A LAW OF LARGE NUMBERS FOR THE ZEROES OF HEINE-STIELTJES POLYNOMIALS

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Abstract. We determine the limiting density of the zeroes of Heine-Stieltjes polynomials (or of any set of points satisfying the conclusion of Heine-Stieltjes theorem) in the thermodynamic limit and use this to prove a strong law of large numbers for the zeroes.

1. Introduction

The *generalized Lamé equation* ([Sz]) is the second-order differential equation given by:

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)\frac{d^2}{dx^2}\phi(x)+2\sum_{\nu=0}^{N}\beta_\nu\prod_{\mu\neq\nu}(x-\alpha_\nu)\frac{d}{dx}\phi(x)=V(x)\phi(x). \tag{1}$$

where $\alpha_0<\alpha_1<\cdots<\alpha_N$ and $\beta_\nu,\ \nu=0,...,N$ are real numbers. We assume that the β_ν are all positive.

It is well known that there are exactly $\sigma(K,N):=\frac{(N+K-1)!}{K!(N-1)!}$ polynomials $V(x)$ of degree $N-1$ such that equation (1) has a polynomial solution $\phi(x)=\phi_K(x)$ of degree K. The polynomials $V(x)$ and the corresponding polynomial solutions $\phi_K(x)$ are respectively called *Van Vleck* and *Heine-Stieltjes* polynomials. For background on the generalized Lamé equation, we refer the reader to [T1, T2, Sz, WW].

Differential equations such as (1) arise naturally when separating variables in certain (quantum) completely integrable systems such as the generalized (real) Gaudin spin chains [HW, K, KKN, KM, Sk].

It is a natural question to determine the distribution of the zeroes of ϕ_K in various asymptotic regimes. Recently, Martinez-Finkelstein and Saff [MS] have studied the density of states of the Heine-Stieltjes zeroes in the regime when N is fixed and $K\to\infty$ (i.e. semi-classical asymptotics). In this paper, we study this question in the *thermodynamic* limit $N\to\infty$.

The combinatorial component of our analysis is based in large part on the following classical result of Heine and Stieltjes [H, St, Sz]:

Theorem 1.1. *(Heine-Stieltjes) Let α_ν and β_ν be as in (1). Given N nonnegative integers $m_1,m_2,\dots,m_N$ such that $m_1+\ldots+m_N=K$, there exists unique Van Vleck and (monic) Heine-Stieltjes polynomials, $V(x)$ and $\phi_K(x)$, such that $\phi_K(x)$ has exactly m_j roots in the interval (α_{j-1},α_j) for $1\le j\le N$. The roots of ϕ_K are all simple, and there are exactly $\sigma(K,N)$ polynomial solutions $\phi(x)$ of (1) of degree K, corresponding to $\sigma(K,N)$ possible choices of the N-tuples $(m_1,m_2,\dots,m_N)$.*

In the paper, we fix N "spin sites" $0<\alpha_0<\cdots<\alpha_N<1$ (without loss of generality, we can map $[0,1]$ to any other interval on the real line since (1) is invariant under translations and scaling). Taking into account the Heine-Stieltjes result, we call the N-tuple $\underline{m}=(m_1,m_2,\dots,m_N)$ a *configuration* of zeroes, and we call $|\underline{m}|:=m_1+\ldots+m_N=K$ the *size* of the configuration. We denote the zeroes of the Heine-Stieltjes polynomial of degree K corresponding to the configuration $\underline{m}$ by $\theta_1(\alpha;\underline{m})<\ldots<\theta_K(\alpha;\underline{m})$. We also denote by l_j the number of the interval in which θ_j lies:

$$\theta_j\in(\alpha_{l_j-1},\alpha_{l_j}). \tag{2}$$

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To study the asymptotic distribution of the zeroes of the Heine-Stieltjes polynomials (in the limit $N \to \infty$), we consider the density of states measure

$$d\rho_{DS}(x; \underline{m}, K, N, \alpha) := \frac{1}{K} \sum_{j=1}^{K} \delta(x - \theta_j(\alpha; \underline{m})), \tag{3}$$

We define the averaged density of states in a similar fashion:

$$d\rho_{DS}^{AV}(x; K, N, \alpha) := \frac{1}{\sigma(K,N)} \sum_{|\underline{m}|=K} \frac{1}{K} \sum_{j=1}^{K} \delta(x - \theta_j(\alpha; \underline{m})). \tag{4}$$

Sections 2 through 5 are devoted to the study of the large N asymptotics of $d\rho_{DS}^{AV}(x; K, N, \alpha)$ and $d\rho_{DS}(x; \underline{m}, N, K, \alpha)$.

Our first result, Proposition 2.1, concerns the asymptotics of the averaged density of states. We show that for any $\phi \in BV([0,1])$ and any $\alpha \in \Lambda^N := \{\alpha \in [0,1]^{N+1} : 0 < \alpha_0 < \cdots < \alpha_N < 1\}$,

$$\left| d\rho_{DS}^{AV}(\phi)(\alpha) - \frac{1}{N+1} \sum_{\nu=0}^{N} \phi(\alpha_\nu) \right| = \mathcal{O}\left(\frac{1}{N}\right), \tag{5}$$

as $N \to \infty$. In (5), we use the shorter notation $d\rho_{DS}^{AV}(\phi)(\alpha)$ to denote $d\rho_{DS}^{AV}(\phi; K, N, \alpha)$. We will use this notation throughout the rest of the text.

In Theorem 4.1, we strengthen this result by giving the following estimate for the variance of $d\rho_{DS}(x; \underline{m}, K, N, \alpha)$ itself:

$$\frac{1}{\sigma(K,N)} \sum_{|\underline{m}|=K} \left[d\rho_{DS}(\phi)(\alpha; \underline{m}) - \frac{1}{N+1} \sum_{\nu=0}^{N} \phi(\alpha_\nu) \right]^2 = \mathcal{O}\left(\frac{1}{K}\right) + \mathcal{O}\left(\frac{1}{N}\right). \tag{6}$$

Thus, as long as the degree K and the number of spin sites N *both* go to infinity, the typical Heine-Stieltjes polynomial has its zeroes distributed uniformly in the interval $[0,1]$. As a corollary to Theorem 6, we arrive at the main result of the paper (Theorem 5.1), a strong law of large numbers for the zeroes of generic Heine-Stieltjes polynomials.

Remarks: (i) In this paper, the only information that we use about the zeros corresponding to the configuration $\underline{m}$ is that there are m_j zeros in the interval (α_{j-1}, α_j); thus our results hold for *any* collection of points as above, not just for zeros of $\phi_K(x)$. In future work, we hope to obtain more precise asymptotic results about the Heine-Stieltjes zeros using the fact that they are in electostatic equilibrium [Sz]. However, we do not use this fact in the current paper.

(ii) It would be interesting to determine whether the zeroes of the joint eigenfunctions, ψ_K, have an invariant asymptotic distribution on $\mathbb{S}^N$ itself relative to the standard round metric. We hope to address this elsewhere.

(iii) We were motivated by the previous work of two of the authors [BT] on the averaged level-spacings of the Heine-Stieltjes zeroes. There are several significant improvements here that we wish point out: (1) By a simple combinatorial monotonicity argument, we get *pointwise* results in the spin sites, $\alpha_0, \ldots, \alpha_N$ and completely avoid averaging over this parameter space. (2) We prove a result about the asymptotic density of states of a "typical" Heine-Stieltjes polynomial (Theorem 4.1); that is, about $d\rho_{DS}$ itself and not just $d\rho_{DS}^{AV}$. (3) We need not assume anything about the relative size of the degree, K, and the dimension, N, in any of our results.

(iv) As a final remark, we would like to point out that the techniques of the present paper allow us to give an alternative proof of Theorem 1.1 in [B].

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2. Mean density of states for the Heine-Stieltjes polynomials

Let $\phi \in BV([0,1])$ with total variation $V(\phi)$. Recall that $\underline{m} = (m_1, \ldots, m_N)$ denotes the configuration with m_j zeroes in (α_{j-1}, α_j); here we consider all the $\sigma(N,K)$ configurations with $|\underline{m}| := m_1 + \ldots + m_N = K$ zeroes. For brevity, we introduce the following notation:

$$d\mu_N(\alpha) := \frac{1}{N+1} \sum_{i=0}^{N} \delta(x - \alpha_i). \tag{7}$$

As a consequence of the notation introduced in Section 1, we can rewrite Equation (4) in the form:

$$d\rho_{DS}^{AV}(\phi)(\alpha) = \frac{1}{\sigma(K,N)} \sum_{|\underline{m}|=K} d\rho_{DS}(\alpha; \underline{m})(\phi). \tag{8}$$

Our first result is about the asymptotics of the averaged measure $d\rho_{DS}^{AV}$:

Proposition 2.1. *Let $\phi \in BV([0,1])$ and choose* any *sequence $K = K(N)$ with $N \to \infty$. Then,*

$$|d\rho_{DS}^{AV}(\phi)(\alpha) - d\mu_N(\alpha)(\phi)| \leq \frac{2V(\phi)}{N}.$$

Proof of Proposition 2.1: The first step is to write $\phi = \phi_+ - \phi_-$ where $\phi_\pm \in BV([0,1])$ are monotone, non-decreasing functions. Since $d\rho_{DS}^{AV}(\phi)(\alpha) = d\rho_{DS}^{AV}(\phi_+)(\alpha) - d\rho_{DS}^{AV}(\phi_-)(\alpha)$, and similarly for $d\mu_N$, it suffices to show that for monotone, non-decreasing ψ

$$|d\rho_{DS}^{AV}(\psi)(\alpha) - d\mu_N(\alpha)(\psi)| \leq \frac{V(\psi)}{N}. \tag{9}$$

To prove (9), we first remark that by the inequalities (2) and monotonicity of ψ,

$$\frac{1}{K\sigma(K,N)} \sum_{|\underline{m}|=K} \sum_{j=1}^{K} \psi(\alpha_{l_j-1}) \leq d\rho_{DS}^{AV}(\psi)(\alpha) \leq \frac{1}{K\sigma(K,N)} \sum_{|\underline{m}|=K} \sum_{j=1}^{K} \psi(\alpha_{l_j}). \tag{10}$$

Changing the order of summation, we get

$$\frac{1}{K\sigma(K,N)} \sum_{|\underline{m}|=K} \sum_{j=1}^{N} m_j \psi(\alpha_{j-1}) \leq d\rho_{DS}^{AV}(\psi)(\alpha) \leq \frac{1}{K\sigma(K,N)} \sum_{|\underline{m}|=K} \sum_{j=1}^{N} m_j \psi(\alpha_j). \tag{11}$$

It is easy to see that for any $0 \leq j \leq N-1$, the expression $\psi(\alpha_j)$ appears on the left-hand side of (11) with the coefficient $\frac{1}{K\sigma(K,N)} \sum_{|\underline{m}|=K} m_{j+1}$. Similarly, for any $1 \leq j \leq N$, the expression $\psi(\alpha_j)$ appears on the right-hand side of (11) with the coefficient $\frac{1}{K\sigma(K,N)} \sum_{|\underline{m}|=K} m_j$.

We next prove the following

Lemma 2.2. *For any $1 \leq j \leq N$,*

$$\sum_{|\underline{m}|=K} m_j = \frac{K \cdot \sigma(N,K)}{N}.$$

Proof of Lemma 2.2: We note that $S = \sum_{|\underline{m}|=K} m_j$ is independent of j by symmetry, so

$$SN = \sum_{j=1}^{N} \sum_{|\underline{m}|=K} m_j = \sum_{|\underline{m}|=K} \sum_{j=1}^{N} m_j = \sum_{|\underline{m}|=K} K = K\sigma(K,N). \tag{12}$$

This finishes the proof. $\square$

Returning to the proof of (9), we apply Lemma 2.2 to (11), we conclude that

$$\frac{1}{N} \sum_{j=0}^{N-1} \psi(\alpha_j) \leq d\rho_{DS}^{AV}(\psi)(\alpha) \leq \frac{1}{N} \sum_{j=1}^{N} \psi(\alpha_j), \tag{13}$$

From the second inequality in (13) and since ψ is monotone non-decreasing, we conclude that

$$d\rho_{DS}^{AV}(\psi) - d\mu_N(\psi) \leq \frac{\psi(\alpha_1) + \ldots + \psi(\alpha_N) - N\psi(\alpha_0)}{N(N+1)} \leq \frac{N(\psi(\alpha_N) - \psi(\alpha_0))}{N(N+1)} \leq \frac{V(\psi)}{N+1}.$$

A similar argument using the first inequality in (13) proves that

$$d\rho_{DS}^{AV}(\psi) - d\mu_N(\psi) \geq \frac{-V(\psi)}{N+1},$$

implying (9). This completes the proof of Proposition 2.1. □

3. Approximation

After proving Proposition 2.1, we want to estimate the *variance*, over the space of all configurations of size K, of the difference between the two measures $d\rho_{DS}(\underline{m})$ and $d\mu_N$. We choose a test function $\phi \in \mathrm{BV}([0,1])$, and wish to estimate

$$\mathrm{Var}(\phi) := \frac{1}{\sigma(K,N)} \sum_{|\underline{m}|=K} \left(\int \phi\, d\rho_{DS}(\underline{m}) - \int \phi d\mu_N \right)^2. \tag{14}$$

First, we rewrite the right-hand side of line (14) as

$$\sum_{|\underline{m}|=K} \left[\frac{1}{\sigma(K,N)K^2} \sum_{i,j=1}^{K} \phi(\theta_i(\alpha;\underline{m}))\phi(\theta_j(\alpha;\underline{m})) - \frac{2}{\sigma(K,N)K} \int \phi\, d\rho_{DS}(\underline{m}) \cdot \int \phi d\mu_N + \left(\int \phi d\mu_N \right)^2 \right].$$

We then use Proposition 2.1 for the sum of the second term in (15) to conclude that

$$\mathrm{Var}(\phi) = \frac{1}{\sigma(K,N)K^2} \sum_{|\underline{m}|=K} \sum_{i,j=1}^{K} \phi(\theta_i(\alpha;\underline{m}))\phi(\theta_j(\alpha;\underline{m})) - \left(\frac{1}{N+1} \sum_{i=0}^{N} \phi(\alpha_i) \right)^2 + \mathcal{O}\left(\frac{1}{N}\right). \tag{15}$$

As in the proof of Proposition 2.1, we write $\phi = \phi_+ - \phi_-$ where $\phi_\pm$ are monotone increasing, and assume for the rest of this section that ψ is monotone non-decreasing function on $[0,1]$. We shall use the estimates

$$\psi(\alpha_{l_j-1}) \leq \psi(\theta_j(\alpha;\underline{m})) \leq \psi(\alpha_{l_j}), \tag{16}$$

that follow from (2) and the monotonicity of ψ.

Next, we apply (16) to every term in the sum $S := \sum_{|\underline{m}|=K} \sum_{i,j=1}^{K} \psi(\theta_i(\alpha;\underline{m}))\psi(\theta_j(\alpha;\underline{m}))$. We get the following bounds for the above sum:

$$\sum_{|\underline{m}|=K} \sum_{i,j=1}^{K} \psi(\alpha_{l_i-1}(\underline{m}))\psi(\alpha_{l_j-1}(\underline{m})) \leq S \leq \sum_{|\underline{m}|=K} \sum_{i,j=1}^{K} \psi(\alpha_{l_i}(\underline{m}))\psi(\alpha_{l_j}(\underline{m})) \tag{17}$$

We rewrite the lower bound in (17) as

$$L(\psi) := \sum_{i=0}^{N} L(i)(\psi(\alpha_i))^2 + \sum_{i<j} L(i,j)\psi(\alpha_i)\psi(\alpha_j), \tag{18}$$

where $L(i)$ denotes the number of times $\psi^2(\alpha_i)$ appears on the left-hand side of (17), and $L(i,j)$ denotes the number of times $\psi(\alpha_i)\psi(\alpha_j)$ appears on the left-hand side of (17).

Similarly, we rewrite the upper bound in (17) as

$$U(\psi) := \sum_{i=0}^{N} U(i)(\psi(\alpha_i))^2 + \sum_{i<j} U(i,j)\psi(\alpha_i)\psi(\alpha_j), \tag{19}$$

where $U(i)$ denotes the number of times $\psi^2(\alpha_i)$ appears on the right-hand side of (17), and $L(i,j)$ denotes the number of times $\psi(\alpha_i)\psi(\alpha_j)$ appears on the right-hand side of (17).

We now proceed to calculate $L(i), U(i), L(i,j), U(i,j)$.

Proposition 3.1. *We have that*

(a) $0 = U(0) = L(N)$.
(b) $U(i) = L(j) = \sigma(K-1, N+2) + \sigma(K-2, N+2)$ *for* $i \neq 0, j \neq N$.
(c) $0 = U(i,0) = L(j,N)$, *for all* $i > 0$ *and* $j < N$.
(d) $U(i,j) = L(i,j) = 2\sigma(K-2, N+2)$, *for all* $i < j$ *not described in (c).*

Proof of Proposition 3.1: The proof of (a) and (c) is clear since there are no θ-s to the right of α_N or to the left of α_0. We recall that m_j denotes the number of zeroes that occur in the interval (α_{j-1}, α_j), $1 \leq j \leq N$, for a fixed configuration $\underline{m}$. The contribution to $U(i), 1 \leq i \leq N$ from this configuration is equal to m_i^2. Thus,

$$U(i) = \sum_{|\underline{m}|=K} m_i^2.$$

Similarly, the contribution from $\underline{m}$ to $L(i), 0 \leq i \leq N-1$, is equal to m_{i+1}^2. This implies,

$$L(i) = \sum_{|\underline{m}|=K} m_{i+1}^2.$$

The same argument shows that the contribution from a configuration $\underline{m}$ to $U(i,j)$ and $L(i,j)$ with $i < j$ are respectively equal to $2m_i m_j$ and $2m_{i+1}m_{j+1}$. Accordingly,

$$U(i,j) = \sum_{|\underline{m}|=K} 2m_i m_j, \qquad L(i,j) = 2\sum_{|\underline{m}|=K} m_{i+1}m_{j+1}.$$

The statements (b) and (d) are now immediate consequence of Lemma 3.2 below. ☐

Lemma 3.2. *For any* $1 \leq i \leq N$, *we have that*

$$\sum_{|\underline{m}|=K} m_i^2 = \sigma(K-1, N+2) + \sigma(K-2, N+2).$$

Moreover, for any $1 \leq i < j \leq N$, *we also have*

$$\sum_{|\underline{m}|=K} m_i m_j = \sigma(K-2, N+2).$$

The proof of this lemma is purely combinatorial and make use of multi-variable generating functions. We prefer to postpone it for the moment and present it at the end of the paper.

The next step consists of showing that $U(\psi)$ and $L(\psi)$ provide an asymptotically good approximation for the sum

$$\sum_{|\underline{m}|=K} \sum_{i,j=1}^{K} \psi(\theta_i(\alpha; \underline{m}))\psi(\theta_j(\alpha; \underline{m})).$$

As the next result shows, this is indeed the case.

Lemma 3.3. *Let* $\Delta(\psi) := (U(\psi) - L(\psi))/(K^2\sigma(K,N))$. *Then,* $\Delta(\psi) = \mathcal{O}(N^{-1})$ *as* $N \to \infty$.

Proof of Lemma 3.3: From (18), (19) and Proposition 3.1, we conclude that

$$\begin{aligned}\Delta(\psi) &= \frac{\sigma(K-1, N+2) + \sigma(K-2, N+2)}{K^2\sigma(K,N)}(\psi^2(\alpha_N) - \psi^2(\alpha_0)) \\ &\quad + \frac{2\sigma(K-2, N+2)(\psi(\alpha_N) - \psi(\alpha_0))}{K^2\sigma(K,N)} \cdot \sum_{i=1}^{N-1} \psi(\alpha_j).\end{aligned}$$

Since by definition $\sigma(K,N) = \binom{N+K-1}{N-1}$, we see that

$$\frac{\sigma(K-2, N+2)}{K^2\sigma(K,N)} = \frac{K-1}{KN(N+1)}, \quad \frac{\sigma(K-1, N+2)}{K^2\sigma(K,N)} = \frac{N+K}{KN(N+1)}. \tag{20}$$

It follows from (20) and the fact that the function ψ is bounded, i.e. $|\psi(\alpha_j)| \leq M$ for some fixed $M > 0$ and all j,

$$\Delta(\psi) \leq 2M^2 \left[\frac{N+2K-1}{KN(N+1)} + \frac{(K-1)N}{KN(N+1)}\right] = \mathcal{O}\left(\frac{2}{N^2} + \frac{1}{KN} + \frac{1}{N}\right) = \mathcal{O}\left(\frac{1}{N}\right).$$

This completes the proof of Lemma 3.3. □

4. Variance

As a consequence of (15) and Lemma 3.3, we see that

$$\mathrm{Var}(\phi_\pm) = \frac{U(\phi_\pm)}{K^2\sigma(K,N)} - \left(\frac{1}{N+1}\sum_{i=0}^{N}\phi_\pm(\alpha_i)\right)^2 + \mathcal{O}\left(\frac{1}{N}\right).$$

The last expression can be rewritten as

$$\mathrm{Var}(\phi_\pm) = S_1 + S_2 + S_3 + \mathcal{O}(1/N),$$

where

$$S_1 := \sum_{i=1}^{N}\left(\frac{\sigma(K-2,N+2)+\sigma(K-1,N+2)}{K^2\sigma(K,N)} - \frac{1}{(N+1)^2}\right)\phi_\pm^2(\alpha_i),$$

$$S_2 := 2\sum_{1\leq i<j\leq N}\left(\frac{\sigma(K-2,N+2)}{K^2\sigma(K,N)} - \frac{1}{(N+1)^2}\right)\phi_\pm(\alpha_i)\phi_\pm(\alpha_j),$$

and

$$-S_3 := \frac{\phi_\pm(\alpha_0)}{(N+1)^2}\sum_{j=0}^{N}\phi_\pm(\alpha_j).$$

It is clear that $S_3 = \mathcal{O}(1/N)$. We proceed to estimate S_1 and S_2. Using (20), we see that

$$S_2 \leq \frac{2M^2N(N-1)}{2}\left(\frac{K-1}{KN(N+1)} - \frac{1}{(N+1)^2}\right) = \mathcal{O}\left(\frac{1}{K} + \frac{1}{N}\right).$$

Finally, using (20) once more, we see that

$$S_1 \leq NM^2\left(\frac{N+2K-1}{KN(N+1)} - \frac{1}{(N+1)^2}\right) = \mathcal{O}\left(\frac{1}{K} + \frac{1}{N}\right).$$

Summing up, we have

$$\mathrm{Var}(\phi_\pm) = \mathcal{O}\left(\frac{1}{K}\right) + \mathcal{O}\left(\frac{1}{N}\right). \tag{21}$$

We now come to the main theorem of this section.

Theorem 4.1. *Let $\phi \in BV([0,1])$. Then, $Var(\phi) \to 0$ as $K \to \infty$ and $N \to \infty$. More precisely, we have*

$$Var(\phi) = \mathcal{O}\left(\frac{1}{K}\right) + \mathcal{O}\left(\frac{1}{N}\right).$$

Proof of Theorem 4.1: Since $\phi = \phi_+ - \phi_-$, it follows from (21) that

$$\begin{aligned}\mathrm{Var}(\phi) &= \mathrm{Var}(\phi_+) + \mathrm{Var}(\phi_-) - 2\mathrm{Cov}(\phi_+,\phi_-)\\ &= \mathcal{O}\left(\frac{1}{K}\right) + \mathcal{O}\left(\frac{1}{N}\right) - 2\mathrm{Cov}(\phi_+,\phi_-),\end{aligned} \tag{22}$$

where $\mathrm{Cov}(\phi_+, \phi_-)$ denotes the covariance of ϕ_+ and ϕ_-. Moreover, we can apply Proposition 2.1 to conclude that

$$\begin{aligned}
\mathrm{Cov}(\phi_+, \phi_-) &= \frac{1}{\sigma(K,N)} \sum_{|\underline{m}|=K} \left[\int \phi_+ d\rho_{DS}(\underline{m}) - \int \phi_+ d\mu_N\right] \left[\int \phi_- d\rho_{DS}(\underline{m}) - \int \phi_- d\mu_N\right] \\
&\leq \frac{2\|\phi\|_0}{\sigma(K,N)} \left| \sum_{|\underline{m}|=K} \left[\int \phi_+ d\rho_{DS}(\underline{m}) - \int \phi_+ d\mu_N\right] \right| \\
&\leq \frac{4\|\phi\|_0 V(\phi)}{N}. \qquad (23)
\end{aligned}$$

The rest of the proof is an immediate consequence of (22) and (23). □

5. Asymptotic distribution of zeroes

There is a rather simple corollary of Theorem 4.1 that we would like to address in this section. Fix a configuration $\underline{m}$ with $|\underline{m}| = K$ and consider the average

$$\overline{\theta}(\alpha; \underline{m}) := \frac{1}{K} \sum_{j=1}^{K} \theta_j(\alpha; \underline{m}). \qquad (24)$$

Recall that a sequence of measurable subsets $U_N; N = 1, 2, \ldots$ of a probability space (X, μ) is said to be of full density if $\lim_{N\to\infty} \mu(U_N) = 1$.

Then, by the Chebyshev inequality it follows from Theorem 4.1 that for any $\alpha \in \Lambda^N$, there exists a full density subset of configurations $D_N(\alpha) \subset \Omega(K, N)$, such that for any $\underline{m} \in D_N(\alpha)$ and $\delta > 0$,

$$\overline{\theta}(\alpha; \underline{m}) = \frac{1}{N+1} \sum_{\nu=0}^{N} \alpha_\nu + \mathcal{O}_\delta\left(\max\{N^{-1/2+\delta}, K^{-1/2+\delta}\}\right). \qquad (25)$$

Let $[0,1]^\infty$ denote the set of all infinite sequences $\{\alpha_j\}_{j=0}^\infty$ with $\alpha_j \in [0,1]$. We embed $[0,1]^{N+1}$ in $[0,1]^\infty$ via the map $(\alpha_0, \ldots, \alpha_N) \mapsto (\alpha_0, \ldots, \alpha_N, 0, 0, \ldots.)$. Similarly, we define $\Lambda^\infty \subset [0,1]^\infty$ to consist of all strictly increasing sequences with $\alpha_0 < \alpha_1 < \cdots$ and $\Lambda^N \subset \Lambda^\infty$ in the same way.

Given any $\sigma \in S_N$, the symmetric group on N-elements, it is clear that the estimate in (25) extends pointwise to $\sigma(\Lambda^N)$ since

$$\frac{1}{N+1} \sum_{\nu=0}^{N} \alpha_{\sigma(\nu)} = \frac{1}{N+1} \sum_{\nu=0}^{N} \alpha_\nu.$$

On $\sigma(\Lambda^N)$ the functions $\theta_j(\alpha; \underline{m})$ are defined in the same way as on Λ^N except that $\alpha_{\sigma(j)}$ replaces α_j for $j = 0, \ldots, N$. So, the estimate (25) holds Lebesgue a.e. on $[0,1]^{N+1}$ and consequently, Lebesgue a.e. on $[0,1]^\infty$ since the $\overline{\theta}(\alpha; \underline{m})$ only depend on the first $N+1$ coordinates $(\alpha_0, \ldots, \alpha_N)$. We delete here the set D consisting of points $\alpha = (\alpha_0, \ldots, \alpha_N) \in [0,1]^{N+1}$ where at least two of the α_j's coincide.

Since the monomials $\alpha_\nu;\ \nu = 0, \ldots, N$ are independent, identically distributed random variables on $[0,1]^\infty$, by the strong law of large numbers,

$$\frac{1}{N+1} \sum_{\nu=0}^{N} \alpha_\nu = \frac{1}{2} + \mathcal{O}_\delta(N^{-1/2+\delta}) \qquad (26)$$

with probability one on $[0,1]^\infty$. Combining (25) and (26) gives:

Theorem 5.1. *Let $N, K \to \infty$. Then, for Lebesgue almost every $\alpha \in [0,1]^\infty$ there exist a full density of configurations, $D_N(\alpha)$, such that for any $\underline{m} \in D_N(\alpha)$, $|\underline{m}| = K$, and any $\delta > 0$,*

$$\overline{\theta}(\alpha; \underline{m}) = \frac{1}{2} + \mathcal{O}_\delta\left(\max\{N^{-1/2+\delta}, K^{-1/2+\delta}\}\right).$$

Remark: It would be interesting to see whether one can prove a central limit theorem for the $\theta_j(\alpha)$'s. For this, we would need an estimate for $\theta_j(\alpha)$ in terms of the monomials $\alpha_j; j = 0, ..., N$ that is better than the bound that one gets from the Heine-Stieltjes theorem. We hope to address this in future work.

6. Proof of Lemma 3.2

The proof will use multi-variable generating functions. We shall actually compute a more general expression (since the proof is just as easy). Given a positive number $p \leq N$, we let $\{i_1, i_2, \ldots, i_p\}$ be a sequence of p distinct elements of $\{1, \ldots, N\}$. We will compute the expression

$$G(i_1, \ldots, i_p; a_1, \ldots, a_p) := \sum_{|\underline{m}|=K} \prod_{j=1}^{p} (m_{i_j})^{a_j}. \tag{27}$$

where $a_j, j = 1, \ldots, p$ is a (positive integer) power to which m_{i_j} is raised. We remark that since $G(i_1, \ldots, i_p; a_1, \ldots, a_p)$ involves the sum over all configurations, we can assume by symmetry that $i_j = j$. So, without loss of generality we shall compute $G(1, \ldots, p; a_1, \ldots, a_p)$.

To evaluate (27), we define a bijection ρ between the set set of configurations of size K and monomials of degree K in N variables as follows:

$$\rho(\underline{m}) := x_1^{m_1} x_2^{m_2} \ldots x_N^{m_N}$$

Here the variable x_j corresponds to the jth interval, and $K = m_1 + m_2 + \ldots + m_N$. We need the following lemma.

Lemma 6.1. *The sum*

$$\sum_{|\underline{m}|=K} \rho(\underline{m})$$

consists of all monomials of total degree K in the infinite series in N variables:

$$F(x_1, \ldots, x_N) := \prod_{i=1}^{N} \frac{1}{(1-x_i)} = \prod_{i=1}^{N} (1 + x_i + x_i^2 + \ldots) = \sum_{m_1, \ldots, m_N = 0}^{\infty} x_1^{m_1} x_2^{m_2} \ldots x_N^{m_N}.$$

We remark that $F(x_1, \ldots, x_N)$ thus defines the generating series for the set of *all* configurations with N intervals.

Proof of Lemma 6.1: Choose $x_1^{m_1}$ from the first bracket $(1 + x_1 + x_1^2 + \ldots)$, $x_2^{m_2}$ from the second bracket etc. □

If we want to compute the *number* of such monomials of degree K, we substitute $x_j = x$ into $F(x_1, \ldots, x_N)$ and compute the coefficient of x^K. The power series becomes

$$F(x, \ldots, x) = 1/(1-x)^N = (1 + x + x^2 + \ldots)^N,$$

and it is well-known that the coefficient of x^K is equal to $\sigma(K, N) = \binom{N+K-1}{N-1}$.

To compute $G(1, \ldots, p; a_1, \ldots, a_p)$, we shall apply a differential operator

$$H = \prod_{j=1}^{p} \left(x_j \frac{\partial}{\partial x_j} \right)^{a_j}$$

to the series $F(x_1, \ldots, x_N)$. We remark that $H[\rho(\underline{m}))] = \prod_{j=1}^{p} (m_j)^{a_j} \rho(\underline{m})$. Summing over all $|\underline{m}| = K$, we see that we need the sum of coefficients of monomials of degree K in the series

$$H[F(x_1, \ldots, x_N)] = \prod_{j=1}^{p} \left(x_j \frac{\partial}{\partial x_j} \right)^{a_j} (1 + x_j + x_j^2 + \ldots) \prod_{j=p+1}^{N} (1 + x_j + x_j^2 + \ldots) \tag{28}$$

Let $F_j(x_j)$ denote the jth factor in (28) for $1 \leq j \leq p$. It is easy to see that

$$\left(z \frac{d}{dz} \right)^q \left(\frac{1}{1-z} \right) := \frac{P_q(z)}{(1-z)^{q+1}},$$

where $P_q(z)$ is a polynomial in z of degree q. The polynomials $P_q(z)$ are computed explicitly at the end of the Appendix. Thus, the expression in (28) is equal to

$$\prod_{j=1}^{p} \frac{P_{a_j}(x_j)}{(1-x_j)^{1+a_j}} \prod_{j=p+1}^{N} \frac{1}{(1-x_j)}. \tag{29}$$

To compute $G(1,\ldots,p;a_1,\ldots,a_p)$, we substitute $x_j = x$ into (29), collect the terms and compute the coefficient of x^K in the resulting power series expansion. It follows from (29) that we shall get the following expression:

$$\left(\prod_{j=1}^{p} P_{a_j}(x)\right) / (1-x)^{N+a_1+\ldots+a_p} \tag{30}$$

The numerator is therefore equal to

$$P(x) := \prod_{j=1}^{p} P_{a_j}(x) := b_0 + b_1 x + b_2 x^2 + \ldots \tag{31}$$

The following proposition is now immediate.

Proposition 6.2. *Let b_j-s be the coefficients of the polynomial P defined by* (31), *and let $A = a_1 + \ldots + a_p$. Then*

$$G(i_1,\ldots,i_p;a_1,\ldots,a_p) = \sum_{j=0}^{K} b_j \sigma(K-j, N+A).$$

Proof of Proposition 6.2: Expand the denominator $1/(1-x)^{N+A}$ in (30) into power series and multiply it by $P(x)$. □.

Finally, to prove Lemma 3.2 we apply Proposition 6.2 in the following two cases: $p = 1, a_1 = 2$; and $p = 2, a_1 = a_2 = 1$. The corresponding polynomials $P(x)$ are computed from the following two identities:

$$\left(y\frac{\partial}{\partial y}\right)\left(\frac{1}{1-y}\right) = \frac{y}{(1-y)^2} \quad \textit{and} \quad \left(y\frac{\partial}{\partial y}\right)^2\left(\frac{1}{1-y}\right) = \frac{y+y^2}{(1-y)^3}.$$

Thus, $P_1(x) = x$ and $P_2(x) = x + x^2$. □

We end this section by computing explicitly the polynomial $P_q(x)$ for $q > 2$. Recall that it is defined by the formula

$$\left(x\frac{d}{dx}\right)^q \left(\frac{1}{1-x}\right) := \frac{P_q(x)}{(1-x)^{q+1}}.$$

We let $P_q(x) = \sum_{r=1}^{q} b_{q,r} x^r$. In the Proposition 6.3 below, we compute the numbers $b_{q,r}$:

Proposition 6.3. *Let $q \geq 1$, and let $b_{q,r}$ be as above. Then*

$$b_{q,r} = \sum_{k=1}^{r} k^q (-1)^{r-k} \binom{q+1}{r-k}$$

Proof of Proposition 6.3: By definition,

$$P_q(x) = (1-x)^{q+1} \cdot \left(x\frac{d}{dx}\right)^q \left(\frac{1}{1-x}\right) = (1-x)^{q+1}\left(\sum_{k=1}^{\infty} k^q x^k\right).$$

The proposition now follows by expanding $(1-x)^{q+1}$ and multiplying the two series. □

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