ON THE SPECTRUM OF GEOMETRIC OPERATORS ON KÄHLER MANIFOLDS

DMITRY JAKOBSON, ALEXANDER STROHMAIER, AND STEVE ZELDITCH

Abstract. On a compact Kähler manifold there is a canonical action of a Lie superalgebra on the space of differential forms. It is generated by the differentials, the Lefschetz operator and the adjoints of these operators. We determine the asymptotic distribution of irreducible representations of this Lie-superalgebra on the eigenspaces of the Laplace-Beltrami operator. Because of the high degree of symmetry the Laplace-Beltrami operator on forms can not be quantum ergodic. We show that after taking these symmetries into account quantum ergodicity holds for the Laplace-Beltrami operator and for the $\mathrm{Spin}^{\mathbb{C}}$-Dirac operators if the unitary frame flow is ergodic. The assumptions for our theorem are known to be satisfied for instance for negatively curved Kähler manifolds of odd complex dimension.

1. Introduction

Properties of the spectrum of the Laplace-Beltrami operator on a manifold are closely related to the properties of the underlying classical dynamical system. For example ergodicity of the geodesic flow on the unit tangent bundle T_1X of a compact Riemannian manifold X implies quantum ergodicity. Namely, for any complete orthonormal sequence of eigenfunctions $\phi_j \in L^2(X)$ to the Laplace operator Δ with eigenvalues $\lambda_j \nearrow \infty$ one has (see [Shn74, Shn93, Zel87, CV85, HMR])

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\le N} |\langle \phi_j, A\phi_j\rangle - \int_{T_1^*X} \sigma_A(\xi) d\mu_L(\xi)|^2 = 0, \tag{1}$$

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for any zero order pseudodifferential operator A, where integration is with respect to the normalized Liouville measure μ_L on the unit-cotangent bundle T_1^*X, and σ_A is the principal symbol of A. Quantum ergodicity is equivalent to the existence of a subsequence ϕ_{j_k} of counting density one such that

$$\lim_{k\to\infty}\langle\phi_{j_k}, A\phi_{j_k}\rangle = \int_{T_1^*X}\sigma_A(\xi)d\mu_L(\xi). \tag{2}$$

In particular, A might be a smooth function on X and the above implies that the sequence

$$|\phi_{j_k}(x)|^2 dV_g \tag{3}$$

converges to the normalized Riemannian measure dV_g in the weak topology of measures. For bundle-valued geometric operators like the Dirac operator acting on sections of a spinor bundle or the Laplace-Beltrami operator the corresponding Quantum ergodicity for eigensections is known in a precise way to relate to the ergodicity of the frame flow on the corresponding manifold [JS]; see also [BoG04, BoG04.2, BO06].

This paper deals with a situation in which the frame flow is not ergodic, namely the case of Kähler manifolds. In this case the conclusions in [JS] do not hold since there is a huge symmetry algebra acting on the space of differential forms. This algebra is the universal enveloping algebra of a certain Lie superalgebra that is generated by the Lefschetz operator, the complex differentials and their adjoints. On the level of harmonic forms this symmetry is responsible for the rich structure of the cohomology of Kähler manifolds and can be seen as the main ingredient for the Lefschetz theorems. Here we are interested in eigensections with non-zero eigenvalues, that is in the spectrum of the Laplace-Beltrami operator acting on the orthogonal complement of the space of harmonic forms. The action of the Lie superalgebra on the orthogonal complement of the space of harmonic forms is much more complicated than on the space of harmonic forms where it basically becomes the action of $sl_2(\mathbb{C})$. In this paper we classify all finite dimensional unitary representations of this algebra and determine the asymptotic distribution of these representations in the eigenspaces. Since the typical irreducible representation of the algebra decomposes into four irreducible representation for $sl_2(\mathbb{C})$ this shows that eigenspaces to the Laplace-Beltrami operator have multiplicities. An important observation in our treatment is that the universal enveloping algebra of this Lie superalgebra is generated by two commuting subalgebras, one of which is isomorphic to the universal enveloping algebra of $sl_2(\mathbb{C})$. This

$sl_2(\mathbb{C})$-action is generated by an operator L_t and its adjoint L_t^* which is going to be defined in section 3.1. This operator can be interpreted as the Lefschetz operator in the directions of the frame bundle which are orthogonal to the frame flow. However, L_t is not an endomorphism of vector bundles, but it acts as a pseudodifferential operator of order zero.

Guided by this result we tackle the question of quantum ergodicity for the Laplace-Beltrami operator on (p,q)-forms. Unlike in the case of ergodic frame flow it turns out that there might be different quantum limits of eigensections on the space of co-closed (p,q)-forms because of the presence of the Lefschetz operator. Our main results establishes quantum ergodicity for the Dirac operator and the Laplace Beltrami operator if one takes the Lefschetz symmetry into account and under the assumption that the $U(m)$-frame flow is ergodic. For example our analysis shows that in case of an ergodic $U(m)$-frame flow for any complete sequence of co-closed primitive (p,q)-forms there is a density one subsequence which converges to a state which is an extension of the Liouville measure and can be explicitly given. For the $\mathrm{Spin}^{\mathbb{C}}$-Dirac operators we show that quantum ergodicity does not hold for Kähler manifolds of complex dimension greater than one. Thus, negatively curved Spin-Kähler manifolds provide examples of manifolds with ergodic geodesic flow where quantum ergodicity does not hold for the Dirac operator. Our analysis shows that there are certain invariant subspaces for the Dirac operator in this case and we prove quantum ergodicity for the Dirac operator restricted to these subspaces provided that the $U(m)$-frame flow is ergodic.

2. Kähler manifolds

Let (X,ω,J) be a Kähler manifold of real dimension $n=2m$. Let g be the metric, $h=g+\mathrm{i}\,\omega$ be the hermitian metric, and ω the symplectic form. As usual let J be the complex structure. A k-frame $(e_1,\dots,e_k)$ for the cotangent space at some point $x\in X$ is called unitary if it is unitary with respect to the hermitian inner product induced by h. Hence, a k-frame $(e_1,\dots,e_k)$ is unitary iff $(e_1,Je_1,e_2,Je_2,\dots,e_k,Je_k)$ is orthonormal with respect to g. A unitary m-frame at a point $x\in X$ is an ordered orthonormal basis for T_x^*X viewed as a complex vector space.

Clearly, the group $U(m)$ acts freely and transitively on the set of unitary m-frames. The bundle U_mX of unitary m-frames is therefore a $U(m)$-principal fiber bundle. Let T_1^*X be the unit cotangent bundle with bundle projection π. Then projection onto the first vector makes

$U_m X$ a principal $U(m-1)$-bundle over $T_1^* X$.

$$\begin{tikzcd} U_m X \arrow[r, "U(m-1)"] \arrow[rr, bend right, "U(m)"'] & T_1^* X \arrow[r, "S^{2m-1}"] & X \end{tikzcd}$$

Transporting covectors parallel with respect to the Levi-Civita connection extends the Hamiltonian flow on $T_1^* X$ to a flow on $U_m X$ which we call the $U(m)$-frame flow (in the literature it is also referred to as the restricted frame flow). This is indeed a flow on $U_m X$ since J is covariantly constant and therefore unitary frames are parallel transported into unitary ones. This flow is the appropriate replacement for the $SO(2m)$-frame flow for Kähler manifolds as it can be shown to be ergodic in some cases, whereas the $SO(2m)$-frame flow never is ergodic for Kähler manifolds

Suppose that X is a negatively-curved Kähler manifold [Bor]. We summarize results that can be found in [Br82, BrG80, BrP74]. We refer the reader to [BuP03, JS, Br82] and references therein for discussion of frame flows on general negatively-curved manifolds. Note that the frame flow is *not* ergodic on negatively-curved Kähler manifolds, since the almost complex structure J is preserved. This is the only known example in negative curvature when the geodesic flow is ergodic, but the frame flow is not. In fact, given an orthonormal k-frame $(e_1, \dots, e_k)$, the functions $(e_i, Je_j), 1 \leq i, j \leq k$ are first integrals of the frame flow.

However, the following proposition was proved in [BrG80]:

Proposition 2.1. *Let X be a compact negatively-curved Kähler manifold of* complex *dimension m. Then the $U(m)$-frame flow is ergodic on $U_m X$ when $m = 2$, or when m is odd.*

3. The Hodge Laplacian and the Lefschetz decomposition

Let $\wedge^* X$ be the complex vector bundle $\wedge^* T^*_{\mathbb{C}} X$, where $T^*_{\mathbb{C}} X$ is the complexification of the co-tangent bundle. Then the Lefschetz operator $L : C^\infty(X; \wedge^* X) \to C^\infty(X; \wedge^* X)$ is defined by exterior multiplication with the Kähler form ω, i.e. $L = \omega \wedge$. Its adjoint L^* is then given by interior multiplication with ω. Is is well known that

$$[L^*, L] := H = \sum_k (m - r) P_r, \tag{4}$$

where P_r is the orthoprojection onto $C^\infty(\wedge^r X)$, and L, L^*, H define a representation of $sl_2(\mathbb{C})$ which commutes with the Laplace operator $\Delta = 2\Delta_{\bar\partial} = 2\Delta_\partial$. The decomposition into irreducible representations on the level of harmonic forms is called the Lefschetz decomposition. We will refer to this decomposition as the Lefschetz decomposition in general. Note that since the Lefschetz operator commutes with Δ each eigenspace decomposes into a direct sum of irreducible subspaces for the $sl_2(\mathbb{C})$ action.

The operators $L, L^*, H, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*, \Delta$ satisfy the following relations (see e.g. [B])

$$
\begin{gathered}
[L, \bar{\partial}^*] = -i\partial,\ [L^*, \partial] = i\bar{\partial}^*,\ [L^*, \bar{\partial}] = -i\partial^*,\ [L, \partial^*] = i\bar{\partial}, \\
[L^*, L] = H,\ [H, L] = -2L,\ [H, L^*] = 2L^*, \\
\{\partial, \partial\} = \{\bar{\partial}, \bar{\partial}\} = \{\partial^*, \partial^*\} = \{\bar{\partial}^*, \bar{\partial}^*\} = 0, \\
[L, \bar{\partial}] = [L, \partial] = [L^*, \bar{\partial}^*] = [L^*, \partial^*] = 0, \\
\{\partial, \bar{\partial}\} = \{\partial, \bar{\partial}^*\} = \{\bar{\partial}, \partial^*\} = 0, \\
\{\partial, \partial^*\} = \{\bar{\partial}, \bar{\partial}^*\} = \frac{1}{2}\Delta.
\end{gathered} \tag{5}
$$

Thus, the operators form a Lie superalgebra with central element Δ (see also [FrGrRe99]). Let $\Delta^{-1}|_{\ker\Delta^\perp}$ the inverse of the Laplace operator on the orthocomplement of the kernel of Δ. We view this as an operator defined in $L^2(X, \wedge^* X)$ by defining it to be zero on $\ker\Delta$ and write Δ^{-1} slightly abusing notation.

3.1. **The transversal Lefschetz decomposition.** The operator $Q := 2\Delta^{-1}\bar{\partial}\partial$ is a partial isometry with initial space $\mathrm{Rg}(\bar{\partial}^*) \cap \mathrm{Rg}(\partial^*)$ and final space $\mathrm{Rg}(\bar{\partial}) \cap \mathrm{Rg}(\partial)$. Hence, Q^*Q is the orthoprojection onto $\mathrm{Rg}(\bar{\partial}^*) \cap \mathrm{Rg}(\partial^*)$ and QQ^* is the orthoprojection onto $\mathrm{Rg}(\bar{\partial}) \cap \mathrm{Rg}(\partial)$. From the above relations one gets

$$[L, Q] = 0, \tag{6}$$

$$[L, Q^*] = 2i\Delta^{-1}(\bar{\partial}\bar{\partial}^* - \partial^*\partial), \tag{7}$$

$$[Q^*, Q] = -2\Delta^{-1}(\bar{\partial}\bar{\partial}^* - \partial^*\partial), \tag{8}$$

from which one finds that

$$[L - iQ, Q^*] = [L - iQ, Q] = 0. \tag{9}$$

We define the transversal Lefschetz operator L_t by

$$L_t := L - iQ. \tag{10}$$

Then clearly $L_t^* = L^* + iQ^*$ and one gets that

$$[L_t^*, L_t] = H_t = H + [Q^*, Q], \tag{11}$$

$$[H_t, L_t] = -2L_t, \quad [H_t, L_t^*] = -2L_t^*, \tag{12}$$

$$[\partial, L_t] = [\partial^*, L_t] = [\bar{\partial}, L_t] = [\bar{\partial}^*, L_t] = 0, \tag{13}$$

and hence, also the transversal Lefschetz operators defines an action of $\mathrm{sl}_2(\mathbb{C})$ on $L^2(X, \wedge^* X)$. Unlike the Lefschetz operator the transversal Lefschetz operator commutes with the holomorphic and antiholomorphic codifferentials.

Denote by $\mathfrak{g}$ the Lie-superalgebra generated by $a, \bar{a}, L, H, a^*, \bar{a}^*, L^*$ and relations

$$
\begin{aligned}
&[L, \bar{a}^*] = -ia,\ [L^*, a] = i\bar{a}^*,\ [L^*, \bar{a}] = -ia^*,\ [L, a^*] = i\bar{a},\\
&[L^*, L] = H,\ [H, L] = -2L,\ [H, L^*] = 2L^*,\\
&\{a, a\} = \{\bar{a}, \bar{a}\} = \{a^*, a^*\} = \{\bar{a}^*, \bar{a}^*\} = 0,\\
&[L, \bar{a}] = [L, a] = [L^*, \bar{a}^*] = [L^*, a^*] = 0,\\
&\{\bar{a}, a\} = \{\bar{a}, a^*\} = \{a, \bar{a}^*\} = 0,\\
&\{a, a^*\} = \{\bar{a}, \bar{a}^*\} = 1.
\end{aligned}
\tag{14}
$$

The subspace of odd elements is spanned by $a, a^*, \bar{a}, \bar{a}^*$, the subspace of even elements is spanned by L, L^* and H. In the following we will denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of this Lie-superalgebra viewed as a unital $*$-algebra, i.e. the unital $*$-algebra generated by the symbols $\{L, L^*, H, a, a^*, \bar{a}, \bar{a}^*\}$ and the above relations.

The relations (3.1) are obtained from the relations (5) by sending a to $\sqrt{2}\Delta^{-1/2}\partial$ and $\bar{a}$ to $\sqrt{2}\Delta^{-1/2}\bar{\partial}$. Therefore, we obtain a $*$-representation of the Lie-superalgebra $\mathfrak{g}$ on the orthogonal complement of the kernel of Δ.

3.2. **The representation theory of $\mathcal{U}(\mathfrak{g})$.** The calculations in the previous section used the relations in $\mathcal{U}(\mathfrak{g})$ only. Hence, they remain valid if we regard $Q = \bar{a}a$ and $L_t := L - iQ$ as elements in the abstract $*$-algebra $\mathcal{U}(\mathfrak{g})$. Hence, L_t, L_t^* generate a subalgebra in $\mathcal{U}(\mathfrak{g})$ which is canonically isomorphic to the universal enveloping algebra of $\mathrm{sl}_2(\mathbb{C})$ and which we therefore denote by $\mathcal{U}(\mathrm{sl}_2(\mathbb{C}))$. Note that $\mathcal{U}(\mathrm{sl}_2(\mathbb{C}))$ commutes with $a, \bar{a}, a^*$ and $\bar{a}^*$. Since $\mathcal{U}(\mathfrak{g})$ is generated by two commuting subalgebras the representation theory for $\mathcal{U}(\mathfrak{g})$ is very simple. The $*$-subalgebra $\mathcal{A}$ generated by a and $\bar{a}$ has the following canonical representation on $\wedge^*\mathbb{C}^2 \cong \mathbb{C}^4$. For an orthonormal basis $\{e, \bar{e}\}$ of $\mathbb{C}^2$ define the action of a by exterior multiplication by $\mathrm{i}\, e$, and the action of $\bar{a}$ by exterior multiplication by $\mathrm{i}\, \bar{e}$. It is easy to see that all non-trivial finite dimensional irreducible $*$-representations of $\mathcal{A}$ are unitarily equivalent to this representation.

Note that the equivalence classes of finite dimensional irreducible $*$-representations of $\mathcal{U}(\mathrm{sl}_2(\mathbb{C}))$ are labeled by the non-negative integers. Denote the Verma-module for the Spin-$\frac{n}{2}$ representation by V_n and the distinguished highest weight vector in V_n by h. Remember that V_n is spanned by vectors of the form $L_t^k h$ with $k = 0, \dots, n$ and we have $L_t^* h = 0$ and $H_t h = nh$.

Now define an action of $\mathcal{U}(\mathfrak{g})$ on $H_n := V_n \otimes \wedge^{*,*}\mathbb{C}^2$ by

$$\begin{aligned} L_t(v \otimes w) &= (L_t v) \otimes w, \\ L_t^*(v \otimes w) &= (L_t^* v) \otimes w, \\ H_t(v \otimes w) &= (H_t v) \otimes w, \\ a(v \otimes w) &= v \otimes aw, \\ \bar{a}(v \otimes w) &= v \otimes \bar{a}w, \\ a^*(v \otimes w) &= v \otimes a^* w, \\ \bar{a}^*(v \otimes w) &= v \otimes \bar{a}^* w, \end{aligned} \tag{15}$$

Clearly, this defines a $*$-representation of $\mathcal{U}(\mathfrak{g})$ on H_n.

Theorem 3.1. *The representations H_n are irreducible and pairwise inequivalent. Any non-trivial finite dimensional irreducible $*$-representation of $\mathcal{U}(\mathfrak{g})$ is unitary equivalent to some H_n.*

Proof. Since $\mathcal{U}(\mathfrak{g})$ is generated by two commuting subalgebras $\mathcal{U}(\mathrm{sl}_2(\mathbb{C}))$ and $\mathcal{A}$ any irreducible $*$-representation of is also an irreducible $*$-representation of $\mathcal{U}(\mathrm{sl}_2(\mathbb{C})) \otimes \mathcal{A}$. If it is finite dimensional it is therefore a tensor product of two finite dimensional irreducible representations of $\mathcal{U}(\mathrm{sl}_2(\mathbb{C}))$ and $\mathcal{A}$. $\square$

Corollary 3.2. *Any non trivial finite dimensional irreducible $*$- representation of $\mathcal{U}(\mathfrak{g})$ decomposes into a direct sum of 4 equivalent modules for the $\mathrm{sl}_2(\mathbb{C})$ action defined by L_t, L_t^*, H_t.*

If h_n is a highest weight vector of V_n then the kernel of L_t^* in the representation H_n is given by $h_n \otimes \wedge^*\mathbb{C}^2$. Using the unitary basis $e, \bar{e}$ for $\mathbb{C}^2$ as before we see that the vectors

$$h_n \otimes 1, h_n \otimes e, h_n \otimes \bar{e}$$

are in the kernel of L^*. Moreover,

$$H(h_n \otimes 1) = (n-1)(h_n \otimes 1), \tag{16}$$

$$H(h_n \otimes e) = n(h_n \otimes e), \tag{17}$$

$$H(h_n \otimes \bar{e}) = n(h_n \otimes \bar{e}). \tag{18}$$

Therefore, in the decomposition of H_n into irreducibles of the $\mathrm{sl}_2(\mathbb{C})$ action defined by L, L^*, H the representations V_n occur with multiplicity at least 2 and the representation V_{n-1} occurs with multiplicity at least 1. The vector $h_n \otimes (e \wedge \bar{e})$ has weight $n + 1$ and therefore, there must be another representation of highest weight greater or equal than $n+1$ occurring. Since

$$4 \dim V_n - 2 \dim V_n - \dim V_{n-1} = \dim V_{n+1}$$

this shows that as a module for the $\mathrm{sl}_2(\mathbb{C})$ action defined by L, L^*, H we have $H_n = V_{n+1} \oplus V_n \oplus V_n \oplus V_{n-1}$.

Corollary 3.3. *Every non-trivial finite dimensional irreducible $*$- representation of $\mathcal{U}(\mathfrak{g})$ is as a module for the $\mathrm{sl}_2(\mathbb{C})$ action defined by L, L^*, H unitarily equivalent to the direct sum $V_{n+1} \oplus V_n \oplus V_n \oplus V_{n-1}$. By convention $V_{-1} = \{0\}$.*

Corollary 3.4. *Let V and W be two finite dimensional $\mathcal{U}(\mathfrak{g})$ modules. Then V and W are unitarily equivalent if and only if they are equivalent as modules for the $sl_2(\mathbb{C})$ action defined by L, L^*, H.*

3.3. **The model representations.** There is another very natural representation ρ of the $*$-algebra $\mathcal{U}(\mathfrak{g})$ which is important for our purposes. This representation will be referred to as the model representation and can be described as follows. Let us view $\mathbb{C}^m \cong \mathbb{R}^{2m}$ as a real vector space with complex structure J. Let $\{e_i\}_{i=1,\dots,m}$ be the standard unitary basis in $\mathbb{R}^{2m}$. Then in the complexification $\mathbb{R}^{2m} \otimes \mathbb{C} = \mathbb{C}^{2m}$ we define

$$w_i = e_i - \mathrm{i}\, J e_i, \tag{19}$$

$$\bar{w}_i = e_i + \mathrm{i}\, J e_i. \tag{20}$$

We define $\rho(L)$ to be the operator of exterior multiplication by $\omega = \frac{\mathrm{i}}{2} \sum_{i=1}^m w_i \wedge \bar{w}_i$ on the space $\wedge^* \mathbb{C}^{2m} = \bigoplus_{p,q} \wedge^{p,q} \mathbb{C}^{2m}$. Let $\pi(L^*)$ be its adjoint, namely the operator of interior multiplication by ω. Let $\rho(a)$ be the operator of exterior multiplication by $\frac{\mathrm{i}}{\sqrt{2}} w_1$ and $\rho(\bar{a})$ be the operator of exterior multiplication by $\frac{\mathrm{i}}{\sqrt{2}} \bar{w}_1$. The operators $\rho(a^*)$ and $\rho(\bar{a}^*)$ are defined as the adjoints of $\rho(a)$ and $\rho(\bar{a})$. This defines a representation ρ of $\mathcal{U}(\mathfrak{g})$ on $\wedge^* \mathbb{C}^{2m}$. This representation decomposes into a sum of irreducibles. Note that $\rho(L_t) = \rho(L - \mathrm{i}\, \bar{a} a)$ is given by exterior multiplication by $\frac{\mathrm{i}}{2} \sum_{i=2}^m w_i \wedge \bar{w}_i$. The restriction of ρ to the two subalgebras generated by $\rho(L), \rho(L^*), \rho(H)$ and $\rho(L_t), \rho(L_t^*), \rho(H_t)$ define representations of $sl_2(\mathbb{C})$. Since the maximal eigenvalue of H is m, only representations of highest weight k with $k \leq m$ can occur in the decomposition of the model representation with respect to the $sl_2(\mathbb{C})$-action by $\rho(L), \rho(L^*), \rho(H)$. Consequently, by Cor 3.3 in the decomposition of the model representation into irreducible representations only the representations H_k with $k \leq m$ can occur.

4. Asymptotic decomposition of Eigenspaces

Since the action of $\mathcal{U}(\mathfrak{g})$ commutes with the Laplace operator Δ on forms each eigenspace

$$V_\lambda = \{\phi \in \wedge^* X : \Delta\phi = \lambda\phi\}$$

with $\lambda \neq 0$ is a $\mathcal{U}(\mathfrak{g})$-module and can be decomposed into a direct sum of irreducible $\mathcal{U}(\mathfrak{g})$-modules. In the previous section we classified all irreducible $*$-representations of $\mathcal{U}(\mathfrak{g})$ and found that they are isomorphic to H_n for some non-negative integer n. Therefore, we may define the function $m_k(\lambda)$ as

$$m_k(\lambda) := \{\text{the multiplicity of } H_k \text{ in } V_\lambda\}, \tag{21}$$

so that

$$V_\lambda \cong \bigoplus_{k=0}^{\infty} m_k(\lambda) H_k \tag{22}$$

Theorem 4.1. *Let X be any compact Kähler manifold of complex dimension m. Then in the decomposition of the eigenspaces of the Laplace-Beltrami operator Δ into irreducible representations of $\mathcal{U}(\mathfrak{g})$ the proportion of irreducible summands of type H_k in $L^2(X; \wedge^* X)$ is in average the same as the proportion of such irreducibles in the model representation of $\mathcal{U}(\mathfrak{g})$ on $\wedge^*\mathbb{C}^{2m}$:*

$$\frac{1}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} m_k(\lambda_j) \sim \frac{1}{\dim(\wedge^*\mathbb{C}^{2m})} m_k(\wedge^*\mathbb{C}^{2m}), \tag{23}$$

where $N(\lambda) = \mathrm{tr}\Pi_{[0,\lambda]}$ and $\Pi_{[0,\lambda]}$ is the spectral projection of the Laplace-Beltrami operator Δ.

We recall that $N(\lambda) \sim \frac{rk(E)vol(X)}{(4\pi)^m\Gamma(m+1)}\lambda^m$ for the Laplacian on a bundle $E \to X$ of rank $rk(E)$ over a manifold X of real dimension $2m$. Note that apart from the fact that we are not dealing with a group but with a Lie superalgebra the action is neither on X, nor on T^*X, but rather on the total space of the vector bundle $\pi^*(\wedge^* X) \to T_1^* X$. The action there leaves the fibers invariant and therefore it is rather different from a group action on the base manifold. The above theorem thus falls outside the scope of the equivariant Weyl laws of articles such as [BH1, BH2, GU, HR, TU]. In fact its conclusion is rather different from the conclusions in these articles as in our case only a fixed number of types of irreducible representations may occur.

Proof. For a compact Kähler manifold $\mathcal{U}(\mathfrak{g})$ acts by pseudodifferential operators on $C^\infty(X; \wedge^* X)$. Therefore, the symbol map defines an action of $\mathcal{U}(\mathfrak{g})$ on each fiber of the bundle $\pi^*(\wedge^* X) \to T_1^* X$. The representation of $\mathcal{U}(\mathfrak{g})$ on each fiber is easily seen to be equivalent to the model representation. Since the maximal eigenvalue of H, acting on $L^2(X; \wedge^* X)$, is m, only representations of highest weight k with $k \leq m$ can occur in the decomposition of $L^2(X; \wedge^* X)$ into irreducible subspaces with respect to the $\mathrm{sl}_2(\mathbb{C})$-action by L^*, L, H. Again, by Cor 3.3 types H_k with $k > m$ cannot occur in the decomposition with respect to the $\mathcal{U}(\mathfrak{g})$-action. Let P_k be the orthogonal projection onto the type H_k in $L^2(X; \wedge^* X)$. Then P_k is actually a pseudodifferential operator of order 0. Namely, the quadratic Casimir operator $\mathcal{C}$ of the $\mathrm{sl}_2(\mathbb{C})$-action by L_t^*, L_t, H_t given by

$$\mathcal{C} = L_t^* L_t + L_t L_t^* + \frac{1}{2} H^2, \tag{24}$$

is a pseudodifferential operator of order 0. On a subspace of type H_k it acts like multiplication by $\frac{k^2}{2} + k$. Therefore, if Q is a real polynomial that is equal to 1 at $\frac{k^2}{2} + k$ and equal to 0 at $\frac{l^2}{2} + l$ for any integer $l \neq k$ between 0 and m it follows that $P_k = Q(\mathcal{C})$. Thus, P_k is a pseudodifferential operator of order 0 and its principal symbol at ξ projects onto the subspace in the fiber $\pi_\xi^*(\wedge^* X)$ which is spanned by the representations of type H_k. Therefore, for every ξ:

$$\frac{1}{\dim(H_k)} \mathrm{tr}(\sigma_{P_k}(\xi)) = m_k(\wedge^* \mathbb{C}^{2m}). \tag{25}$$

Applying Karamatas Tauberian theorem to the heat trace expansion

$$\mathrm{tr}(P_k e^{-\Delta t}) = (4\pi)^{-m} \mathrm{Vol}(X) \left(\int_{T_1^* X} \mathrm{tr}(\sigma_{P_k}(\xi)) d\xi \right) t^m + O(t^{m - \frac{1}{2}}). \tag{26}$$

gives

$$\frac{1}{N(\lambda)} \sum_{j : \lambda_j \leq \lambda} \mathrm{tr}(\Pi_{[0,\lambda]} P_k) \sim m_k(\wedge^* \mathbb{C}^{2m}) \dim(H_k) \frac{1}{\dim(\wedge^* \mathbb{C}^{2m})}. \tag{27}$$

After dividing by $\dim(H_k)$ this reduces to the statement of the theorem. $\square$

Remark 4.2. *A natural question is whether, for generic Kähler metrics, the eigenspaces of the Laplace-Beltrami operator are irreducible representations of the Lie superalgebra $\mathfrak{g}$ and of complex conjugation. Such irreducibility is suggested by the heuristic principle of 'no accidental degeneracies', i.e. in generic cases, degeneracies of eigenspaces should be entirely due to symmetries (see* [Zel90] *for some results and* *references). Cor. 5.3 would then suggest that for a generic Kähler manifold the spectrum of Δ on the space of primitive co-closed (p,q)-forms should be simple for fixed p and q.*

5. Quantum ergodicity for the Laplace-Beltrami operator

We will now investigate the question of quantum ergodicity for the Laplace-Beltrami operator on a compact Kähler manifold X and we keep the notations from the previous sections. As shown in [JS] this question is intimately related to the ergodic decomposition of the tracial state on the C^*-algebra $C(X;\pi^* \wedge^* X)$. The transversal Lefschetz decomposition plays an important role here.

5.1. **Ergodic decomposition of the tracial state.** On the space of (p,q)-forms denote by $P_{p,q}$ the projection onto the space of transversally-primitive forms, i.e. onto the kernel of L_t^*. Let $P_{p,q,k}$ be the projection onto the range of $L_t^k P_{p-k,q-k}$. The operators

$$P_1 = P_{\partial\bar\partial} = 4\Delta^{-2}\partial\bar\partial\bar\partial^*\partial^* = QQ^*, \tag{28}$$

$$P_2 = P_{\partial^*\bar\partial^*} = 4\Delta^{-2}\partial^*\bar\partial^*\bar\partial\partial = Q^*Q, \tag{29}$$

$$P_3 = P_{\partial\bar\partial^*} = 4\Delta^{-2}\partial\bar\partial^*\bar\partial\partial^*, \tag{30}$$

$$P_4 = P_{\partial^*\bar\partial} = 4\Delta^{-2}\partial^*\bar\partial\bar\partial^*\partial \tag{31}$$

are projections onto the ranges of the corresponding operators. We have

$$P_H + \sum_{i=1}^{4} P_i = 1 \tag{32}$$

where P_H is the finite dimensional projection onto the space of harmonic forms. Using the transversal Lefschetz decomposition we obtain a further decomposition

$$\sum_{k=0}^{\min(p,q)} P_{p,q,k}P_H + \sum_{k=0}^{\min(p,q)} \sum_{i=1}^{4} P_{p,q,k}P_i = 1 \tag{33}$$

where each of the subspaces onto which $P_{p,q,k}P_i$ projects is invariant under the Laplace operator.

Note that the principal symbols of these projections are invariant projections in $C(T_1^*X, \pi^*\mathrm{End}(\wedge^{p,q}X))$ and the above relation gives rise to a decomposition of the tracial state ω_{tr} on $C(T_1^*X, \pi^*\mathrm{End}(\wedge^{p,q}X))$

defined by

$$\omega_{tr}(a) := \frac{1}{\mathrm{rk}(\wedge^{p,q}X)} \int_{T_1^*X} \mathrm{tr}(a(\xi))d\xi \tag{34}$$

into invariant states. Thus, the tracial state is not ergodic. However, if the $U(m)$-frame flow is ergodic this decomposition turns out to be ergodic.

Proposition 5.1. *Suppose that the $U(m)$-frame flow on U_mX is ergodic. Let P be one of the projections*

$$P_{p,q,k}P_i,$$
$$1 \leq i \leq 4,$$
$$0 \leq k \leq \min(p,q).$$

Then the state ω_P on $C(T_1^*X;\pi^*\mathrm{End}(\wedge^{p,q}X))$ defined by $\omega_P(a) := c_P\omega_{tr}(\sigma_P a)$ is ergodic. Here $c_P = \omega_{tr}(\sigma_P)^{-1}$.

Proof. The bundle $\wedge^{p,q}X$ can be naturally identified with the associated bundle $U_mX \times_{\hat{\rho}_1} \wedge^{p,q}\mathbb{C}^{2m}$, where $\hat{\rho}_1$ is the representation of $U(m)$ on

$$\wedge^{p,q}\mathbb{C}^{2m} = \wedge^p\mathbb{C}^m \otimes \wedge^q\overline{\mathbb{C}}^m.$$

obtained from the canonical representation on $\mathbb{C}^m$. The pull back $\pi^* \wedge^{p,q} X$ of $\wedge^{p,q}X$ can analogously be identified with the associated bundle

$$U_mX \times_{\hat{\rho}} \wedge^{p,q}\mathbb{C}^{2m}, \tag{35}$$

where $\hat{\rho}$ is the restriction of $\hat{\rho}_1$ to the subgroup $U(m-1)$. Since the first vector in $\mathbb{C}^m$ is invariant under the action of $U(m-1)$ we have the decomposition

$$\wedge^{p,q}\mathbb{C}^{2m} = \wedge^{p,q}\mathbb{C}^{2m-2} \oplus \wedge^{p-1,q}\mathbb{C}^{2m-2} \oplus \wedge^{p,q-1}\mathbb{C}^{2m-2} \oplus \wedge^{p-1,q-1}\mathbb{C}^{2m-2}$$

into invariant subspaces. The projections onto these subspaces in each fiber is exactly given by the principal symbols σ_{P_i} of the projections P_i. The representation of $U(m-1)$ on $\wedge^{p',q'}\mathbb{C}^{2m-2}$ may still fail to be irreducible. However, it is an easy exercise in representation theory (c.f. [FuHa91], Exercise 15.30, p. 226) to show that the kernel of $\sigma_{L_t^*}$ in each fiber is an irreducible representation of $U(m-1)$. Thus, σ_P projects onto a sub-bundle F of $\pi^* \wedge^{p,q} X$ that is associated with an irreducible representation ρ of $U(m-1)$, i.e.

$$F \cong U_mX \times_\rho V_\rho. \tag{36}$$

This identification intertwines the $U(m)$-frame flow on U_mX and the flow β_t. To show that the state ω_P is ergodic it is enough to show that any positive β_t-invariant element f in $\sigma_P L^\infty(T_1^*X, \pi^*\mathrm{End} \wedge^{p,q} X)\sigma_P$ is proportional to σ_P (see [JS], Appendix). Under the above identification f gets identified with a function $\hat{f} \in L^\infty(U_m X; V_\rho)$ which satisfies

$$\hat{f}(xg) = \rho(g)\hat{f}(x)\rho(g)^{-1}, \quad x \in U_m X, g \in U(m-1). \tag{37}$$

If such a function is invariant under the $U(m)$-frame flow it follows from ergodicity of the $U(m)$-frame flow that it is constant almost everywhere. So almost everywhere $\hat{f}(x) = M$, where M is a matrix. By the above transformation rule M commutes with $\rho(g)$. Since ρ is irreducible it follows that M is a multiple of the identity matrix. Thus, $\hat{f}$ is proportional to the identity and consequently, f is proportional to σ_P. □

Applying the abstract theory developed in [Zel96] the same argument as in [JS] can be applied to obtain

Theorem 5.2. *Let P be one of the projections*

$$\begin{gathered} P_{p,q,k}P_i, \\ 1 \le i \le 4, \\ 0 \le k \le \min(p,q). \end{gathered}$$

and let (ϕ_j) be an orthonormal basis in $\mathrm{Rg}(P)$ *with*

$$\begin{gathered} \Delta\phi_j = \lambda_j\phi_j, \\ \lambda_j \nearrow \infty. \end{gathered} \tag{38}$$

If the $U(m)$-frame flow on $U_m X$ is ergodic, then quantum ergodicity holds in the sence that

$$\frac{1}{N}\sum_{j=1}^{N} |\langle\phi_j, A\phi_j\rangle - \omega_P(\sigma_A)| \to 0, \tag{39}$$

for any $A \in \Psi\mathrm{DO}^0_{cl}(X, \wedge^{p,q}X)$.

Since for co-closed forms primitivity and transversal primitivity are equivalent there is a natural gauge condition that manages without the above heavy notation.

Corollary 5.3. *Let ϕ_j be a complete sequence of primitive co-closed (p,q)-forms such that*

$$\begin{gathered} \Delta\phi_j = \lambda_j\phi_j, \\ \lambda_j \nearrow \infty. \end{gathered} \tag{40}$$

Then, if the $U(m)$-frame flow on U_mX is ergodic, quantum ergodicity holds in the sence that

$$\frac{1}{N}\sum_{j=1}^{N}|\langle\phi_j, A\phi_j\rangle - \omega_P(\sigma_A)| \to 0, \tag{41}$$

for any $A \in \Psi\mathrm{DO}^0_{cl}(X, \wedge^{p,q}X)$, where $P = P_{p,q,0}P_2$ is the orthogonal projection onto the space of primitive co-closed (p,q)-forms.

6. Quantum ergodicity for Spin$^{\mathbb{C}}$-Dirac operators

In this section we consider the quantum ergodicity for Dirac type operators rather than Laplace operators. The complex structure on Kähler manifolds gives rise to the so-called canonical and anti-canonical Spin$^{\mathbb{C}}$- structures. The spinor bundle of the latter can be canonically identified with the bundle $\wedge^{0,*}X$ in such a way that the Dirac operator gets identified with the so-called Dolbeault Dirac operator. Other Spin$^{\mathbb{C}}$- structures (e.g. the canonical one) can then be obtained by twisting with a holomorphic line bundle. Let us quickly describe the construction of the twisted Dolbeault operator.

Let L be a holomorphic line bundle. Then the twisted Dolbeault complex is given by

$$\cdots \xrightarrow{\bar\partial} \wedge^{0,k-1}X \otimes L \xrightarrow{\bar\partial} \wedge^{0,k}X \otimes L \xrightarrow{\bar\partial} \cdots$$

This is an elliptic complex and the twisted Dolbeault Dirac operator is defined by

$$D = \sqrt{2}(\bar\partial + \bar\partial^*). \tag{42}$$

As mentioned above this operator is the Dirac operator of a Spin$^{\mathbb{C}}$-structure on X where the spinor bundle is identified with $S = \wedge^{0,*}X \otimes L$. Note that Spin structures on X are in one-one correspondence with square roots of the canonical bundle $K = \wedge^{n,0}TX$, i.e. with holomorphic line bundles L such that $L \otimes L = K$. In this case the Dirac operator D is exactly the twisted Dolbeault Dirac operator.

The twisted Dolbeault Dirac operator is a first order elliptic formally self-adjoint differential operator. It is therefore self-adjoint on the domain $H^1(X; \wedge^{0,*}X \otimes L)$ of sections in the first Sobolev space. As D is a first order differential operator its spectrum is unbounded from both sides.

The Dolbeault Laplace operator is given by $2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) = D^2$ and will be denoted by Δ^L. The Hodge decomposition is

$$C^\infty(X; \wedge^{0,k} X \otimes L) = \ker(\Delta_k^L) \oplus \bar{\partial} C^\infty(X; \wedge^{0,k-1} X \otimes L) \oplus \bar{\partial}^* C^\infty(X; \wedge^{0,k+1} X \otimes L). \tag{43}$$

Note that the Dirac operator leaves $\ker(\Delta_k^L)$ invariant since it commutes with Δ^L. Moreover, D maps $\bar{\partial} C^\infty(X; \wedge^{0,k-1} X \otimes L)$ to $\bar{\partial}^* C^\infty(X; \wedge^{0,k} X \otimes L)$ and $\bar{\partial}^* C^\infty(X; \wedge^{0,k} X \otimes L)$ to $\bar{\partial} C^\infty(X; \wedge^{0,k-1} X \otimes L)$. Therefore, the subspaces

$$\mathcal{H}^k = \bar{\partial} C^\infty(X; \wedge^{0,k-1} X \otimes L) \oplus \bar{\partial}^* C^\infty(X; \wedge^{0,k} X \otimes L) \tag{44}$$

are invariant subspaces for the Dirac operator. The orthogonal projections Π_k onto the closures of $\mathcal{H}^k$ are clearly zero order pseudodifferential operators which commute with the Dirac operator.

Let $\overline{\Psi\mathrm{DO}^0_{cl}(X; S)}$ be the norm closure of the $*$-algebra of zero order pseudodifferential operators in $\mathcal{B}(L^2(X, S))$. Then the symbol map extends to an isomorphism

$$\overline{\Psi\mathrm{DO}^0_{cl}(X; S)}/\mathcal{K} \cong C(T_1^* X, \pi^* \mathrm{End}(S)). \tag{45}$$

By theorem 1.4 in [JS] $\overline{\Psi\mathrm{DO}^0_{cl}(X; S)}$ is invariant under the automorphism group $\alpha_t(A) := e^{-i(\Delta^L)^{1/2} t} A e^{+i(\Delta^L)^{1/2} t}$ and the induced flow β_t on $C(T_1^* X, \pi^* \mathrm{End}(S))$ is the extension of the geodesic flow defined by parallel translation along the fibers.

As in the analysis for the Laplace-Beltrami operator we have to consider the tracial state

$$\omega_{tr}(a) = \frac{1}{\mathrm{rk}(S)} \int_{T_1^* X} \mathrm{tr}(a(\xi)) d\xi, \tag{46}$$

As already remarked in [JS] this state is not ergodic with respect to β_t since it has a decomposition

$$\omega_{tr}(a) = \frac{1}{2}\omega_+(a) + \frac{1}{2}\omega_-(a), \tag{47}$$

where

$$\omega_\pm(a) = \omega_{tr}((1 \pm \sigma_{\mathrm{sign}(D)})a) \tag{48}$$

On $\mathrm{Spin}^{\mathbb{C}}$-manifolds with ergodic frame flows the states $\omega_\pm$ were shown in [JS] to be ergodic. On Kähler manifolds of complex dimension greater than one they are not ergodic since we have a further decomposition

$$\omega_\pm(a) = \sum_k \omega_\pm(\sigma_{\Pi_k} a) \tag{49}$$

into invariant states.

Proposition 6.1. *Suppose that the $U(m)$-frame flow on U_mX is ergodic. Then the states $\omega_\pm^k := c_k \omega_\pm(\sigma_{\Pi_k} a)$ are ergodic with respect to the group β_t. Here $c_k := \omega_\pm(\sigma_{\Pi_k})^{-1}$.*

Proof. Let R be one of the projections $\frac{1 \pm \mathrm{sign}(D)}{2}\Pi_k$ and let σ_R be its principal symbol. Hence, σ_R is a projection in

$$C(T_1^*X, \pi^*\mathrm{End}(S)) \cong C(T_1^*X, \pi^*\mathrm{End}(\wedge^{0,*}X)). \tag{50}$$

We need to show that $a \to \omega_{tr}(\sigma_R)^{-1}\omega_{tr}(\sigma_R a)$ is an ergodic state. As in the proof of Proposition 5.1 this is equivalent to showing that any positive element in $L^\infty(T_1^*X, \pi^*\mathrm{End}(\wedge^{0,k}X))\sigma_R$ is proportional to σ_R. A positive element in $L^\infty(T_1^*X, \pi^*\mathrm{End}(\wedge^{0,k}X))\sigma_R$ is also in

$$\sigma_R L^\infty(T_1^*X, \pi^*\mathrm{End}(\wedge^{0,k}X))\sigma_R = L^\infty(T_1^*X, \mathrm{End}F),$$

where F is the sub-bundle of $\pi^* \wedge^{0,k} X$ onto which σ_R projects. Since σ_R is β_t-invariant the flow clearly restricts to a flow on the sub-bundle $\mathrm{End}F$ of $\pi^*\mathrm{End}(\wedge^{0,k}X)$. We will show that under the stated assumptions an invariant element in $L^\infty(T_1^*X, \mathrm{End}F)$ is proportional to the identity in $L^\infty(T_1^*X, \mathrm{End}F)$. Note that $\pi^* \wedge^{0,k} X$ is naturally identified with an associated bundle

$$\pi^* \wedge^{0,k} X \cong U_mX \times_{\wedge^k \tilde{\rho}} \wedge^k \overline{\mathbb{C}}^m, \tag{51}$$

where $\tilde{\rho}$ is the restriction of the anticanonical representation of $U(m)$ on $\bar{\mathbb{C}}^m$ to $U(m-1)$. Here we view U_mX as a $U(m-1)$-principal fiber bundle over T_1^*X. Note that $\wedge^k\tilde{\rho}$ is not irreducible but splits into a direct sum of two irreducible representations. This corresponds to the splitting $\wedge^k(\bar{\mathbb{C}}^{m-1} \oplus \bar{\mathbb{C}}) = \wedge^{k-1}\bar{\mathbb{C}}^{m-1} \oplus \wedge^k\bar{\mathbb{C}}^{m-1}$. Under the above correspondence the projections onto the sub-representations are exactly the principal symbols of the projections onto $\mathrm{Rg}(\bar{\partial})$ and $\mathrm{Rg}(\bar{\partial}^*)$. One finds that F is associated with a representation ρ of $U(m-1)$

$$F \cong U_mX \times_\rho V_\rho, \tag{52}$$

where ρ is equivalent to $\wedge^k\hat{\rho}$ and $\hat{\rho}$ is the anticanonical representation of $U(m-1)$. Therefore, ρ is irreducible. Hence, elements in $f \in L^\infty(T_1^*X, \mathrm{End}F)$ can be identified with functions $\hat{f} \in L^\infty(U_mX, \mathrm{End}V_\rho)$ that satisfy the transformation property

$$\hat{f}(xg) = \rho(g)\hat{f}(x)\rho(g)^{-1}, \quad x \in U_mX, g \in U(m-1). \tag{53}$$

This identification intertwines the pullback of the frame flow with β_t. Now exactly in the same way as in the proof of Proposition 5.1 we conclude that an invariant element in $L^\infty(T_1^*X, \mathrm{End}F)$ must be a multiple of the identity. Thus, the corresponding state is ergodic. $\square$

The above theorem gives rise to an ergodic decomposition of the tracial state on the C^*-algebra of continuous sections of $\pi^*\mathrm{End}(S)$ which is different from the decomposition obtained from Prop. 5.1. The advantage of this decomposition is that it is more suitable to study quantum ergodicity for the Dirac operator. Namely, the decomposition (47) corresponds to the splitting into negative energy and positive energy subspaces of the Dirac operator. Thus, if we are interested in quantum limits of eigensections with positive energy we need to decompose the state ω_+ into ergodic components. This is achieved by Prop. 6.1.

In the same way as in [JS] one obtains

Theorem 6.2. *Let X be a Kähler manifold and let L be a holomorphic line bundle. Let D be the associated $Spin^{\mathbb{C}}$-Dirac operator and let $L^2_+(X,S)$ be the positive spectral subspace of D. Suppose that (ϕ_j) is an orthonormal basis in $\Pi_k L^2_+(X,S)$ such that*

$$D\phi_j = \lambda_j \phi_j, \quad \lambda_j \nearrow \infty. \tag{54}$$

If the $U(m)$-frame flow on $U_m X$ is ergodic, then

$$\frac{1}{N}\sum_{j=1}^{N} |\langle \phi_j, A\phi_j\rangle - \omega_k(\sigma_A)| \to 0, \tag{55}$$

for any $A \in \Psi\mathrm{DO}^0_{cl}(X,S)$. Here ω_k is the state on $C(T_1^*X, \pi^*\mathrm{End}(S))$ defined by

$$\omega_k(a) = C\int_{T_1^*X} \mathrm{tr}\left((1+\sigma_{\mathrm{sign}(D)}(\xi))\sigma_{\Pi_k}(\xi)a(\xi)\right) d\xi, \tag{56}$$

where integration is with respect to the normalized Liouville measure and C is fixed by the requirement that $\omega_k(1)=1$.

This shows that quantum ergodicity for the Dirac operators holds only after taking the symmetry Π_k into account. The states ω_k differ for different k. Therefore, Dirac operators on a Kähler manifolds of complex dimension greater than one are never quantum ergodic in the sense of [JS].

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Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Str. West, Montréal QC H3A 2K6, Canada.
E-mail address: `jakobson@math.mcgill.ca`

Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire, LE11 3TU, UK
E-mail address: `a.strohmaier@lboro.ac.uk`

Johns Hopkins University, Department of Mathematics, 404 Krieger Hall, 3400 N. Charles Street, Baltimore, MD 21218
E-mail address: `zelditch@math.jhu.edu`