology).

To analyse the trace, we write out (with $A = Op^{ave}(a)$)

$$T_\mu^* A T_\mu = \sum_{j,k} \sqrt{c_j c_k} T_{g_j}^* A T_{g_k}$$

and consider the individual traces

$$Tr\ e^{i\theta D} \Pi_{\gamma_o} \sqrt{c_j c_k} T_{g_j}^* A T_{g_k}.$$

It is clear that T_g is an FIO associated to the graph of the canonical transformation χ_g on T^*S^n lifted from T_g on S^n. Hence $T_{g_j}^* A T_{g_k}$ is an FIO associated to $\chi_{g_j^{-1} g_k}$. Also, $e^{i\theta D}$ is associated to G^θ, the geodesic flow, and Π_{γ_o} is the restriction to $\gamma_o \times \gamma_o$. Hence the singularities of the trace occur at values of θ for which the fixed point set $Fix(G^\theta \chi_{g_j^{-1} g_k}|_{\gamma_o})$ is non-empty. It is clear that no fixed points can occur unless $g_j^{-1} g_k$ fixes γ_o. Since we may assume with no loss of generality that $g_j\gamma_o$ are distinct geodesics, $g_j^{-1} g_k$ will only fix γ_o when $j = k$. Moreover, when $j = k$, the only θ for which $Fix(G^\theta \chi_{g_j^{-1} g_k}|_{\gamma_o})$ is non-empty is $\theta = 0$. The fixed point set is obviously clean, and so by the composition formula [B-G, Theorem 7.5] we have

$$Tr\ e^{i\theta D} \Pi_{\gamma_o} \sqrt{c_j c_k} T_{g_j}^* A T_{g_k} \in I^*(S^1, T_0^*(S^1))$$

with principal symbol $\delta_{jk} c_k \int_{\gamma_o} a \circ T_{g_k} ds$. Summing over (j, k) we get that

$$\Upsilon_\mu(\theta) \in I^*(S^1, T_0^*(S^1)), \qquad \sigma_{\Upsilon_\mu} = \sum_k c_k \int_{\gamma_o} a \circ T_{g_k} ds = \int_{G(S^n, can)} a d\rho.$$

It then follows by [B-G, Ch. 12] that

$$\langle \Phi_k^\mu, a \rangle \to \int_{G(S^n, can)} a d\rho,$$

completing the proof of the Lemma 3.1 and hence of the Theorem 1.1. □

4. Scarring on a closed geodesic: Proof of Theorem 1.4

Let us begin by discussing joint eigenfunctions of Hecke-Laplace operators in the framework of states on algebras of pseudodifferential and Fourier integral operators.

4.1. **Hecke operators.** Hecke operators are finite Radon transforms associated to finite Riemannian covering diagrams

$$\begin{array}{ccc} & (\tilde{M}, \tilde{g}) & \\ \pi \swarrow & & \searrow \rho \\ (M, g) & & (M, g) \end{array}$$

where, we emphasize, both ρ and π are covers of $\tilde{M} \to M$. The diagram gives rise to an isometric correspondence

$$C : M \to M, \qquad C(x) = \rho\pi^{-1}\{x\}$$

and to the associated Hecke operator

$$T_C := \rho_* \pi^*.$$

Our first observation is that T_C is an FIO (Fourier Integral operator), associated to the homogeneous canonical relation

$$\Lambda_C := N^*(gr(C)) \subset T^*(M) \times T^*(M)$$

with $gr(C) \subset M \times M$ the graph of the correspondence C and with N^* the conormal bundle.

We may describe this canonical relations in a more useful way, freely identifying cotangent and tangent objects by the metric. First, the covers ρ and π extend (by their derivatives) to the tangent (or cotangent) bundles $T^*\tilde{M} \to T^*M.$; we continue to denote them by ρ, π. We then have a map:

$$\pi \times \rho : T^*(\tilde{M}) \to T^*(M) \times T^*(M)$$

whose image is easily seen to be a diffeomorphism onto its image, equal to Λ_C. It commutes with the action of $\mathbf{R}^+$ on the cotangent bundles, so we will usually slice the action to get the map

$$\pi \times \rho : S^*(\tilde{M}) \to S\Lambda_C$$

with $S\Lambda_C$ the unit length elements in Λ_C.

4.2. **Algebras, states and classical limits.** We now reformulate the questions raised in the introduction from a C* algebraic point of view as in [Z1]. This begins with the fact that on any compact manifold M the C* algebra $\Psi^o(M)$ of bounded pseudodifferential operators (in the norm topology) fits into the exact sequence

$$0 \to \mathcal{K} \to \bar{\Psi}^o \to C(S^*M) \to 0$$

with $\mathcal{K}$ the compact operators. Since the linear functionals Φ_j are states on Ψ^o and since any weak limit ν annihilates $\Psi^{-1}(M)$ (the -1st order operators) and hence annihilates $\mathcal{K}$, ν may be identified with a state on $C(S^*M)$, i.e. with a probability measure on S^*M. Moreover, invariance of Φ_j under α_t implies that ν is invariant under the quotient automorphism, the geodesic flow.

To take Hecke operator T_C into account, we will extend the algebra Ψ^o to the algebra

$$\mathcal{A} := \bigvee_{\{C\}} I^o(M \times M, \Lambda_C)$$

generated by FIO's associated to the canonical relations Λ_C. The elements of $I^o(M \times M, \Lambda_C)$ may all be expressed in the form

$$B = \rho_* A \pi^*$$

with $A \in \Psi^o(\tilde{M})$. In particular, this space includes operators of the form $T_C A$ and AT_C with $A \in \Psi^o(M)$. The wave group still defines an automorphism α_t of this algebra and the functionals Φ_j still define invariant states.

Since any weak limit of the Φ_j's, as functionals on $\mathcal{A}$, annihilates compact operators, it defines a functional on the symbol algebra $\mathcal{S}$ corresponding to $\mathcal{A}$. This symbol algebra may be described as follows: any polynomial in the above FIO's is associated to a union of the canonical relations Λ_C. Hence its symbol consists of a collection of homogeneous 1/2-densities on the associated Λ_C's. Since they are graphs, each Λ_C carries a canonical volume 1/2-density, and hence we may identify the symbols on them with the scalar coefficient of this half-density. Thus the symbol is a collection of functions σ_C, one on each component Λ_C. If Λ_C has components (as it will below) we may regard symbols as a collection of functions on the components.

Via the maps $\pi \times \rho$ above, each symbol σ_C may be lifted back to the space $S^*(\tilde{M})$. Thus we can regard the symbol as a collection of functions σ_C on $S^*(\tilde{M})$. It follows that any weak limit Φ of the Φ_j's on $\mathcal{A}$ correspond to a collection of measures ν_C, one on each component of $S^*(\tilde{M})$. From the invariance of Φ_j under the automorphisms $\alpha_t(A) = U_t^* A U_t$, it follows that each ν_g is invariant under the geodesic flow G^t of $\tilde{g}$ on $S^*(\tilde{M})$. Note that the limit measures ν_C's correspond to sparse subsequences of ϕ_j's: Indeed $\nu_C(T_C)$ is the limit eigenvalue of T_C along the subsequence. So regarding the Φ_j's as functionals on the larger algebra splits up the eigenfunctions into much smaller subsequences than on Ψ^o.

Our problem is to pin down ν, the limit measure on $S^*(M)$. The following proposition answers Question (1) in the introduction, i.e. describes the symmetry property of the limit measures.

Proposition 4.1. *Let $\{\phi_{j_k}\}$ be a subsequence of eigenfunctions for which the Φ_{j_k}'s have a (unique) weak limit on $\mathcal{A}$. Then, for any Hecke operator T_C,*

$$\rho_* \nu_C = \pi_* \nu_C = \nu_C(1) \nu.$$

Proof: We have:

$$\Phi_j(AT_C) = \Phi_j(T_C A) = \Phi_j(T_C)\Phi_j(A).$$

Passing to the limit Φ_∞ along a subsequence as $j_k \to \infty$, we get

$$\Phi_\infty(AT_C) = \Phi_\infty(T_C A) = \Phi_\infty(T_C)\Phi_\infty(A).$$

Identifying Φ_∞ with a collection of measures ν_C as above, and noting that the symbol of AT_C, qua function on $S^*(\tilde{M})$ is $\pi^*(\sigma_A)$ while that of $T_C A$ is $\rho^*(\sigma_A)$, we get the equation

$$\nu_C(\rho^*\sigma_A) = \nu_C(\pi^*\sigma_A) = \nu_C(1)\nu(\sigma_A)$$

which is equivalent to the stated formula. □

Remark 4.2. *Had T_C been an isometry, the 'conjugation' $\alpha_C(A) = T_C^* A T_C$ would have been an automorphism, $\nu_C(1)$ would have equaled 1, and the maps π (resp. ρ) would have been id (resp. C). Hence the proposition would have established the invariance of ν under translation by C.*

We now specialize to the case of Hecke operators on $M = (S^n, can)$. The main simplification is that each term T_{C_j} acts on M so that all the covers discussed above are trivial. We therefore introduce the situation in this level of generality.

4.3. **Hecke operators for trivial covers.** We will assume in this section that

$$\tilde{M} = M \times \{-N, \dots, N\} = M \times M \cdots \times M \qquad \text{2N times.}$$

We will assume moreover that M is equipped with a metric g and N isometries $C_j : M \times M$ and define

$$\pi(x, j) = x, \qquad \rho(x, j) = C_j^{sgn(j)} x$$

where $C_j^{sgn(j)} = C_j$ if $j > 0$ and $= C_j^{-1}$ if $j < 0$. The Hecke operator T_C is then given by

$$T_C f(x) = \sum_{j=1}^{N} f(C_j x) + f(C_j^{-1} x).$$

As discussed in the previous section, the limit measure ν_C associated to a weakly convergent sequence $\{\Phi_{j_k}\}$ may be identified with a set of measures ν_{C_k} on the $2N$ components of $S^*\tilde{M}$. It is easy to see what they are: Indeed, consider an operator of the form $\rho_* A \pi^*$ with A a pseudodifferential operator on $\tilde{M}$ which is zero except on the component $M \times \{k\}$. Let a be its principal symbol, viewed as a function on S^*M. Then we have

$$\nu_{C_k}(a) = \lim_{m\to\infty} \langle A T_{C_k} \phi_{j_m}, \phi_{j_m} \rangle.$$

By Egorov, we also have

$$\nu_{C_k}(a \circ C_k) = \lim_{m\to\infty} \langle T_{C_k} A \phi_{j_m}, \phi_{j_m} \rangle.$$

Hence the invariance principle may be summarized in the form:

$$\sum_{k=1}^{N} \nu_{C_k} + \nu_{C_k^{-1}} = \sum_{k=1}^{N} \nu_{C_k} \circ T_{C_k} + \nu_{C_k^{-1}} \circ T_{C_k^{-1}} = \mu_\infty(C)\nu. \tag{6}$$

It implies the following semi-invariance properties, which in part are phase space versions of inequalities in [R-S]

Proposition 4.3. *Let ν be a classical limit of the states $\{\Phi_j\}$ on Ψ^o and let ν_{C_k} be the classical limit measures described above, i.e. $\nu = lim_{m\to\infty}\Phi_{j_m}$ and $\nu_{C_k}(a) = \lim_{m\to\infty}\langle AT_{C_k}\phi_{j_m}, \phi_{j_m}\rangle$. Then:*

(a) $\nu_{C_k} << \nu$ *and* $\nu_{C_k} << \nu\circ T_{C_k}$.

(b) *If the limit eigenvalue* $\mu_\infty := lim_{k\to\infty}\mu_k(C) \neq 0$, *we have*

$$\nu \ll \nu\circ T_C.$$

(c) *If* $\mu_\infty = 0$ *then:*

$$\nu \ll \nu\circ T_{\tilde{C}}$$

with $T_{\tilde{C}} = T_{C_1^{-2}} + \sum_{j\neq\pm1} T_{C_1^{-1}C_j}$.

Proof of Proposition 4.3:

(a) We have

$$|\nu_{C_k}(|a|^2)| = \lim_{m\to\infty} |\langle A^*AT_{C_k}\phi_{j_m}, \phi_{j_m}\rangle|.$$

But

$$\begin{aligned} |\langle AT_{C_k}\phi_{j_m}, A\phi_{j_m}\rangle| &\leq ||AT_{C_k}\phi_{j_m}||\ \ ||A\phi_{j_m}|| = \\ &= |\langle T^*_{C_k}A^*AT_{C_k}\phi_{j_m}, \phi_{j_m}\rangle|^{\frac{1}{2}}|\langle A^*A\phi_{j_m}, \phi_{j_m}\rangle|^{\frac{1}{2}}. \end{aligned}$$

Denoting the matrix element by Φ_{j_m}, this says

$$|\Phi_{j_m}(A^*AT_{C_k})| \leq |\Phi_{j_m}(T^*_{C_k}A^*AT_{C_k})|^{\frac{1}{2}}|\Phi_{j_m}(A^*A)|^{\frac{1}{2}}.$$

Passing to the limit and applying Egorov to $T^*_{C_k}A^*AT_{C_k}$ we get

$$|\nu_{C_k}(|a|^2)| \leq \nu(|a|^2\circ C_k)^{\frac{1}{2}}\nu(|a|^2)^{\frac{1}{2}}.$$

Letting $a = \chi_\Omega$ with $\Omega\subset S^*M$ a regular measurable set we get

$$|\nu_{C_k}(\Omega)| \leq \nu(C_k(\Omega))^{\frac{1}{2}}\nu(\Omega)^{\frac{1}{2}}.$$

It is obvious from this that $\nu_{C_k}(\Omega) = 0$ if either $\nu(\Omega) = 0$ or if $\nu(C_k(\Omega) = 0$. □

(b) From the last line in (a) we get

$$\begin{aligned} |\mu_\infty(C)\nu(\Omega)| = |\sum\nu_{C_k}(\Omega)| &\leq \sum|\nu_{C_k}(\Omega)| \\ &\leq |\nu(\Omega)|^{\frac{1}{2}}\sum\nu(C_k(\Omega))^{\frac{1}{2}} \end{aligned}$$

hence

$$|\mu_\infty(C)|\cdot|\nu(\Omega)|^{\frac{1}{2}} \leq \sum\nu(C_k(\Omega))^{\frac{1}{2}}$$

or

$$|\mu_\infty(C)|^2|\nu(\Omega)| \leq 2N\sum\nu(C_k(\Omega).$$

Hence $\nu << \nu\circ T_C$ as long as $\mu_\infty(C)\neq 0$. □

(c) On the other hand, suppose $\mu_\infty(C) = 0$. Pick one 'branch' say C_1 of the correspondence and define a new one,

$$\tilde{C} := \{C_1^{-2}, C_1^{-1}C_2, C_1^{-1}C_2^{-1}, \dots, C_1^{-1}C_N\}$$

giving rise to a new Hecke operator

$$T_{\tilde{C}}f(x) = f(C_1^{-2}x) + \sum_{i=2}^N f(C_iC^{-1}x) + f(C_i^{-1}C_1^{-1}x). \tag{7}$$

From the fact that

$$\begin{aligned} \phi_{j_k}(x) = -\big[&\phi_{j_k}(C_1^{-2}x) + \phi_{j_k}(C_2C_1^{-1}x) + \phi_{j_k}(C_2^{-1}C_1^{-1}x) + \cdots + \\ &\phi_{j_k}(C_NC_1^{-1}x) + \phi_{j_k}(C_N^{-1}C_1^{-1}x)\big] + o(1)\phi_{j_k}(C_1^{-1}x) \end{aligned} \tag{8}$$

we get

$$\Phi_{j_k}(A) = -(AT_{\tilde{C}}\phi_{j_k}, \phi_{j_k}) + o(1) = \Phi_{j_k}(AT_{\tilde{C}}) + o(1) = \Phi_{j_k}(T^*_{\tilde{C}}A) + o(1)$$

hence by a repetition of the argument of part (a),

$$\nu(\Omega) \leq 2N\nu(\tilde{C}(\Omega)).$$

□

We will actually need a more complete collection of inequalities of these kinds. To see why, let us suppose that there exists a subsequence $\{\Phi_{j_m}\}$ converging to $\nu = \delta_\gamma$ as functionals on $\Psi^o(M)$. By passing to a subsequence if necessary we may suppose that the limit measures ν_{C_j} are also well defined. By part (a) of the proposition, we have $\nu_{C_j} \ll \delta_\gamma$. Since ν_{C_j} is flow invariant and δ_γ is an ergodic measure, it follows that $\nu_{C_j} = a_j\delta_\gamma$ for some complex constant a_j. By part (a) again, we have $\nu_{C_j} \ll \delta_{C_j\gamma}$ and also $\nu_{C_j} \ll \delta_\gamma$, which is a contradiction unless $C_j\gamma = \gamma$. Under the assumptions of Theorem 1.4, there can be at most two terms (say C_1 and C_1^{-1}) fixing γ. Hence the invariance principle (6) simplifies to $\nu_{C_1} + \nu_{C_1^{-1}} = \mu_\infty(C)\delta_\gamma$, with all other terms zero and with $\nu_{C_1} = a\delta_\gamma$. This is a strong conclusion regarding the eigenfunctions $\{\phi_{j_m}\}$, but not apparently strong enough for a contradiction. However, we easily get a contradiction by following this line of reasoning a bit further: Consider the larger collection of limit measures

$$\nu_{(i_1,\dots,i_n;j_1,\dots,j_l)}(a) := \lim_{m\to\infty}\langle T_{C_{i_1}}\dots T_{C_{i_n}} Op(a) T_{C_{j_1}}\dots T_{C_{j_l}}\phi_{j_m}, \phi_{j_m}\rangle.$$

associated to thesequence $\{\phi_{j_m}\}$ of eigenfunctions. Here, the indices i_k, j_k run over $\{-N,\dots,N\}$, with negative indices indicating adjoints such as $T_{-j} = T_j^*$. By passing to a subsequence of $\{\phi_{j_m}\}$, we may assume that any finite collection of such limit measures is well-defined. We have:

Proposition 4.4. *In the above notation,*

$$\nu_{(i_1,\dots,i_n;j_1,\dots,j_l)} \ll \nu\circ(C_{j_1}\circ\dots\circ C_{j_l}), \qquad \nu_{(i_1,\dots,i_n;j_1,\dots,j_l)} \ll \nu\circ(C_{i_n}^{-1}\circ\dots\circ C_{i_1}^{-1}).$$

Proof: This follows as in the previous proposition by bounding the inner products by $||Op(a)T_{C_{j_1}}\dots T_{C_{j_l}}\phi_{j_m}||$ and by $||Op(a)^*\circ(T^*_{C_{i_1}}\circ\dots\circ T^*_{C_{i_n}})\phi_{j_m}||$. □

4.4. **Completion of proof of Theorem 1.4.** We now complete the proof that the joint $T_C - \Delta$ eigenfunctions cannot scar on a closed geodesic γ.

As in the previous section we suppose that there exits a subsequence $\{\Phi_{j_m}\}$ converging to $\nu = \delta_\gamma$ as functionals on $\Psi^o(M)$; with no loss of generality we may suppose any finite subcollection of $\nu_{(i_1,\dots,i_n;j_1,\dots,j_l)}$'s are also well-defined.

Case 1: $\mu_\infty(C) \neq 0$. By part (b) of the Proposition 4.3, $\gamma \subset C_1\gamma \cup C_1^{-1}\gamma \cup C_2\gamma \cup \dots \cup C_n\gamma \cup C_n^{-1}\gamma$. It follows that at least one C_j must fix γ. Such C_j must be unique and we will arrange the indices so $j = 1$.

We then consider the limit formula

$$\langle T_C Op(a) T_C \phi_{j_m}, \phi_{j_m}\rangle \to \mu_\infty(C)^2\nu(a). \tag{9}$$

We may write out the left side as

$$\langle T_C Op(a)T_C\phi_{j_m}, \phi_{j_m}\rangle = \sum_k \langle T^*_{C_k} Op(a) T_{C_k}\phi_{j_m}, \phi_{j_m}\rangle + \langle T_{C_1} Op(a) T_{C_1}\phi_{j_m}, \phi_{j_m}\rangle \tag{10}$$

$$+\langle T^*_{C_1} Op(a) T^*_{C_1}\phi_{j_m}, \phi_{j_m}\rangle + \sum{}' \langle T_{C_i} Op(a) T_{C_j}\phi_{j_m}, \phi_{j_m}\rangle$$

where in the last sum at least one of indices i, j is not equal to 1, and the indices are not opposites to each other.

Lemma 4.5. *Assume that the limiting measure ν of the joint $T_C - \Delta$ eigenfunctions satisfies $\nu = \delta_\gamma$ and that γ is fixed by W, W^{-1} where W is a word of length at most two in the free group $\mathcal{F}$ generated by the symbols C_j (and that W is not a power of another element in $\mathcal{F}$). Let $0 \le n, l \le 2$ and let $-N \le i_1, \dots, i_n, j_1, \dots, j_l \le N$. Assume that the word*

$$C_{i_1} \dots C_{i_n} C_{j_1} \dots C_{j_l}$$

does not reduce to a power of W in $\mathcal{F}$. Then

$$\nu_{i_1,\dots,i_n;j_1,\dots,j_l} \;=\; 0.$$

Proof: We let $\mu = \nu_{i_1,\dots,i_n;j_1,\dots,j_l}$. It follows from Proposition 4.4, the ergodicity of delta measures on closed geodesics and the flow invariance of μ that that μ is a multiple of the delta measure on the geodesic $C_{j_1} \dots C_{j_n}\gamma$ on the one hand, and that it is a multiple of the delta measure on the geodesic $C_{i_n}^{-1} \dots C_{i_1}^{-1}\gamma$ on the other hand. If $\mu \ne 0$ then

$$C_{i_1} \dots C_{i_n} C_{j_1} \dots C_{j_l}\gamma \;=\; \gamma.$$

But that, together with the assumption on the word $C_{i_1} \dots C_{i_n} C_{j_1} \dots C_{j_l}$, contradicts the assumption (ii) of Theorem 1.4. □

It follows from Lemma 4.5 (applied for $W = C_1$) that the last term in (10) tends to zero as $m \to \infty$. On the other hand, the first sum tends to $\int_\gamma T_C(a)ds$. And the other C_1-terms tend to $b\int_\gamma ads$ for some constant b. Thus we have

$$\int_\gamma T_C(a)ds + b\int_\gamma ads = \mu_\infty(C)^2 \int_\gamma ads$$

or more simply (for time reversal invariant symbols)

$$2\sum_{j=2}^{N} \int_{C_j\gamma} ads = b' \int_\gamma ads.$$

Clearly this is impossible. The contradiction shows that the eigenfunctions cannot scar on the closed geodesic if $\mu_\infty(C) \ne 0$ and finishes the proof in case 1. □

Case 2: $\mu_\infty(C) = 0$. We shall consider two cases:

Case 2A: γ is fixed by one of the C_j-s, say C_1. Consider the expression

$$\langle T^*_{\tilde{C}} A T_{\tilde{C}} \phi_{j_m}, \phi_{j_m}\rangle \tag{11}$$

By (8),

$$\langle T^*_{\tilde{C}} A T_{\tilde{C}} \phi_{j_m}, \phi_{j_m}\rangle \;\sim\; \langle A\phi_{j_m}, \phi_{j_m}\rangle \;\to\; \nu(a) \tag{12}$$

as $m \to \infty$.

Expanding (11) and using (12), we get

$$\langle T_{C_1^2} A T_{C_1^{-2}} \phi_{j_m}, \phi_{j_m}\rangle + \sum_{j \ne \pm 1} \langle T_{C_j^{-1}C_1} A T_{C_1^{-1}C_j} \phi_{j_m}, \phi_{j_m}\rangle + \tag{13}$$

$$\sum_{j\ne\pm1} \Big(\langle T_{C_1^2} A T_{C_1^{-1}C_j} \phi_{j_m}, \phi_{j_m}\rangle + \langle T_{C_j^{-1}C_1} A T_{C_1^{-2}} \phi_{j_m}, \phi_{j_m}\rangle \Big) +$$

$$\sum_{j,k\ne\pm1; j\ne -k} \langle T_{C_jC_1} A T_{C_1^{-1}C_k} \phi_{j_m}, \phi_{j_m}\rangle \;\to\; \nu(a)$$

By the assumption of case 2A,

$$\nu \;=\; \nu \circ C_1^{-2} \;=\; \nu \circ C_1^2 \tag{14}$$

Passing to the limit, we see that the first term in (13) cancels the right-hand side. The last two sums in (13) converge to zero by Lemma 4.5 (with $W = C_1$), so we get

$$\sum_{j \neq \pm 1} \int_\gamma T_{C_1^{-1}C_j}(a)\, ds \;=\; 0. \tag{15}$$

Changing variables, we get (for time reversal invariant symbols)

$$\sum_{j=2}^{N} \int_{C_j \gamma} a\, ds \;=\; 0.$$

As in the case $\mu_\infty(C) \neq 0$, this contradicts the assumptions of Theorem 1.4 and completes the proof in case 2A. □

Case 2B: γ is not fixed by any of the C_j-s. It follows then from part (c) of Proposition 4.3 that γ is fixed by one of $C_1^{-1}C_j$-s (for $j \neq \pm 1$). By the assumption (i) of the Theorem 1.4 and by the definition (7) of $T_{\tilde{C}}$ there is at most one such term, say $C_1^{-1}C_2$. Then

$$\nu \;=\; \nu \circ C_1^{-1} C_2$$

By considering the expression (11) and using Lemma 4.5 with $W = C_1^{-1}C_2$ we arrive to the contradiction exactly as in the case 2A. This completes the proof in case 2B. Theorem 1.4 is now proved. □

Remark 4.6. *One can generalize Theorem 1.4 to prove "non-scarring" of the joint $T_C - \Delta$ eigenfunctions to a measure $\nu = a\omega + b\delta_\gamma$, where ω is the Liouville measure and where a and $b > 0$ are positive constants, $a + b = 1$. To modify the proof, one replaces the measure ν throughout by its singular part in the Lebesgue decomposition with respect to ω.*

Finally, we remark that the assumptions of Theorem 1.4 are satisfied for a "generic" choice of rotations $(C_1, \ldots, C_N) \in SO(n+1)^N$. Indeed, the assumptions basically require that certain products of C_j-s don't lie in stabilizers of some closed geodesics. The stabilizer of a closed geodesic on S^n is isomorphic to $SO(2) \times SO(n-1)$ and has codimension $2n-2$ in $SO(n+1)$. So, the N-tuples $(C_1, \ldots, C_N)$ which don't satisfy the assumptions lie in a finite union of subsets of $SO(n+1)^N$ of large codimension and the assertion follows. When $n = 2$ or 3 the stabilizer of a closed geodesic on S^n is commutative, so the assumption (i) of Theorem 1.4 holds if the corresponding words don't commute.

5. Level spacing: Proof of Theorem 1.5

Let (M, g) denote a compact, negatively curved manifold with Laplacian Δ. As discussed in the introduction, our interest is in the distribution of spacings between normalized eigenvalues of Δ. The normalized operator in dimension n is $\Delta^{\frac{n}{2}}$, whose eigenvalues $\{\lambda_j^{\frac{n}{2}}\}$ have unit mean level spacing by Weyl's law. We will assume that the spacings $x_j = \lambda_j^{\frac{n}{2}} - \lambda_{j-1}^{\frac{n}{2}}$ are asymptotically distributed according to a measure μ on $[0, \infty]$.

Our purpose is to show that addition of a smooth perturbation $V \in \Psi^{-\frac{n}{2}}(M)$ does not change the spacings distribution. The new eigenvalue problem is

$$(\Delta + V)\varphi + \lambda\varphi \;=\; 0.$$

Let $\tilde{\lambda}_j$ be the new eigenvalues and ψ_j the new eigenfunctions. Since $\Delta + V$ has the same principal symbol as Δ, the Weyl law for the $\tilde{\lambda}_j$-s is the same as for Δ, and hence the renormalized operator is $(\Delta + V)^{\frac{n}{2}}$. Without loss of generality we can assume that $\int_{S^*M} V(x)dxd\xi = 0$ (since adding a constant to V doesn't change the spacings).

Theorem 5.1. *The consecutive spacings between the $\tilde{\lambda}_j$-s have the same limiting distribution, μ.*

Proof of Theorem 5.1. We consider the perturbed Laplacian (for $0 \le t \le 1$)

$$A(t) \;=\; \Delta + tV. \tag{16}$$

By standard results in perturbation theory (cf. [Rel, §2.2]), one can choose the eigenvalue branches so that the eigenvalues $\lambda_j(t)$ are real analytic functions of t. Let $x_j(t) = \lambda_j(t)^{\frac{n}{2}} - \lambda_{j-1}(t)^{\frac{n}{2}}$. We want to prove that the limiting distribution of $x_j(t)$-s is μ for every t; putting $t = 1$ will then finish the proof.

Accordingly, let $f \in C_o^\infty([0,\infty))$ be a test function. It suffices to show that

$$\lim_{N\to\infty} \frac{|\sum_{j=1}^N f(x_j(t)) - f(x_j)|}{N} \;=\; 0. \tag{17}$$

To do that, we first differentiate $\sum_{j=1}^N f(x_j(t))$ with respect to t and get

$$\frac{n}{2}\sum_{j=1}^N f'(x_j(t))[\lambda_j(t)^{\frac{n}{2}-1}\lambda_j'(t) - \lambda_{j-1}^{\frac{n}{2}-1}(t)\lambda_{j-1}'(t)].$$

The last sum is majorized by a constant multiple of

$$\sum_{j=1}^N \lambda_j(t)^{\frac{n}{2}-1}|\lambda_j'(t)|.$$

By standard results in perturbation theory (cf. [Rel, §2.2]), for each $0 \le t \le 1$ there exists an orthonormal basis of eigenfunctions $\varphi_j(t)$ such that

$$\lambda_j'(t) \;=\; (V\varphi_j(t), \varphi_j(t))$$

(since $A'(t) = V$ for all t). Hence

$$\lambda_j(t)^{\frac{n}{2}-1}|\lambda_j'(t)| = |(\Delta^{\frac{n}{2}-1}V\varphi_j(t), \varphi_j(t))|.$$

The operator $\Delta^{\frac{n}{2}-1}V$ is a pseudodifferential operator of order 0 whose principal symbol has mean value zero. Since the principal symbol of $A(t)$ has ergodic bicharacteristic flow, the sum tends to zero for each t. To complete the proof of the theorem, we only need to check that the rate at which it goes to zero is uniform for $t \in [0,1]$.

Although it is by no means necessary to assume the metric is negatively curved– any ergodic geodesic flow would do– we can essentially quote the proof of uniformity from the estimates of [Z3]. In fact they show that

$$\frac{\sum_{j=1}^N |(\Delta^{\frac{n}{2}-1}V\varphi_j(t), \varphi_j(t))|}{N} \;\ll\; (\log N)^{-1/2}, \tag{18}$$

uniformly in t, so that perturbations of logarithmically higher order would not affect the level spacings either. To prove (18) from [Z3], we note that the only change is that the evolution operator $exp(is\sqrt{\Delta})$ in [Z3] gets replaced by the perturbed evolution operator $U_t(s) := exp(is\sqrt{\Delta + tV})$. Since this is still a Fourier integral operator with the same canonical relation (hence phases) as in the unperturbed case, the only impact of the perturbation is on the amplitudes. The estimates on the amplitudes used in [Z3], which were quoted from Volovoy ([Vo]), came from a detailed analysis of the eikonal and transport equations over long time intervals. The eikonal equation is unchanged from the unperturbed case, while some of the transport equations acquire an additional term coming from tV. Since the equations are solved by integration and the perturbed evolution operator $U_t(s)$ is constructed in the form $U_t(s/N)^N$ it is evident that the perturbed amplitudes depend smoothly on t. Moreover, the exponential dependence on s entered thru the exponential growth rate of the geodesic flow and this aspect is unchanged by addition of tV. Hence Volovoy's estimates on the amplitudes apply easily to the perturbed case (in fact, they were already stated for exponentials of positive elliptic pseudodifferential operators $exp(itP)$ such as for $P = \sqrt{\Delta + tV}$.) Otherwise the argument in [Z3] is unchanged by the perturbation.

It follows that the decay rate in (18) is uniform in t. Integrating (18) with respect to t, we prove (17) and hence Theorem 5.1. □

As mentioned in the introduction, Theorem 5.1 has some relevance to the well-known conjecture that the level spacings distribution in the chaotic case is *universal*, i.e. is independent of the specific features of the system. One aspect of this universality is that the level spacings should be independent of the particular choice of quantization of the Hamiltonian, since the principal symbol alone determines the nature of the classical flow. This is somewhat problematic in that the subprincipal terms are large enough to shift the eigenvalues by more than the mean level spacing. How then can the dynamics of the principal symbol alone dictate the form of the level spacings distribution?

As a small contribution to this difficult problem we observe that at least in dimension two, the level spacings distribution is independent of the non-invariant aspects of the quantization (i.e. those aspects which depend on the choice of local coordinates and quantization of symbols to pseudodifferential operators).

Corollary 5.2. *Suppose that* $\dim(M) = 2$ *and that* P, Q *are two second order positive elliptic pseudodifferential operators with the same principal and subprincipal symbols. Suppose also that their common bicharacteristic flow is ergodic with exponential estimates. Then the level spacings distributions of* P *and* Q *are the same.*

Recall here that the principal and subprincipal symbols are invariant under change of local coordinates, choice of pseudodifferential calculus, etc. [D-H]. Hence $P - Q$ is of order 0 and Theorem 5.1 applies.

References

[Ar] V. I. Arnold: *Modes and quasimodes*, Funk. Anal. i Prilozh. 6 (1972), 12–20.

[Bl] P. Bleher: *Trace formula for quantum integrable systems, lattice-point problem, and small divisors*, these Proceedings.

[BKS] P. Bleher, D. Kosygin and Y. Sinai: Distribution of energy levels of quantum free particle on the Liouville surface and trace formulae, Comm. Math. Phys. 170 (1995), 375–403.

[B-G] L. Boutet de Monvel and V. Guillemin: *The spectral theory of Toeplitz operators*, Princeton Univ. press, 1981.

[B-D] T. Bröcker and T. Dieck: *Representations of Compact Lie Groups*, 2nd ed, Springer, 1995.

[CV1] Y. Colin de Verdiere: *Quasi-Modes sur les varietes Riemanniennes*, Inv. Math. 43 (1977), 15–52.

[CV2] Y. Colin de Verdiere: *Spectre conjoint d'opérateurs pseudo-différentiels qui commutent II. Le cas intégrable*, Math. Z. 171 (1980), 51–73.

[CV3] Y. Colin de Verdiere: *Ergodicité et fonctions propres du laplacien*, Comm. Math. Phys. 102 (1985), 497–502.

[CV4] Y. Colin de Verdiere: *Distribution de points sur une sphere*, Sem. Bourbaki 41 (1988-89) no. 703, in Asterisque 177-78 (1989), 83–93.

[C-P] Y. Colin de Verdiere and B. Parisse: *Équilibre instable en régime semi-classique. I. Concentration microlocale*, Comm. PDE 19 (1994), 1535–1563.

[Con] B. Connes: Sur les coefficients des séries trigonométriques convergents sphériquement, C. R. Acad. Sci. Paris 283A (1975), 159–161.

[D] J. Duistermaat: *On global action-angle coordinates*, Comm. Pure Appl. Math. 33 (1980), 687–706.

[D-G] J. Duistermaat and V. Guillemin: *The spectrum of positive elliptic operators and periodic geodesics*, Inv. Math. 29 (1975), 39–79.

[D-H] J. Duistermaat and L.Hörmander: *Fourier Integral Operators. II*, Acta Math. 128 (1972), 183–269.

[G] V. Guillemin: *Some microlocal aspects of analysis on compact symmetric spaces*, Annals of Math. St. 93, Princeton Univ. Press, 1979.

[G-S] V. Guillemin and S. Sternberg: *Homogeous quantization and multiplicities of group representations*, J. Funct. An. 47 (1982), 344–380.

[J1] D. Jakobson: *Quantum limits on flat tori*, Annals of Math 145(2)(1997), 235–266; announced in ERA-AMS, 1(2)(1995), 80–86.

[J2] D. Jakobson: *Quantum Unique Ergodicity for Eisenstein Series on* $PSL_2(\mathbf{Z})\backslash PSL_2(\mathbf{R})$, Annales de l'Institut Fourier 44 (1994), 1477–1504.

[J3] D. Jakobson: *Equidistribution of cusp forms on* $PSL_2(\mathbf{Z})\backslash PSL_2(\mathbf{R})$, Annales de l'Institut Fourier 47 (1997), 967–984.

[K-M-S] D. Kosygin, A. Minasov and Y. Sinai: *Spectra of Laplace Beltrami operators on Liouville surfaces*, Russ. Math. Surv. 48 (1993), 1–126.

[L-L] L.D.Landau and E.M.Lifshitz, *Quantum Mechanics*, Pergamon 3rd ed. (1977).

[L-S] W. Luo and P. Sarnak: *Quantum ergodicity of eigenfunctions on* $PSL_2(\mathbf{Z})\backslash\mathbf{H}$, IHES Publ. 81 (1995), 207–237.

[Rel] F. Rellich: *Perturbation theory of eigenvalue problems*, Gordon and Breach, 1969.

[R-S] Z. Rudnick and P. Sarnak: *The Behaviour of Eigenstates of Arithmetic Hyperbolic Manifolds*, Comm. Math. Phys. 161 (1994), 195–213.

[SQC] P. Sarnak: *Arithmetic quantum chaos*, Israel Math. Conf. Proc. 8 (1995), 183–236.

[Shi] G. Shimura: *Introduction to the arithmetic theory of automorphic functions*, Publ. Math. Soc. Japan 11, Princeton Univ. Press, 1994.

[Shn1] A. Shnirelman: *Ergodic Properties of Eigenfunctions*, Uspekhi Mat. Nauk 29 (1974), 181–182.

[Shn2] A. Shnirelman: *On the Asymptotic Properties of Eigenfunctions in the Regions of Chaotic Motions*, addendum to V. F. Lazutkin's book *KAM Theory and Semiclassical Approximations*, Springer, 1993.

[Ta] M. E. Taylor: *Noncommutative harmonic analysis*, Math. Surveys and Monographs 22, AMS, 1986.

[T1] J. Toth: *Various quantum mechanical aspects of quadratic forms*, J. Fucnt. Anal. 130 (1995), 1–42.

[T2] J. Toth: *Eigenfunction Localization in the quantized rigid-body*, J. Diff. Geom. 43 (1996), 844–858.

[Va] J. VanderKam, L^∞ *norms and quantum ergodicity on the sphere*, IMRN 7 (1997), 329–347.

[Vo] A. Volovoy: *Improved two term asymptotics for the eigenvalue distribution function*, Comm. PDE 15 (1990), 1509–1563.

[Z1] S. Zelditch, *Quantum ergodicity of* C^* *dynamical systems*, Comm. Math. Phys. 177 (1996), 507–528.

[Z2] S. Zelditch: *Quantum transition amplitudes for ergodic and for completely integrable systems*, J. Funct. An. 94 (1990), 415–436.

[Z3] S. Zelditch: *On the rate of quantum ergodicity, I. Upper bounds*, Comm. Math. Phys. 160 (1994), 81–92.

[Z4] S. Zelditch: *A Szego limit formula for conjugates of ΨDO's by Hecke operators*, J. Funct. An. 77 (1988), 198–210.

[Z5] S. Zelditch: *Quantum Ergodicity on the Sphere*, Comm. Math. Phys. 146 (1992), 61–71.

[Z6] S. Zelditch: *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, *Duke Math. Jour.* 55 (1987), 919–941.

[Z7] S. Zelditch: *Mean Lindelöf hypothesis and equidistribution of cusp forms and Eisenstein series*, Jour. of Funct. Anal. 97 (1991), 1–49.

[Zyg] A. Zygmund: *On Fourier coefficients and Transforms of functions of two variables*, Studia Mathematica, 50 (1974), 189–201.

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