QUASI-SYMMETRY OF L^p NORMS OF EIGENFUNCTIONS

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Abstract. We study quasi-symmetry properties of L^p norms of positive and negative parts of eigenfunctions of Laplacians.

The behavior of L^p norms of eigenfunctions of the Laplacian on Riemannian manifolds has been a subject of active investigations. The rate of growth of L^∞ norms was studied in [Hor, D-G, Vol, Iv, Gr]; in [Ber, Don] for negatively curved manifolds (see also [I-S, R-S, SSE, ABST]); and by Colin de Verdiére ([CV]) and others for manifolds with integrable geodesic flows (see [Bo1, Bo2, T, T-Z1, T-Z2, Va, Zyg]). Seeger and Sogge ([S-S, So]) gave an improved upper bound for the rate of growth of L^p norms of eigenfunctions.

There are many other interesting questions about the statistical behavior of eigenfunctions on manifolds (cf. [J-N-T]). In particular, in view of the predictions of the random wave conjectures of quantum chaos ([Be], [HR], [ABST]) it seems natural to investigate the relationship between positive and negative parts of real eigenfunctions on Riemannian manifolds. In the paper [Nad] the second author studied quasi-symmetry relation between positive and negative parts of the distribution function of an eigenfunction of the Laplacian on a Riemannian manifold. He considered the volume of a domain on which an eigenfunction has constant sign, as well as the size of positive and negative extrema of eigenfunctions. Here we want to study quasi-symmetry properties of L^p norms of positive and negative parts of eigenfunctions.

Let M be a smooth compact manifold, φ a nonconstant real eigenfunction of the Laplacian. We define φ_+ and φ_- by

$$\begin{aligned} \varphi_+ &= \varphi \cdot \chi(\{\varphi \geq 0\}) \\ \varphi_- &= \varphi \cdot \chi(\{\varphi \leq 0\}) \end{aligned} \tag{1}$$

Theorem 1. *For any $p \geq 1$ there exists $C > 0$, depending only on p and the manifold M, such that for any nonconstant eigenfunction φ of the Laplacian,*

$$1/C \leq ||\varphi_+||_{L^p}/||\varphi_-||_{L^p} \leq C$$

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For $p = 1$ the ratio in Theorem 1 is equal to 1 since $\int \varphi = 0$, while for $p = \infty$ the Theorem was proven in [Nad]. The exact value of C for $p = \infty$ was considered in [Kr].

We remark that by symmetry it suffices to prove that there exists $C > 0$ such that

$$||\varphi_-||_{L^p} \ \leq \ C \ ||\varphi_+||_{L^p} \ . \tag{2}$$

Another inequality in Theorem 1 can then be proved by a similar argument.

We first assume that the n-dimensional manifold M is divided into n-dimensional cells $\cup_{i \in I} Q_i$ (which we shall call *cells*) with disjoint interiors such that the closure $\bar{Q}$ of each cell Q intersects at most C_1 other cells (we can take $C_1 \approx 3^n$), which we shall call the *neighbors* of Q. The size of the cells(which will depend on the eigenvalue corresponding to φ) will be specified later. We have

$$\int_M |\varphi_\pm|^p \ = \ \sum_{i \in I} \int_{Q_i} |\varphi_\pm|^p \tag{3}$$

Given a positive constant D we say that a cell Q is D-*good* if

$$\int_Q |\varphi_-|^p \ \leq \ D \int_Q |\varphi_+|^p.$$

Otherwise, we shall call a cell D-*bad*. We shall prove the following

Claim 2. *Given $D_2 > 1$, there exists $D_1 > 0$ such that every D_1-bad cell Q_1 has a neighbor Q_2 such that*

$$\int_{Q_2} |\varphi|^p \ > \ D_2 \int_{Q_1} |\varphi|^p. \tag{4}$$

We remark that D_2 is assumed to be strictly greater than one.

Before proving the Claim, we shall prove the following

Proposition 3. *If we choose $D_2 > C_1$, the Claim 2 implies Theorem 1.*

Proof. We define an oriented graph G whose vertices will be cells in our partition of M. All directed edges $Q_1 \to Q_2$ of G originate at bad cells Q_1. For each bad cell Q_1 we choose *exactly one* of its neighbors Q_2 such that (4) holds for Q_1 and Q_2 (the existence of *at least one* such neighbor is guaranteed by Claim 2), and make $Q_1 \to Q_2$ a (directed) edge of G.

We remark that the graph G cannot have *directed* loops $Q_1 \to Q_2 \to \ldots \to Q_n \to Q_1$ (otherwise we would have

$$\int_{Q_1} |\varphi|^p \ < \ \frac{1}{D_2^n} \int_{Q_1} |\varphi|^p$$

which is impossible since $D_2 > 1$). We next prove that G cannot have any *undirected* loops. Let γ be such a loop. There will be at least two edge orientation changes along γ, say at Q_i and Q_j: $Q_1 \to Q_2 \to \ldots \to Q_{i-1} \to Q_i \leftarrow Q_{i+1} \leftarrow \ldots \leftarrow Q_j \to Q_{j+1} \to \ldots$. One of these changes will correspond to *two* directed edges originating at a bad cell (say Q_j). But by the definition of G exactly *one* directed edge originates at any bad cell.

The contradiction implies that the graph G is actually a *forest* (i.e. its path components are contractble).

By the definition of the graph G no arrow can originate at a good cell, so two different good cells always belong to different connected components of G. Moreover, every path in G must terminate (since G is finite), so every connected component of G (which is a tree) contains a unique *good* cell (since an edge originates at every bad cell by the definition of G). We remark that a connected component of G may consist of a single good cell.

Given a *good* cell Q, we define its *basin* $B(Q)$ to be the union of all cells which lie in its connected component. Every *bad* cell Q' belongs to the unique basin (since there is only *one* edge originating at Q'). Denote by $T_1, T_2, \dots, T_k$ the trees that are connected components of G, and by $\{Q_1, Q_2, \dots, Q_k\}$ the corresponding good cells. The basin $B_j = B(Q_j)$ is given by

$$B_j \ = \ \cup_{Q \in T_j} Q.$$

Since G is the union of its connected components T_j, and since the equality (3) holds, in order to prove (2) it suffices to show that there exists $C > 0$ such that for every $1 \le j \le k$,

$$\sum_{Q \in B_j} \int_Q |\varphi_-|^p \ \le \ C \sum_{Q \in B_j} \int_Q |\varphi_+|^p . \tag{5}$$

We shall prove that there exists a constant C_2 (depending on C_1 and D_2) such that for every basin $B_j = B(Q_j)$ the following inequality holds:

$$\sum_{Q \in B_j} \int_Q |\varphi|^p \ \le \ C_2 \int_{Q_j} |\varphi|^p . \tag{6}$$

Since Q_j itself is a D_1-good cell, the inequality (5) will follow from the inequalities (6) and

$$\int_{Q_j} |\varphi_-|^p \ \le \ D_1 \int_{Q_j} |\varphi_+|^p .$$

with $C = C_2(D_1 + 1)$.

We proceed to prove (6). To do that, we define the *distance* $d(Q, Q_j)$ from a bad cell Q which belongs to the basin of Q_j to Q_j to be the length of the (directed) path from Q to Q_j (such path is unique since the connected component of Q_j is a tree); the distance from Q_j to itself is defined to be zero. We define the *sphere* $S(Q_j, r)$ to be the set of all cells $Q \in T_j$ whose distance to Q_j is equal to r. Let I_j be defined by

$$I_{j,r} \ := \ I_r \ := \sum_{Q \in S(Q_j, r)} \int_Q |\varphi|^p \tag{7}$$

Then

$$\sum_{Q \in B(Q_j)} \int_Q |\varphi|^p \ = \ I_0 + I_1 + \dots$$

Let us denote by A_j the integral

$$\int_{Q_j} |\varphi|^p$$

To estimate I_r, we first remark that since each cell has at most C_1 neighbors, the number of cells in $S(Q_j, r)$ is at most C_1^r. Also, by the definition of the graph G, for any cell $Q \in S(Q_j, r)$ we have

$$\int_Q |\varphi|^p \ < \ \frac{A_j}{D_2^r}$$

Accordingly,

$$I_0 + I_1 + I_2 + \dots \ \leq \ A_j \left(1 + \frac{C_1}{D_2} + \left(\frac{C_1}{D_2}\right)^2 + \dots\right).$$

If we choose D_2 so that $C_1/D_2 < 1$, the inequality (6) will hold with

$$C_2 \ = \ \frac{1}{1 - (C_1/D_2)}$$

This finishes the proof of the (5), and hence Proposition 3 is proved. □

Lemma 4. *Proposition 3 remains true even if we don't assume that the interiors of the n-dimensional cells Q_i are disjoint (but we still assume that each cell Q_i intersects at most C_1 other cells).*

Proof of Lemma 4. We need to estimate

(i) $\int_M |\varphi_+|^p$ from below;
(ii) $\int_M |\varphi_-|^p$ from above.

We first note that $\int_M |\varphi_-|^p$ is clearly less than

$$\sum_i \int_{Q_i} |\varphi_-|^p \tag{8}$$

We define the oriented graph G, the trees T_j and the basins B_j as in the proof of Proposition 3. As before, the sum (8) is dominated by

$$\left(1 + \frac{1}{1 - (C_1/D_2)}\right) \sum_{j=1}^{k} \int_{Q_j} |\varphi_+|^p, \tag{9}$$

where the sum is taken over all *good* cubes Q_j.

We shall next show that

$$C_1 \cdot \int_{\cup_j Q_j} |\varphi_+|^p \geq \sum_{j=1}^{k} \int_{Q_j} |\varphi_+|^p.$$

Clearly, this would give a lower bound for (i). By the assumption of the Lemma, each good cell intersects at most C_1 other cells. It follows that each point in $\{x \in \cup_j Q_j | \varphi(x) \geq 0\}$ contributes at most C_1 times to the sum (9), proving the last inequality. Since we have already given an upper bound for the integral in (ii), Lemma 4 is now proved. □

We next turn to the proof of Claim 2. We first prove a lemma about an elliptic differential operator L defined on a ball B_2 of radius two centered at the origin in $\mathbf{R}^n$ by

$$L = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_i b_i(x)\frac{\partial}{\partial x_i} + c(x). \tag{10}$$

We assume that a_{ij}, b_i, c are smooth functions, that

$$\lambda^{-1}||\xi||^2 \leq \sum_{i,j} a_{ij}\xi_i\xi_j \leq \lambda||\xi||^2$$

for some $\lambda > 0$, and that $c(x) \leq 0$.

Lemma 5. *Let $u(x)$ be the solution of the equation $Lu = 0$ such that $u(0) = 0$, and let $1 \leq p \leq \infty$. Then there exist a monotonically increasing function $\rho(t)$ defined for $t \geq 0, \rho(0) = 0$ depending on p, ellipticity constant λ, C^2 norms of a_{ij}, b_i, c such that if*

$$||u^+||_{L_p(B_1)} \;<\; t||u^-||_{L_p(B_1)}$$

for some $t > 0$, then

$$||u||_{L_p(B_1)} \;<\; \rho(t)||u||_{L_p(B_2)} \tag{11}$$

Here B_1 denotes a ball of radius one centered at the origin.

Remark 6. *One can probably strengthen the conclusion of Lemma 5, replacing the C^2 norm (on which the final constant depends) by L^∞ norm, but C^2 norm is sufficient for our purposes.*

Proof of Lemma 5. Assume for contradiction that there exists $t > 0$ and a sequence $u_k, k = 1, 2, \ldots$ such that $Lu_k = 0$ in B_2, $u_k(0) = 0$ and

$$\begin{cases} ||u_k||_{L_p(B_2)} & = 1, \\ ||u_k||_{L_p(B_1)} & > t, \\ \dfrac{||u_k^+||_{L_p(B_1)}}{||u_k^-||_{L_p(B_1)}} & \to \; 0 \end{cases} \tag{12}$$

as $k \to \infty$.

Let $G(x, y)$ denote the Green's function of L defined in $B_{2-\varepsilon}, \varepsilon > 0$. Let $0 < \delta < \varepsilon/2$ and $y \in B_{2-\delta}$. Then $\partial G(x, y)/\partial n < C(\varepsilon)$ for $x \in B_{2-\varepsilon}$. Hence for every $\delta > 0$

$$||u||_{L_\infty(B_{2-\delta})} \;\leq\; C(\delta)||u||_{L_p(B_2)} \tag{13}$$

Let $u_k \to u$ weakly in B_2. Then u is a weak and hence also a strong solution of $Lu = 0$ in B_2, $u(0) = 0$. We shall prove that

Claim 7. $u \not\equiv 0$.

Assuming the Claim, we can prove Lemma 5. Indeed, conditions (12) imply that for all $k \geq 1$

$$||u_k^+||_{L_p(B_1)} + ||u_k^-||_{L_p(B_1)} > t > 0.$$

Since the ratio of the two terms above goes to zero by (12), we conclude that

$$||u_k^+||_{L_p(B_1)} \to 0$$

as $k \to \infty$. Hence $u \le 0$ in B_1. Since $u(0) = 0$ it follows from the strong maximum principle ([G-T]) that $u \equiv 0$ in B_1, and by unique continuation $u \equiv 0$ in B_2. But we have shown before that $u \not\equiv 0$. Contradiction finishes the proof of the Lemma.

Proof of Claim 7. The conditions (12) imply that the norms $||u_k||_{L^p(B_2)}$ are uniformly bounded above by 1. Since C^1-norms of u_k in $B_{2-2\delta}$ are uniformly bounded, we can choose a subsequence of u_k so that $u_k \to u$ strongly in $B_{2-2\delta}$ along that subsequence Also, (12) imply that

$$||u_k^-||_{L_p(B_1)} > t/2 > 0$$

for k large enough. This result, together with strong convergence in $B_{2-2\delta}$, imply that $u \not\equiv 0$, proving the Claim and completing the proof of Lemma 5.

□

We next extend Lemma 5 as follows:

Lemma 8. *The conclusions of Lemma 5 hold if we assume that $|c(x)| < K$ where K is an absolute constant.*

Proof. We first note that we can modify the proof of Lemma 5 to prove the same statement for the *cylinders*

$$Q_j \;=\; \tilde{B}_j \times [-j, j], \qquad j = 1, 2$$

instead of the balls B_j.

Next, let v be a solution of $Lv = 0$ where L is given by (10) with $|c(x)| \le K$. Define a new operator L_1 in $\mathbf{R}^{n+1}$ by

$$L_1 \;=\; \frac{\partial^2}{\partial x_{n+1}^2} \;+\; L.$$

We also define a new function u in $\mathbf{R}^{n+1}$ by

$$u(x_1, \dots, x_n, x_{n+1}) \;=\; e^{\sqrt{K} x_{n+1}}\, v(x_1, \dots, x_n).$$

It follows that $(L_1 - K)u = 0$, and since $|c(x)| \le K$, the assumptions of Lemma 5 are satisfied, so we can apply it to the function u. Assume now that v_k is a sequence of eigenfunctions satisfying (12). We note that $\operatorname{sgn} u(x_1, \dots, x_{n+1}) = \operatorname{sgn} v(x_1, \dots, x_n)$. Denote by $\tilde{B}_1, \tilde{B}_2$ the balls of radius 1 and 2 in $\mathbf{R}^n$ centered at the origin, and by $Q_j, j = 1, 2$ the cylinders defined above. It follows that

$$||u||_{L_p(Q_j)} \;=\; ||v||_{L_p(\tilde{B}_j)}\, \Big(\int_{-j}^{j} e^{-pz\sqrt{K}}\, dz\Big)^{1/p}$$

and that similar equalities hold for u^+ and u^-. The Lemma now follows from the generalization of Lemma 5 described in the beginning of the proof.

□

Proof of Theorem 1. Given an eigenfunction φ with a *large enough* eigenvalue λ, we want to divide M into n-dimensional cells Q_i of diameter $< c_1/\sqrt{\lambda}$ and inradius $> c_2/\sqrt{\lambda}$ such that each Q_i lies inside a coordinate chart on M; $h_i(Q_i) = B_i$ is a ball in $\mathbf{R}^n$ (where h_i is the corresponding coordinate function) for all i, φ vanishes in every cell, and each Q_i intersects at most C_1 other cells for some $C_1 > 0$.

If we do that, then let $y = h_i(x), \psi(y) := \varphi(h_i^{-1}(y))$ for $x \in Q_i$. By previous remarks, ψ vanishes in V_i. M is compact, so we can assume that the Jacobian of h is uniformly bounded. Also, since $\operatorname{diam}(Q_i) < c_1/\sqrt{\lambda}$, the change of variables $z = \sqrt{\lambda}y$ transforms $\psi(y)$ into a function $g(z)$ for which the assumptions of Lemma 8 hold. Theorem 1 then follows by Lemma 4.

It remains to be shown that we can divide M into $\cup Q_i$ as indicated above. It is well known (see, for example, [Bru]) there exists $c_2 > 0$ such that that any nonconstant eigenfunction of the Laplacian on M changes its sign in a ball or radius $c_2/\sqrt{\lambda}$ on M. Also, we can realize M as a (finite) simplicial complex $\mathcal{C}$ whose simplexes are supported in coordinate charts on M. Let $h : U_j \to V_j \subset \mathbf{R}^n$ be such a coordinate neighborhood.

Consider the cubic lattice with the side $t/\sqrt{\lambda}$ in V_j, and "pull it back" to M via h^{-1}. Denote the resulting n-dimensional cells in M by P_{ij}. We also cover V_j by balls centered at the vertices of the corresponding cubic lattice. Denote by Q_{ij} the pullbacks of those balls into M by h_{-1}. By choosing λ large enough, we can ensure that each V_j is covered by a subset of $\cup_k Q_{kj}$ of $\cup_i Q_{ij}$ and that each Q_{kj} intersects at most C_1 other Q_{kj}-s where $C_1 \geq 3^n$ depends on the simplicial complex $\mathcal{C}$ only. It then follows from the fact that Jacobian of h is uniformly bounded that one can choose $t > 0$ so that the partition $M = \cup_{ij} Q_{ij}$ will have the required properties. This finishes the proof of Theorem 1. $\square$

Conclusion. Many questions about the relationship between positive and negative parts of an eigenfunction remain unanswered. One of the interesting questions, in the authors' opinion, is whether $||\varphi_+||_p/||\varphi_-||_p \to 1$ as $\lambda \to \infty$ for $1 < p < \infty$ on a given manifold?

We remark that for $p = \infty$ zonal spherical harmonics provide an example of a sequence of eigenfunctions with $||\varphi_+||_\infty/||\varphi_-||_\infty > C > 1$. Indeed, consider the highest weight spherical harmonic which is proportional to $P_m(\cos\phi)$, where P_m is the m-th Legendre polynomial, ϕ is the latitude on S^2, and m is even (for odd m, the ratio is equal to 1). Then $||\varphi_+||_\infty = 1$ and $||\varphi_-||_\infty = \mu_2(m)$ where $\mu_2(m)$ is the absolute value of the first *minimum* of $P_m(x)$, since the size of the r-th relative maximum of $|P_m(x)|$ decreases with r for m fixed as x decreases from 1 to 0 (cf. [Sz1]). Also, it is known that for a *fixed* r, the size of the r-th maximum $\mu_r(m)$ of $|P_m(x)|$ *decreases* as a function of m as $m \to \infty$ (the maximum of P_m is equal to 1). This conjecture of Todd that was proved by Cooper ([Co]) for large m and by Szegö ([Sz2]) for arbitrary m. Accordingly, the ratio $||\varphi_+||_\infty/||\varphi_-||_\infty$ increases as the weight $m \to \infty$. The fact that the ratios are uniformly bounded above was proved in [Ar] (it also follows from [Nad]); the limit of the ratio as $m \to \infty$ is given in [Kr].

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