EXTREMAL METRIC FOR THE FIRST EIGENVALUE ON A KLEIN BOTTLE

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ABSTRACT. The first eigenvalue of the Laplacian on a surface can be viewed as a functional on the space of Riemannian metrics of a given area. Critical points of this functional are called extremal metrics. The only known extremal metrics are a round sphere, a standard projective plane, a Clifford torus and an equilateral torus. We construct an extremal metric on a Klein bottle. It is a metric of revolution, admitting a minimal isometric embedding into a sphere $\mathbb{S}^4$ by the first eigenfunctions. The corresponding covering torus is a bipolar surface for the Lawson's $\tau_{3,1}$-torus. We conjecture that an extremal metric for the first eigenvalue on a Klein bottle is unique, and hence it provides a sharp upper bound for λ_1 on a Klein bottle of a given area. We present numerical evidence and prove the first results towards this conjecture.

1. Introduction and main results

1.1. **Extremal metrics for the first eigenvalue.** Let M be a closed surface of genus γ and let g be the Riemannian metric on M. Denote by Δ the Laplace-Beltrami operator on M, and by λ_1 the smallest positive eigenvalue (the fundamental tone) of the Laplacian. How large can λ_1 be on such a surface? It was proved in [H], [YY], [LY] that

$$\lambda_1 \mathrm{Area}(M) \leq \mathrm{const}(\gamma),$$

where the constant grows linearly with γ. However, for $\gamma \geq 1$ bounds obtained in this way have no reason to be *sharp*. In order to study sharp upper bounds we recall the following

Definition 1.1.1. *A metric g on a surface is called $\lambda_1 -$ maximal if for any metric $\tilde{g}$ of the same area $\lambda_1(\tilde{g}) \leq \lambda_1(g)$.*

In other words, a λ_1-maximal metric is a global maximum of the functional $\lambda_1 : g \to \mathbb{R}$. Consider critical points of this functional.

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Definition 1.1.2. (see [EI2]). *An extremal metric for the first eigenvalue is a critical point g_0 of the functional $\lambda_1 : g \to \mathbb{R}$, i.e. for any analytic deformation g_t of the Riemannian metric g_0 in the class of metrics of fixed area $\lambda_1(g_t) \le \lambda_1(g_0) + o(t)$ as $t \to 0$.*

Note that the functional λ_1 does not have local minima [EI2].

1.2. **Extremal metrics and minimal immersions.** Only four examples of extremal metrics for the first eigenvalue are known: (i) standard metric on $\mathbb{S}^2$, (ii) standard metric on $\mathbb{RP}^2$, (iii) flat equilateral torus and (iv) Clifford torus. Moreover, it was proved that there are no other extremal metrics on these three surfaces ([MR], [EI1], [EI2]). Metrics (i)-(iii) are λ_1-maximal ([H], [LY], [N1]), (iv) is just a local extremum.

The following remarkable property holds for extremal metrics for the first eigenvalue. Any surface with an extremal metric admits a minimal isometric immersion by the first eigenfunctions into a round sphere of a certain dimension. In all examples (i)–(iv) the dimension is equal to $\text{mult}(\lambda_1) - 1$, where $\text{mult}(\lambda_1)$ is the multiplicity of the first eigenvalue. There is a vast literature on relations between extremal metrics and minimal immersions, see [B], [LY], [MR], [N1], [EI1], [EI2].

1.3. **Extremal metric on a Klein bottle.** It is proved in [N1], that on a Klein bottle there exists a λ_1-maximal (and hence an extremal) metric, which is a metric of revolution with $\text{mult}(\lambda_1) = 5$. However, no example of an extremal metric on a Klein bottle has been known. Our main result is an explicit construction of such a metric.

Theorem 1.3.1. *A metric of revolution*

$$g_0 = \frac{9 + (1 + 8\cos^2 v)^2}{1 + 8\cos^2 v} \left(du^2 + \frac{dv^2}{1 + 8\cos^2 v} \right), \tag{1.3.2}$$

$0 \le u, v < \pi$, is an extremal metric for the first eigenvalue on a Klein bottle $\mathbb{K}$. The surface $(\mathbb{K}, g_0)$ admits a minimal isometric embedding into a sphere $\mathbb{S}^4$ by the first eigenfunctions. The first eigenvalue of the Laplacian for this metric has multiplicity 5 and satisfies the equality

$$\lambda_1 \text{Area}(\mathbb{K}, g_0) = 12\pi E(2\sqrt{2}/3), \tag{1.3.3}$$

where $E(\cdot)$ is a complete elliptic integral of the second kind.

Remark. An extremal metric on a Klein bottle must be a metric of revolution since any conformal diffeomorphism of an extremal metric is an isometry ([MR], [EI1]), and any metric on a Klein bottle is conformally equivalent to a flat metric which is invariant under a natural

$\mathbb{S}^1$-action (see section 2.1). The condition $\text{mult}(\lambda_1) = 5$ follows from the following argument. It is shown in [EI1] that $\text{mult}(\lambda_1) > 3$ for an extremal metric on any surface but a sphere. On the other hand, on a Klein bottle $\text{mult}(\lambda_1) \le 5$ ([N2]), and the case $\text{mult}(\lambda_1) = 4$ has been excluded in [N1].

We prove Theorem 1.3.1 in section 3.

Remark. It is shown in [EI3] that the metric (i) on $\mathbb{S}^2$, the metric (ii) on $\mathbb{RP}^2$ as well as the metrics (iii) and (iv) on $\mathbb{T}^2$ are critical points for the functional $\operatorname{Tr} e^{-t\Delta}$ (the trace of the heat kernel) at *any* time $t > 0$. It is also shown there that on the Klein bottle there exists *no* metric that is critical for $\operatorname{Tr} e^{-t\Delta}$ for all t. Since for $t >> 0$ the first eigenvalue dominates the heat trace, extremal metrics for the first eigenvalue are natural candidates to be also critical points of the heat trace for large t. Comparing this with Theorem 1.3.1 one may conjecture that there exists $t = t_{crit}$, such that g_0 is a critical point of the heat trace for $t > t_{crit}$ but not for $t < t_{crit}$. It would be interesting to understand if such a transition indeed takes place.

1.4. **Interpretation in the language of minimal surfaces.** The Klein bottle $(\mathbb{K}, g_0)$ constructed in Theorem 1.3.1 has the following surprising interpretation in terms of $\mathbb{S}^1$-equivariant minimal surfaces in $\mathbb{S}^4$. Equivariant minimal immersions of tori into spheres is a classical subject in minimal surfaces (see [HL], [U]). In particular, $\mathbb{S}^1$-equivariant minimal immersions of tori into $\mathbb{S}^4$ have been studied in [FP].

Theorem 1.4.1. *The double cover of $(\mathbb{K}, g_0)$ is an $\mathbb{S}^1$-equivariant minimal torus in $\mathbb{S}^4$, which is a bipolar surface of Lawson's $\tau_{3,1}$-torus.*

In section 4 we prove Theorem 1.4.1 and recall the definitions of Lawson's tori and bipolar surfaces (see also [L]). Interestingly enough, the interpretation of g_0 as a metric on a bipolar surface allows us to simplify the explicit formula for g_0 (cf. (1.3.2) and (3.3.2)).

1.5. **Towards a sharp upper bound for the first eigenvalue.** Combining Theorem 1.3.1 with the existence of a λ_1-maximal metric on a Klein bottle proved in [N1], we make the following

Conjecture 1.5.1. *The metric g_0 is a unique extremal metric on a Klein bottle, and in particular it is the λ_1-maximal metric. This implies the following sharp upper bound for the first eigenvalue on a Klein bottle:*

$$\lambda_1(g)\text{Area}(\mathbb{K}, g) \le 12\pi E(2\sqrt{2}/3) \approx 13.365\pi, \tag{1.5.2}$$

with an equality attained only for $g = g_0$.

Recall that the estimate of [LY] gives just

$$\lambda_1(g)\mathrm{Area}(\mathbb{K}, g) \leq 48\pi.$$

Remark. It is claimed in Theorem 3 in [N1] that

$$\lambda_1(g)\mathrm{Area}(\mathbb{K}, g) \leq 8\pi^2/\sqrt{3},$$

the right-hand side being the supremum for λ_1Area on a torus. However, the proof of this claim is incorrect: it relies on the assumption that the first eigenvalue on a Klein bottle is also the first eigenvalue on the covering torus. Though it is an eigenvalue on a torus, it might be not the *first* eigenvalue. In particular, for $(\mathbb{K}, g_0)$ the first eigenvalue is the *third* eigenvalue on the corresponding torus (see Proposition 3.4.1). Note, however, that indeed $12\pi E(2\sqrt{2}/3) < 8\pi^2/\sqrt{3}$.

In order to prove Conjecture 1.5.1 one has to study the nonlinear systems of ODEs (3.1.2) or (3.1.1) that are crucial in the proof of Theorem 1.3.1. We need to show that there are no initial conditions $0 < p < 1$ except for $p = \sqrt{3/8}$ (which corresponds to the metric g_0) admitting periodic solutions with the required number of zeros (see Condition A in section 2.3). We discuss numerical evidence and prove the first results towards Conjecture 1.5.1 in section 5. However, there are serious difficulties in finding a rigorous proof of this conjecture, see section 5.7.

2. A system of ODEs for the extremal metric

2.1. **Preliminaries.** We realize the Klein bottle $\mathbb{K}$ as a fundamental domain in $\mathbb{R}^2$ for the group of motions generated by $(x, y) \to (x + \pi, -y), (x, y) \to (x, y + a)$, where $a > 0$ is a conformal parameter. $\mathbb{K}$ has a double cover, the torus $\mathbb{T}^2$, which is the fundamental domain $\mathbb{R}^2$ for the group of motions generated by $(x, y) \to (x + 2\pi, y), (x, y) \to (x, y + a)$. The functions on $\mathbb{K}$ can thus be thought of as functions on $\mathbb{T}^2$ satisfying the symmetry condition

$$f(x, y) = f(x + \pi, -y). \tag{2.1.1}$$

If we expand the functions on $\mathbb{T}^2$ into Fourier series in x, we can easily see that the functions in $L^2(\mathbb{T}^2)$ satisfying (2.1.1) can be expanded in the series of functions of the form

$$\begin{gathered}\{\phi(y)\sin(2kx), \phi(y)\cos(2kx) : \\ \phi(y) = \phi(-y), \phi(y + a) = \phi(y)\},\end{gathered} \tag{2.1.2}$$

and of the form

$$\begin{gathered}\{\psi(y)\sin(x(2k + 1)), \psi(y)\cos(x(2k + 1)) : \\ \psi(y) = -\psi(-y), \psi(y + a) = \psi(y)\},\end{gathered} \tag{2.1.3}$$

where $k \in \mathbb{Z}$.

As mentioned in section 1.3, it follows from [N1] that an extremal metric for the first eigenvalue on a Klein bottle is necessarily a metric of revolution and the multiplicity of λ_1 for this metric is equal to 5.

Hence without loss of generality we may assume that our metric is invariant under the $\mathbb{S}^1$ action $(x, y) \to (x + t, y)$, $0 \le t \le \pi$, and is given by $\hat{g}_0 = f(y)(dx^2 + dy^2)$, where $f(y) = f(y + a) = f(-y) > 0$ is the conformal factor. The area of the Klein bottle is equal to

$$\mathrm{Area}(\mathbb{K}) = \pi \int_0^a f(y)dy.$$

The Laplacian on $\mathbb{K}$ is given by

$$\Delta = -\frac{1}{f(y)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \tag{2.1.4}$$

Let λ_1 denote the first nonzero eigenvalue of Δ. We want to determine the conformal class (i.e. the value of a) that maximizes the product λ_1Area. Since our metric is rotationally invariant, the operator $\partial/\partial x$ commutes with Δ, so we can find a joint basis of eigenfunctions of the form (2.1.2) and (2.1.3).

2.2. **First eigenfunctions.** By Courant's nodal domain theorem, any eigenfunction in the first eigenspace should have exactly two nodal domains. Our eigenfunctions have the form

$$\varphi_k(y)\cos(kx), \quad \varphi_k(y)\sin(kx), \tag{2.2.1}$$

where $\varphi_k(-y) = (-1)^k \varphi_k(y), \varphi_k(y + a) = \varphi_k(y)$. For k odd, $\varphi_k(0) = 0$ so φ_k vanishes at least once. Also, φ_0 must vanish at least once since the corresponding eigenfunction can't have constant sign. Let φ_k vanish m_k times in the period $[0, a)$.

We can choose the fundamental domain for the Klein bottle to be the set $X = [y, y + \pi] \times [-a/2, a/2]$, with y/π irrational (to avoid vanishing on the vertical sides), and with the appropriate boundary identifications. The nodal set of an eigenfunction (2.2.1) consists of a grid with k distinct vertical lines and m_k distinct horizontal lines. It is easy to show that such an eigenfunction has at least k nodal domains: indeed, k vertical lines divide the set X into $k + 1$ vertical strips, and of those only the two boundary strips are glued into one by side identifications. Therefore, by Courant's nodal domain theorem we must have $k \le 2$.

Substituting into (2.1.4) and taking into account that $\mathrm{mult}(\lambda_1) = 5$, we conclude that the eigenspace corresponding to λ has a basis of eigenfunctions of the form

$$\begin{cases} \varphi_0(y), & \varphi_0(-y)=\varphi_0(y), \varphi_0''=-\lambda f\varphi_0; \\ \cos(x)\varphi_1(y), & \varphi_1(-y)=-\varphi_1(y), \varphi_1''=(1-\lambda f)\varphi_1; \\ \sin(x)\varphi_1(y), & \\ \cos(2x)\varphi_2(y), & \varphi_2(-y)=\varphi_2(y), \varphi_2''=(4-\lambda f)\varphi_2; \\ \sin(2x)\varphi_2(y). & \end{cases} \tag{2.2.2}$$

Here all functions of y are periodic with period a and λf is an unknown positive function. Since an extremal metric necessarily admits a minimal isometric immersion into a sphere (in our case of dimension 4), we get two more conditions on the functions φ_0, φ_1 and φ_2 (cf. [N1]):

$$\varphi_0^2+\varphi_1^2+\varphi_2^2=1. \tag{2.2.3}$$

$$(\varphi_0')^2+(\varphi_1')^2+(\varphi_2')^2=\varphi_1^2+4\varphi_2^2=\lambda f/2. \tag{2.2.4}$$

We can now substitute for λf in the second and the third equations in (2.2.2), getting the following system of second order equations for φ_1 and φ_2 (where λf has been eliminated):

$$\begin{cases} \varphi_1''=(1-2(\varphi_1^2+4\varphi_2^2))\varphi_1; \\ \varphi_2''=(4-2(\varphi_1^2+4\varphi_2^2))\varphi_2. \end{cases} \tag{2.2.5}$$

2.3. **Zeros of the first eigenfunctions.** We use the Courant's nodal domain theorem once again (see previous section) to get a condition on the number of zeros of φ_0, φ_1 and φ_2. Since each of these functions is a non-trivial solution of a second order differential equation (2.2.2), it is impossible that φ_k and φ_k' vanish simultaneously for $k=0,1,2$. Periodicity then implies that the number of zeros m_k for any φ_k is an even number. Recalling that each eigenfunction has exactly two nodal domains, and taking into account boundary identifications as in the previous section, we get $m_0=m_1=2$ and $m_2=0$.

Condition A (zeros). φ_0 and φ_1 should have exactly *two* zeros in the period, while φ_2 *should not vanish.*

2.4. **First integrals.** It is straightforward to check that the following expressions are the first integrals for the system (2.2.2), (2.2.3) (cf.

[U]):

$$\begin{cases} E_0 := \varphi_0^2 + (\varphi_0\varphi_1' - \varphi_1\varphi_0')^2 + (\varphi_0\varphi_2' - \varphi_2\varphi_0')^2/4, \\ E_1 := \varphi_1^2 + (\varphi_1\varphi_2' - \varphi_2\varphi_1')^2/3 - (\varphi_1\varphi_0' - \varphi_0\varphi_1')^2, \\ E_2 := \varphi_2^2 - (\varphi_2\varphi_0' - \varphi_0\varphi_2')^2/4 - (\varphi_2\varphi_1' - \varphi_1\varphi_2')^2/3. \end{cases} \tag{2.4.1}$$

In the verification of this fact, one uses (2.2.3) and its consequence $\varphi_0\varphi_0' + \varphi_1\varphi_1' + \varphi_2\varphi_2' = 0$. In fact, all these integrals are equivalent: one can show that $E_0 + E_1 + E_2 = 1 = E_0 + 3E_1/4$, $E_2 = -E_1/4$. Hence E_j's define just one independent first integral. We make use of the different expressions (2.4.1) in section 5.

Let us evaluate E_1 at $y = 0$.

$$\varphi_1(0) = 0 = \varphi_0'(0) = \varphi_2'(0) \tag{2.4.2}$$

It follows that $\varphi_0(0)^2 + \varphi_2(0)^2 = 1$ and that

$$\varphi_1'(0)^2 = 4\varphi_2(0)^2. \tag{2.4.3}$$

Substituting into the expression for E_1 we find that

$$E_1 = \frac{4}{3}\varphi_2(0)^2(4\varphi_2(0)^2 - 3). \tag{2.4.4}$$

Remark. Alternatively, one can start with the system (2.2.5) (or similar systems involving just φ_0, φ_1, or just φ_0, φ_2) and deduce the following equivalent expressions for the first integrals (2.4.1):

$$\begin{cases} (\varphi_1')^2 + 4(\varphi_2')^2 + (\varphi_1^2 + 4\varphi_2^2)^2 - \varphi_1^2 - 16\varphi_2^2 := \kappa_0, \\ (\varphi_0')^2 - 3(\varphi_2')^2 + 2\varphi_0^2 + 6\varphi_2^2 - (\varphi_0^2 - 3\varphi_2^2)^2 := \kappa_1, \\ 4(\varphi_0')^2 + 3(\varphi_1')^2 + 32\varphi_0^2 + 21\varphi_1^2 - (4\varphi_0^2 + 3\varphi_1^2)^2 := \kappa_2. \end{cases} \tag{2.4.5}$$

One can show that $\kappa_2 - 3\kappa_0 - 4\kappa_1 = 12$ and that $\kappa_0 + \kappa_1 = 1$, so $\kappa_0 + \kappa_2 = 16$. One can also show that $E_1 = \frac{1}{3}\kappa_0$. For certain applications it is more convenient to use the expressions (2.4.5) rather than (2.4.1); however, we shall not use the expressions (2.4.5) in this paper.

3. Proof of Theorem 1.3.1

3.1. **A system for φ_0 and φ_1.** The initial conditions in system (2.2.5) can be parametrized as follows:

$$\begin{cases} \varphi_1'' = (1 - 2\varphi_1^2 - 8\varphi_2^2)\varphi_1 \\ \varphi_2'' = (4 - 2\varphi_1^2 - 8\varphi_2^2)\varphi_2 \\ \varphi_1(0) = 0,\ \varphi_1'(0) = 2p \\ \varphi_2(0) = p,\ \varphi_2'(0) = 0, \end{cases} \tag{3.1.1}$$

where $0 \leq p \leq 1$ is a parameter of the system. Moreover, φ_0 and φ_1 both have two zeros on the period, while φ_2 has constant sign.

The corresponding system for the functions φ_0, φ_1 reads:

$$\begin{cases} \varphi_0'' = (8\varphi_0^2 + 6\varphi_1^2 - 8)\varphi_0 \\ \varphi_1'' = (8\varphi_0^2 + 6\varphi_1^2 - 7)\varphi_1 \\ \varphi_1(0) = 0, \ \varphi_1'(0) = 2p \\ \varphi_0(0) = \sqrt{1-p^2}, \ \varphi_0'(0) = 0, \end{cases} \tag{3.1.2}$$

Note that in (3.1.1) and (3.1.2) initial conditions are determined by (2.4.3) modulo signs. However, changing the signs of the initial conditions may only result in changing the signs of the solutions (in other words, we will get the same eigenfunctions possibly multiplied by -1). Therefore, we may consider only non-negative initial conditions in (3.1.1) and (3.1.2).

3.2. **Solution for $p = \sqrt{3/8}$.** Our objective is to find values of p such that the system has periodic solutions satisfying Condition A, namely that both φ_0 and φ_1 have exactly two zeros on the period. We find a candidate from a numerical experiment: $p = \sqrt{3/8}$. Note that this value of p is exactly the minimum of the first integral E_1 and hence as follows from [FP] it corresponds to a periodic solution. We discuss this in more detail in section 4.

Set $p = \sqrt{3/8}$. Let us look for φ_0 and φ_1 in the following form:

$$\varphi_0(y) = \sqrt{\frac{5}{8}} \cos\theta(y), \ \varphi_1(y) = \frac{1}{\sqrt{2}} \sin\theta(y), \ \theta(0) = 0, \ \theta'(0) = \sqrt{3} \tag{3.2.1}$$

Such a change of variables is also motivated by numerical experiments, suggesting that

$$2\varphi_1^2 + 8/5\varphi_0^2 = 1. \tag{3.2.2}$$

Initial conditions for θ are prescribed by the initial conditions for φ_1, φ_0.

Of course, in principle, such an ansatz could make our system overdetermined: note that instead of two variables φ_0, φ_1 we now have one variable θ. However, as shown below, for this particular choice of constants this does not happen.

Indeed, we have:

$$8\varphi_0^2 + 6\varphi_1^2 = 5\cos^2\theta + 3\sin^2\theta = 5 - 2\sin^2\theta,$$

and hence (3.1.2) can be rewritten as

$$\begin{cases} (\theta')^2 - \theta''\dfrac{\cos\theta}{\sin\theta} = 2 + 2\sin^2\theta \\ (\theta')^2 + \theta''\dfrac{\sin\theta}{\cos\theta} = 3 + 2\sin^2\theta \\ \theta(0) = 0,\ \theta'(0) = \sqrt{3}. \end{cases} \tag{3.2.3}$$

Subtracting the second equation from the first we get

$$\theta'' = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta \tag{3.2.4}$$

Multiplying by θ' and integrating gives

$$(\theta')^2 = 3 + \sin^2\theta. \tag{3.2.5}$$

Exactly the same equation one gets if (3.2.4) is substituted into (3.2.3) and hence the whole system yields to (3.2.5) with an initial condition $\theta(0) = 0$, implying

$$y = \frac{1}{2}\int_0^\theta \frac{d\theta}{\sqrt{1 - \frac{1}{4}\cos^2\theta}}$$

From this equation we can deduce periodicity conditions. The functions φ_0, φ_1 are periodic in θ with the period 2π. Hence, the period a is equal to

$$\frac{1}{2}\int_0^{2\pi} \frac{d\theta}{\sqrt{1 - \frac{1}{4}\cos^2\theta}} = 2\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \frac{1}{4}\cos^2\theta}} =$$
$$2\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \frac{1}{4}\sin^2\theta}} = 2K(1/2),$$

where $K(\cdot)$ is a complete elliptic integral of the first kind. Hence, $a = 2K(1/2)$.

Let us now compute $\lambda\mathrm{Area}(\mathbb{K})$ for this metric (even without computing the metric explicitly – this will be done in the next section). Taking into account (2.2.4) we have:

$$\lambda\mathrm{Area}(\mathbb{K}) = \lambda\pi\int_0^a f(y)dy = 2\pi\int_0^{2K(1/2)} 4 - 3\varphi_1^2(y) - 4\varphi_0^2(y)dy =$$
$$2\pi\int_0^{2K(1/2)} 4 - 5/2\cos^2\theta - 3/2\sin^2\theta dy =$$
$$2\pi\int_0^{2\pi}(5/2 - \cos^2\theta)y'(\theta)d\theta = 2\pi\int_0^{2\pi}\frac{5/2 - \cos^2\theta}{\sqrt{4 - \cos^2\theta}}d\theta =$$
$$2\pi\left(\int_0^{2\pi}\sqrt{4 - \cos^2\theta}d\theta - 3/2\int_0^{2\pi}\frac{d\theta}{\sqrt{4 - \cos^2\theta}}\right) =$$

$$2\pi\left(8\int_0^{\pi/2}\sqrt{1-1/4\cos^2\theta}\,d\theta-3\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-1/4\cos^2\theta}}\right)=$$
$$2\pi(8E(1/2)-3K(1/2))=12\pi E(2\sqrt{2}/3).$$

The last equality follows from an identity relating the complete elliptic integrals of the first and the second kind (see [Erd], p. 319). This proves the assertion (1.3.3) in Theorem 1.3.1 up to the fact that λ is the first eigenvalue (see section 3.4).

3.3. **The eigenfunctions.** In this section we find explicitly the eigenfunctions corresponding to the value $p=\sqrt{3/8}$, and the corresponding metric $\hat{g}_0$. We do this using the relation (3.2.2) between φ_0 and φ_1, which implies a similar equation for φ_2 and φ_1: $4\varphi_2^2-\varphi_1^2=3/2$. This allows to transform our system into three separate equations on $\varphi_0,\varphi_1,\varphi_2$:

$$\varphi_0''=16/5\varphi_0^3-5\varphi_0,\ \ \varphi_1''=-2\varphi_1-4\varphi_1^3,\ \ \varphi_2''=7\varphi_2-16\varphi_2^3.$$

We then reduce them to first order equation:

$$(\varphi_0')^2=8/5\varphi_0^4-5\varphi_0^2+20/8,$$
$$(\varphi_1')^2=-2\varphi_1^2-2\varphi_1^4+3/2$$
$$(\varphi_2')^2=7\varphi_2^2-8\varphi_2^4-3/2.$$

Each of these equation can be solved in terms of elliptic functions (see also [WW], section 20.6). Finally we get:

$$\varphi_0(y)=\sqrt{5/8}\left(1-\frac{3}{2\wp(y;73/12,-595/216)-\frac{1}{6}}\right),\qquad (3.3.1)$$
$$\varphi_1(y)=\frac{1}{\sqrt{2}}\left(-1+\frac{2}{\wp(y+\frac{K(1/2)}{2};-8/3,28/27)+\frac{2}{3}}\right),$$
$$\varphi_2(y)=\sqrt{3/8}+\frac{1}{4}\left(\frac{\sqrt{3/2}}{\wp(y;193/12,2681/216)+\frac{11}{12}}\right),$$

where $\wp(y;\gamma_1,\gamma_2)$ is a Weierstrass $\wp$-function with invariants γ_1,γ_2.

It can be checked directly (analytically or using *Mathematica*) that $\varphi_0,\varphi_1,\varphi_2$ satisfy the Condition A of section 2.2.

Using (3.2.2) we find that the normalized metric $\hat{g}_0=\lambda f(y)(dx^2+dy^2)$ (though it differs by a normalization factor λ from the metric defined in the beginning of section 2.2, we denote it also $\hat{g}_0$) is given by

$$\lambda f(y)=2(\varphi_1^2(y)+4\varphi_2^2(y))=5-\frac{16}{5}\varphi_0^2(y),\qquad (3.3.2)$$

The metric $\hat{g}_0$ is conformally equivalent to a flat metric on $\mathbb{K}$ corresponding to the lattice $(x, y) \to (x+\pi, -y)$, $(x, y) \to (x, y+2K(1/2))$. It still remains to show that this metric coincides (up to a dilatation) with the metric g_0 defined by (1.3.2). We postpone this until section 4.

3.4. **Why λ is the first eigenvalue?** To complete the proof of Theorem 1.3.1 we need to show that the eigenvalue $\lambda = 1$ of the normalized metric $\hat{g}_0$ (3.3.2) is the *first* eigenvalue of the Laplacian (the eigenvalue equals 1 due to the choice of normalization (3.3.2)). Note that though Condition A is a necessary condition for the first eigenfunctions, apriori it is not sufficient.

Proposition 3.4.1. *The eigenvalue $\lambda = 1$ corresponding to the eigenfunctions $\{\varphi_0(y),\ \varphi_1(y)\cos x,\ \varphi_1(y)\sin x,\ \varphi_2(y)\cos 2x,\ \varphi_2(y)\sin 2x)\}$, is the first eigenvalue of the Laplacian on a Klein bottle $(\mathbb{K}, \hat{g}_0)$.*

Proof. We prove this proposition with the help of oscillation theorems of Haupt and Sturm (see [CL], [BeL]). As was mentioned in section 2.2, due to Courant's nodal domain theorem the first eigenvalue on a Klein bottle of revolution can be obtained only from one of the three periodic Sturm-Liouville equations (2.2.2). We need to show that none of the these equations has an eigenfunction corresponding to an eigenvalue $\lambda < 1$ and satisfying the Condition A as well as the parity conditions (we are interested only in even eigenfunctions of the first and the third equation, and only in odd eigenfunctions of the second equation).

Equations (2.2.2) are subject to a theorem of Haupt, stating that each eigenvalue problem has a sequence of eigenfunctions

$$\psi_0, \psi_1, \psi_2, \ldots, \psi_{2n-1}, \psi_{2n}, \cdots$$

such that ψ_0 does not have zeros and ψ_{2n-1}, ψ_{2n} have exactly $2n$ zeros. Let us study the equations (2.2.2) for φ_0, φ_1 and φ_2 separately.

The easiest case is φ_2 – it has no zeros, and hence corresponds to the smallest eigenvalue of the Sturm-Liouville problem.

To handle φ_1, we note that since it is odd, it should have zeros and therefore 0-th eigenvalue for this problem is automatically out of question (indeed, this eigenvalue $\tilde{\lambda} \approx 0.2517$ is the first eigenvalue on the corresponding torus which covers our Klein bottle). Since φ_1 has exactly two zeros, it is either ψ_1 or ψ_2. Assume there exists another odd solution $\tilde{\varphi}_1$ of the eigenvalue problem with exactly 2 zeros, and the corresponding $\tilde{\lambda} < \lambda$. Then by Sturm theorem, between each zero of $\tilde{\varphi}_1$ there should be zeros of φ_1, but since both of them vanish at 0, this will mean that φ_1 should have at least 3 zeros, while it has only 2, and we get a contradiction. Indeed, numerically one can see that there exists another (but even) eigenfunction with 2 zeros. For the normalized problem it corresponds to $\tilde{\lambda} \approx 1.31$.

A similar argument works for φ_0. Note that in this case the 0-th eigenfunction is just a constant, and hence is also not relevant. Similarly, φ_0 is either the first or the second eigenvalue. Assume there is another even eigenfunction with exactly 2 zeros corresponding to some $\tilde{\lambda} < \lambda$. Note that since it is even and periodic with period 1 it should be symmetric with respect to the mid-point of the period $y = 1/2$ (indeed, $\varphi_0(x) = \frac{\varphi_0(x)+\varphi_0(1-x)}{2}$), in particular its zeros have this symmetry — as do the zeros of φ_0. On the other hand, due to Sturm theorem, between each zero of $\tilde{\varphi}_0$ there should be a zero of φ_0, or in other words zeros of φ_0 and $\tilde{\varphi}_0$ should interlace, but this contradicts the fact that they have the symmetry property (symmetry implies that two zeros of one eigenfunction are between two zeros of the other). Numerically one can observe that in reality $\tilde{\varphi}_0$ is an odd function corresponding to $\tilde{\lambda} \approx 0.7768$. This completes the proof of the proposition. □

3.5. **End of the proof of Theorem 1.3.1.** Let us summarize the results of sections 3.1 – 3.4. We have constructed a metric of revolution $\hat{g}_0$ on a Klein bottle, admitting an isometric embedding into $\mathbb{S}^4$ by the first eigenfunctions. The first eigenvalue for this metric satisfies the equality

$$\lambda_1 \mathrm{Area}(\mathbb{K}, \hat{g}_0) = 12\pi E(2\sqrt{2}/3),$$

where $E(\cdot)$ is a complete elliptic integral of the second kind. Hence to complete the proof we just need to show that the metric $\hat{g}_0$ is indeed an extremal metric for the first eigenvalue. We use Proposition 1.1 of [EI2], implying that if the isometric immersion is given by a complete set of the first eigenfunctions (i.e. if the eigenfunctions form a basis of the corresponding eigenspace), then the metric is extremal for λ_1. This is clearly the case for us, since we have used all five eigenfunctions to construct the immersion, and 5 is the maximal possible multiplicity for the first eigenvalue on a Klein bottle. It follows from the explicit formulas for the eigenfunctions in section 3.3 that it is in fact an embedding. To complete the proof of Theorem 1.3.1 it remains to show that the metric $\hat{g}_0$ (3.3.2) coincides up to a dilatation with the metric g_0 (1.3.2). This is done in section 4.2 while proving Theorem 1.4.1. □

4. The extremal metric and $\mathbb{S}^1$-equivariant immersions

4.1. **Bipolar surfaces and Lawson tori.** In this section we follow [L] and [Ken]. Let $\mu : M \to \mathbb{S}^3 \subset \mathbb{R}^4$ be a minimal isometric immersion of a surface M into $\mathbb{S}^3$. A Gauss map $\mu^* : M \to \mathbb{S}^3$ is defined pointwise as the image of the unit normal in $\mathbb{S}^3$ translated to the origin in $\mathbb{R}^4$. The image $\mu^*(M)$ is called a *polar variety.*

Let $\tilde{\mu} = \mu \wedge \mu^*$ — the exterior product of μ and μ^*. It is a vector in $\wedge^2\mathbb{R}^4 = \mathbb{R}^6$, and one can check ([L]) that it defines a minimal immersion of M into $\mathbb{S}^5$: $\tilde{\mu} : M \to \mathbb{S}^5 \subset \mathbb{R}^6$. The minimal surface $\tilde{M} = \tilde{\mu}(M)$ in $\mathbb{S}^5$ is called a *bipolar surface* to M. The metric on $\tilde{M}$ is given by $ds^2 = (2-\kappa)d\sigma^2$, where $d\sigma^2$ is the metric on M and κ is the Gaussian curvature on M ([L]).

Let $M = \tau_{m,k}$ $(m \geq k \geq 1)$ be a Lawson's torus, that is a minimal torus defined by a doubly periodic immersion $\mu : \mathbb{R}^2 \to \mathbb{S}^3$,

$$\mu(u,v) = (\cos mu \cos v, \sin mu \cos v, \cos ku \sin v, \sin ku \sin v). \tag{4.1.1}$$

One may check that the bipolar surface for $\tau_{m,k}$ is a minimal torus in $\mathbb{S}^4$ ([L]). The metric on a bipolar surface for $\tau_{m,k}$ is given by (see [Ken]):

$$ds^2 = \frac{(k^2+(m^2-k^2)\cos^2 v)^2 + m^2k^2}{k^2+(m^2-k^2)\cos^2 v}\left(du^2 + \frac{dv^2}{k^2+(m^2-k^2)\cos^2 v}\right).$$

4.2. **Proof of Theorem 1.4.1.** Note that for $m=3$, $k=1$ the metric above is exactly the metric g_0 (1.3.2). Therefore in order to show that the metric on the covering torus for $(\mathbb{K}, \hat{g}_0)$ defines a bipolar surface to the $\tau_{3,1}$-torus one needs to check that $\hat{g}_0$ coincides (up to a dilatation) with (1.3.2). This result can be deduced from the arguments of ([FP]). Indeed, due to (2.2.2) the metric g_0 determines a $(2,1)$-equivariant minimal immersion in $\mathbb{S}^4$. Moreover, the first integral E_1 (H_2 in the notations of [FP]) achieves its minimum for $p = \sqrt{3/8}$, see section 3.2. Hence, as mentioned in ([FP], p. 274), this metric defines a bipolar surface for the Lawson's torus in $\mathbb{S}^3$ corresponding to $(2+1, 2-1)$ circular action, that is exactly the torus $\tau_{3,1}$. The equation (3.2.2) defining the extremal metric is equivalent to equation (11) in [FP] by setting $z := \varphi_2$, $w := \varphi_1$. Therefore, metric $\hat{g}_0$ indeed coincides with $\hat{g}_0$ up to a rescaling. This completes the proof of Theorem 1.4.1 and also finishes the last step of the proof of Theorem 1.3.1. $\square$

Remark. In fact, it can be verified directly that $2\hat{g}_0 = g_0$. It is a lengthy calculation in elliptic functions. Indeed, set $x = u$ and

$$z(v) = \int_0^v \frac{dv}{\sqrt{1+8\cos^2 v}} = \frac{1}{3}\int_0^v \frac{dv}{\sqrt{1-\frac{8}{9}\sin^2 v}}$$

Then in the (x,z) variables metric (1.3.2) becomes conformal. Note also that the relation above implies $\cos v = \mathrm{cn}(3z, 2\sqrt{2}/3)$, the corresponding Jacobi elliptic function. Set $y = 2z + \frac{K(1/2)}{2}$ (note that

$2K(1/2) = \frac{4}{3}K(2\sqrt{2}/3)$ is the period of $\mathrm{cn}(3z, 2\sqrt{2}/3))$. Taking into account (3.3.1) and (3.3.2) we arrive to the following identity that it suffices to check:

$$\frac{(1+8\mathrm{cn}^2(3z,\frac{2\sqrt{2}}{3})^2+9)}{1+8\mathrm{cn}^2(3z,\frac{2\sqrt{2}}{3})^2} = 10 - \left(\frac{24\wp(y;\frac{73}{12},-\frac{595}{216})-38}{12\wp(y;\frac{73}{12},-\frac{595}{216})-1}\right)^2.$$

The clue to this identity is the following relation between the Jacobi and the Weierstrass elliptic functions (see [Erd], 13.16.5):

$$\mathrm{cn}^2\left(3z,\frac{2\sqrt{2}}{3}\right) = \frac{12\wp(2z;\frac{73}{12},-\frac{595}{216})-10}{12\wp(2z;\frac{73}{12},-\frac{595}{216})+17}.$$

The remainder of the argument is a rather straightforward application of formulas from section 13.13 of [Erd].

5. Towards a sharp upper bound for the first eigenvalue

5.1. **Two intervals of the parameter.** The aim of section 5 is to present numerical evidence for Conjecture 1.5.1 and to prove the first result in that direction (Theorem 5.2.1). Our ultimate goal is to show that there are no extremal metrics corresponding to the values of the parameter $0 < p < 1$ except for $p = \sqrt{3/8}$. It turns out that the dynamics of the solutions differs for $0 < p < \sqrt{3}/2$ and $\sqrt{3}/2 \le p < 1$. We study these two intervals separately. In the latter case we prove the absence of extremal metrics (sections 5.2 and 5.3). For $0 < p < \sqrt{3}/2$ we present a purely numerical argument (section 5.6) and explain the nature of difficulties in proving Conjecture 1.5.1 (section 5.7).

Initial conditions of (3.1.2) and (3.1.1) are parametrized by values of $0 < p < 1$. We shall be using first integrals (2.4.1):

$$E_1 = (4/3)p^2(4p^2-3),\ E_2 = (-1/3)p^2(4p^2-3),\ E_0 = 1 - p^2(4p^2-3).$$

The periodic solution corresponds to $p^2 = 3/8$ which is the minimum of E_1. We want to show that there are no other periodic solutions satisfying Condition A.

The value $p^2 = 3/4, E_1 = E_2 = 0, E_0 = 1$ corresponds to a separatrix of some sort, the behavior of the solutions changes (section 5.3).

5.2. **Ruling out the interval $1 > p > \sqrt{3}/2$.** In this section we show that there are no periodic solutions of (3.1.1) satisfying Condition A, if $\sqrt{3}/2 < p < 1$, and hence $E_1 > 0, E_2 < 0, 0 < E_0 < 1$.

Theorem 5.2.1. *Assume that $\sqrt{3}/2 < p < 1$. Then the system* (2.2.2) *doesn't have periodic solutions satisfying Condition A.*

We shall prove Theorem 5.2.1 by showing that the solutions of the system (3.1.1) "rotate" around the origin in the (φ_1, φ_2)-plane. In other words, if we introduce polar coordinates in (3.1.1), the angle will be monotone increasing. This fact implies that the function φ_2 vanishes at *some* point on any periodic orbit, contradicting the condition A in section 2.3. We use the condition (2.2.3) to introduce spherical coordinates in the system (2.2.2) and use the integrals E_1 and E_2 to rule out the initial conditions $\sqrt{3}/2 < p < 1$.

We introduce the following spherical coordinates in (2.2.2):

$$\begin{cases} \varphi_0 = \cos\psi, \\ \varphi_1 = \sin\psi\sin\theta, \\ \varphi_2 = \sin\psi\cos\theta. \end{cases} \tag{5.2.2}$$

Taking into account parity conditions in (2.2.2), we find that ψ is an even function, while θ is an odd function.

Differentiating once, we find that

$$\varphi_0' = -\sin\psi\cdot\psi', \varphi_1' = \cos\psi\cdot\psi'\sin\theta + \sin\psi\cos\theta\cdot\theta',$$
$$\varphi_2' = \cos\psi\cdot\psi'\cos\theta - \sin\psi\sin\theta\cdot\theta'. \tag{5.2.3}$$

It is easy to see that ψ and θ satisfy the following initial conditions:

$$\begin{gathered} \theta(0) = 0, \theta'(0) = 2, \\ \psi'(0) = 0, \psi(0) = \arcsin p. \end{gathered} \tag{5.2.4}$$

We next express the integrals E_0, E_1, E_2 in terms of θ, ψ and their derivatives. An elementary calculation using (5.2.2) and (5.2.3) gives the following identities:

$$\begin{cases} \varphi_0\varphi_1' - \varphi_1\varphi_0' = \psi'\sin\theta + \dfrac{\sin(2\psi)}{2}\cos\theta\cdot\theta'; \\ \varphi_0\varphi_2' - \varphi_2\varphi_0' = \psi'\cos\theta - \dfrac{\sin(2\psi)}{2}\sin\theta\cdot\theta'; \\ \varphi_1\varphi_2' - \varphi_2\varphi_1' = -\sin^2\psi\cdot\theta'. \end{cases} \tag{5.2.5}$$

We now substitute (5.2.2) and (5.2.5) into (2.4.1). We obtain the following identities for E_1 and E_2:

$$E_1 = \sin^2\theta(\sin^2\psi - (\psi')^2) - \frac{\psi'\theta'\sin(2\psi)\sin(2\theta)}{2}$$
$$+\frac{(\theta')^2\sin^4\psi}{3} - \left(\frac{\theta'\cos\theta\sin(2\psi)}{2}\right)^2. \tag{5.2.6}$$

$$E_2 = \cos^2\theta(\sin^2\psi - (\psi')^2/4) + \frac{\psi'\theta'\sin(2\psi)\sin(2\theta)}{8} - \frac{(\theta')^2\sin^4\psi}{3} - \left(\frac{\theta'\sin\theta\sin(2\psi)}{4}\right)^2. \quad (5.2.7)$$

If we now assume that $\theta' = 0$, the two expressions simplify to

$$\begin{cases} E_1 = \sin^2\theta(\sin^2\psi - (\psi')^2), \\ E_2 = \cos^2\theta(\sin^2\psi - (\psi')^2/4). \end{cases} \quad (5.2.8)$$

We now prove the main theorem of this section.

Proof of Theorem 5.2.1. Assume that there exists a periodic orbit such that $\varphi_2 \neq 0$ (this is one of the requirements of condition A of section 2.3). Then this orbit has a point satisfying $\theta' = 0$, since θ is the angle in polar coordinates in (φ_1, φ_2)-plane, and the orbit is a compact set lying in the upper half-plane. Now, if $\sqrt{3}/2 < p < 1$, we have $E_1 > 0, E_2 < 0$. We evaluate those integrals at a point where $\theta' = 0$. Substituting into (5.2.8), we see that

$$\begin{cases} \sin^2\psi - (\psi')^2 > 0, \\ \sin^2\psi - (\psi')^2/4 < 0. \end{cases} \quad (5.2.9)$$

It is clear that (5.2.9) leads to a contradiction. This finishes the proof of Theorem 5.2.1. □

5.3. **The separatrix $p = \sqrt{3}/2$.** If $p = \sqrt{3}/2$, the first integral E_1 vanishes. The solutions are given explicitly by the following formulas:

$$\varphi_0(y) = (3\cos(\theta(y)) - 1)/4, \varphi_1(y) = \sqrt{3}\sin(\theta(y))/2, \quad (5.3.1)$$

where

$$\theta(y) = \pi - 4\,\mathrm{arccot}(e^y).$$

One can observe that these solutions are not periodic. As $y \to \infty$, we get the upper half of an ellipse in the plane φ_0, φ_1.

In fact, since for $p = \sqrt{3}/2$ the integral $E_1 = 0$, if there existed a corresponding minimal isometric immersion of a Klein bottle into $\mathbb{S}^4$ it would be superminimal (see [FP]). However, as indicated in the appendix of [MR], the only superminimal surface immersed into $\mathbb{S}^4$ by the first eigenfunctions is the standard sphere (this result is attributed to N. Ejiri).

5.4. **Dynamics in the (φ_0, φ_1)-plane for $0 \le p < \sqrt{3}/2$.** We are left to check that the solution for $p = \sqrt{3/8}$ is the only one in the interval $0 < p < \sqrt{3}/2$. For the first integrals, this interval corresponds to $E_1 < 0, E_2 > 0, 1 < E_0 < 25/16$.

Proposition 5.4.1. *For $0 < p < \sqrt{3}/2$, functions φ_2 and $\varphi_1\varphi_0' - \varphi_0\varphi_1'$ don't vanish. Moreover, the function φ_2 is bounded away from ± 1.*

Proof. We recall from (2.4.1) that

$$E_2 = \varphi_2^2 - (\varphi_2\varphi_0' - \varphi_0\varphi_2')^2/4 - (\varphi_2\varphi_1' - \varphi_1\varphi_2')^2/3. \tag{5.4.2}$$

Since $E_2 > 0$ for $0 < p < \sqrt{3}/2$, the positive term φ_2^2 in the preceding formula cannot vanish, proving the 1st part of the proposition.

We recall from (2.4.1) that

$$E_1 = \varphi_1^2 + (\varphi_1\varphi_2' - \varphi_2\varphi_1')^2/3 - (\varphi_1\varphi_0' - \varphi_0\varphi_1')^2 \tag{5.4.3}$$

Since $E_1 < 0$ for $0 < p < \sqrt{3}/2$, the negative term $-(\varphi_1\varphi_0' - \varphi_0\varphi_1')^2$ in the preceding formula cannot vanish, proving the 2nd part of the proposition.

Finally, if $\varphi_2 = \pm 1$ then $\varphi_0 = \varphi_1 = 0$, contradicting the fact that $\varphi_1\varphi_0' - \varphi_0\varphi_1' \neq 0$. □

Corollary 5.4.4. *For $0 < p < \sqrt{3}/2$, the solutions of the system* (3.1.2) *"rotate" around the origin in the (φ_0, φ_1)-plane. In other words, if we introduce polar coordinates in* (3.1.2)*, the angle will be monotone increasing.*

Proof. The angle in polar coordinates in the (φ_0, φ_1)-plane is given by $\theta = \arctan(\varphi_1/\varphi_0)$, and

$$\theta' = (\varphi_0\varphi_1' - \varphi_1\varphi_0')/(\varphi_0^2 + \varphi_1^2).$$

Proposition 5.4.1 now implies that $\theta' \neq 0$. □

Using Corollary 5.4.4 we conclude that the condition A in section 2.3 (i.e. that φ_0, φ_1 both have two zeros in the period) means that a periodic orbit should make *one* turn around the origin. The periodic solution corresponding to $p = \sqrt{3/8}$ does exactly that (the orbit in that case is the ellipse $10\varphi_1^2 + 8\varphi_0^2 = 5$).

5.5. **Intersection angle.** Consider the first (for $y > 0$) intersection of the trajectory on the (φ_0, φ_1)-plane with the φ_0-axis. Let $y(p)$ be the intersection point, $\alpha(p)$ be the angle of the intersection. In this section we establish

Proposition 5.5.1. *If p corresponds to an extremal metric, $\alpha(p) = \pi/2$, or, equivalently, $\varphi_0'(y(p)) = 0$.*

Proof. We know from (2.2.2)

$$\varphi_0(-y) = \varphi_0(y), \varphi_1(-y) = -\varphi_1(y),$$

i.e. that the solution for $y > 0$ and the solution for $y < 0$ are symmetric with respect to the φ_0-axis.

Assume now that for some $p > 0$ the system (3.1.2) has a periodic solution with period $a(p)$. We have

$$\varphi_0(y(p)) = \varphi_0(-y(p)), \quad \varphi_1(y(p)) = -\varphi_1(-y(p)) = 0.$$

The periodicity condition together with the condition A imply that $y(p) = a(p)/2$ and that

$$\varphi_0(y(p) + t) = \varphi_0(-y(p) + t), \quad \varphi_1(y(p) + t) = \varphi_1(-y(p) + t).$$

But since φ_0 is an even function, we also have

$$\varphi_0(y(p) + t) = \varphi_0(-y(p) + t) = \varphi_0(y(p) - t).$$

The last equality implies $\varphi_0'(y(p)) = 0$. □

5.6. **Ruling out the interval** $(0, \sqrt{3}/2)$ **numerically.** To rule out the interval $(0, \sqrt{3/2})$ we use the following

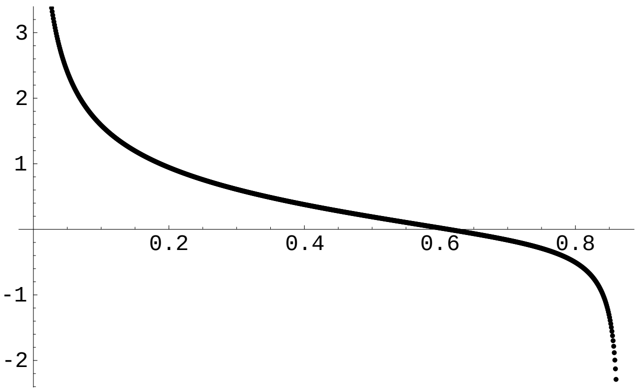
Conjecture 5.6.1. *The angle $\alpha(p)$ is a monotone function for $p \in (0, \sqrt{3/2})$.*

We check this conjecture numerically for $p \in (\delta, \sqrt{3/2} - \delta)$ for small $\delta > 0$ using *Mathematica*.

We have included the values of $\cot(\alpha(p))$ for 999 values of p, $p = \frac{\sqrt{3}j}{2000}$, $1 \le j \le 999$. Those values were computed using a *Mathematica* program. The differential equation solved by that program is obtained by first writing the system of two second order differential equations for the variables (ψ, θ) corresponding to the change of variables

$$\varphi_2 = \cos\psi, \varphi_1 = \sin\psi\sin\theta, \varphi_0 = \sin\psi\cos\theta,$$

then rewriting that system using θ as an independent variable (we can do that for $0 < p < \sqrt{3}/2$ by Corollary 5.4.4).
The results are shown below:



Graph of $\cot(\alpha(p))$ *for* $0 < p < \sqrt{3}/2$. *The only zero occurs for* $p = \sqrt{\frac{3}{8}} \approx 0.612$.

Clearly, since $\cot(\alpha(p))$ is monotone, the same is true for $\alpha(p)$.

We next prove the following

Proposition 5.6.2. *Conjecture 5.6.1 implies Conjecture 1.5.1.*

Proof. Since the angle $\alpha(p)$ is monotone, it takes the value $\alpha(p) = \pi/2$ only once. This happens exactly for $p = \sqrt{3}/8$, so by Proposition 5.5.1, the only extremal metric on the interval $(0, \sqrt{3}/2)$ is the metric g_0. $\square$

5.7. **Difficulties in proving Conjecture 1.5.1.** One would like to have a computer-assisted proof of Conjecture 5.6.1, or of a weaker statement (that still implies Conjecture 1.5.1) that the conclusion of Proposition 5.5.1 only holds for $p = \sqrt{3/8}$. The main obstacle to finding a rigorous (even a computer-assisted) proof seems to be that the systems (3.1.2) and (3.1.1) are lacking stability, and therefore estimates for the dependence of the solutions on the initial conditions are very rough. Accordingly, one has to make numerical measurements with the step not 10^{-3} as in the previous section, but a dozen of orders of magnitude smaller. Otherwise it seems impossible to control the behavior of the solutions between the two measurements. However, such precision seems to be beyond the reach of existing numerical software.

A similar difficulty occurs near the ends of the interval $(0, \sqrt{3}/2)$. In [JNP] it is shown that

$$\lim_{p \to 0} y(p) = \lim_{p \to \sqrt{3}/2} y(p) = \infty,$$

in fact that $y(p) \to \infty$ as $c|\ln p|$ for an explicit constant c. Consequently, if the system (3.1.2) for $p > 0$ has a periodic solution with period $a(p)$, then

$$\lim_{p \to 0} a(p) = \lim_{p \to \sqrt{3}/2} a(p) = \infty.$$

It is also shown that there exists an explicit $M > 0$ such that for any $a > M$ and for any metric g_a on $\mathbb{K}$ with the conformal class a, we have (in the notation of section 1),

$$\lambda_1(g_a)\mathrm{Area}(\mathbb{K}^2, g_a) < 12\pi E(2\sqrt{2}/3).$$

Altogether this implies the existence of a computable constant $\delta > 0$ such that an extremal metric for λ_1Area cannot be attained for $p < \delta$ and $p > \sqrt{3}/2 - \delta$. However, the value of δ we could obtain is way too small for being useful in a computer-assisted proof.

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