GEOMETRIC PROPERTIES OF EIGENFUNCTIONS

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ABSTRACT. We give an overview of some new and old results on geometric properties of eigenfunctions of Laplacians on Riemannian manifolds. We discuss the properties of nodal sets and critical points, the number of nodal domains, as well as asymptotic properties of eigenfunctions in the high energy limit (such as weak* limits, the rate of growth of L^p norms, and the relationship between positive and negative parts of eigenfunctions). We also discuss isoperimetric inequalities for eigenvalues.

1. INTRODUCTION

It is well-known that on a compact Riemannian manifold M one can choose an orthonormal basis of $L^2(M)$ consisting of eigenfunctions φ_j of Δ satisfying

$$\Delta\varphi_j + \lambda_j\varphi_j = 0, \tag{1}$$

where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ are the eigenvalues. The purpose of this survey paper is to present some recent (and not so recent) results about the asymptotics of Laplace eigenfunctions on compact manifolds. We focus here mainly on results about the nodal sets, asymptotic L^p bounds and the problem of determining weak-limits of expected values (i.e. quantum chaos). This is not a comprehensive survey of the field of spectral asymptotics. We have mainly discussed topics related to eigenfunctions that overlap with the interests of the authors. Also, we do not discuss eigenvalue statistics, the vast literature surrounding the wave trace formula and results in random matrix theory. For these and other topics, we refer the reader to [De, Dui-G, G2, G3, Z12] and the references therein. As a final disclaimer, it should be pointed out that we have also not touched on the vast physics literature on eigenfunctions: for this we refer the reader, e.g. to [B-B-C-S, C-C, Ef, G-V-Z, Gutz] and the references therein.

2. NODAL SETS.

The behavior of nodal lines of the eigenfunctions of Laplacians was first investigated by German physicist E.F.F. Chladni ([Chl, Ul]). He visualized the nodal lines of a vibrating metal plate by stroking with a violin bow metal plates covered with sand, and systematically recorded the resulting patterns. Lord Rayleigh in his classical treatise *On the theory of sound* [Ray] showed for different regions how nodal lines changed when the region did. After Rayleigh there were many attempts to understand intricate patterns of nodal lines of eigenfunctions. The first classical

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results were Sturm's theorem for an oscillating string [C-H] and a theorem due to Courant [C-H] about the number of nodal domains (connected components of the complement of the nodal set):

Theorem 2.1. *The n-th eigenfunction can have at most n nodal domains.*

Pleijel ([Pl]) improved Courant's estimate. Recently, Blum, Gnutzman and Smilansky ([Blu-Gn-Sm]) studied the limiting distribution of the variable ν_j/j, where ν_j is the number of the nodal domains of φ_j. They have determined the limiting distribution for surfaces with integrable geodesic flows (cf. section 6) where the variables separate.[1] They have also considered Dirichlet problem for planar billiards with ergodic billiard flow, and studied the limiting distribution of the number of nodal domains and of nodal line intersections with the boundary (comparing predictions from the random wave theory (cf. section 5) with numerical computations of eigenfunctions). Eigenfunctions with *few* nodal domains were constructed in [C-H] and [Lew]; see also section 3.

We next discuss local and metrical structure of the nodal sets of oscillating membranes, eigenfunctions of Δ on Riemannian manifolds, solutions of Shrödinger equations etc. In the 2-dimensional case, the local structure is described by the following theorem of Bers [Bers]:

Theorem 2.2. *If $\varphi(x) = 0$, then in any neighborhood of x the nodal line is either a smooth curve, or an intersection of n smooth curves at equal angles.*

For eigenfunctions on a n-dimensional Riemannian manifold, one can give only a metric description of the nodal set. Let Γ be the nodal set of an eigenfunction of a Shrödinger or a more general elliptic operator H. Under very weak assumptions on the regularity of the coefficients of H, Γ has finite $(n-1)$-dimensional Hausdorff measure [Har-Sim].

Let Γ_0 be the set of critical points lying on Γ, and let $\Gamma_1 = \Gamma \backslash \Gamma_0$. Then one has additional information on the structure of sets Γ_0 and Γ_1. Indeed, according to Schauder's estimates , Γ_1 is smooth enough provided the coefficients of H are regular enough. Rather unexpectedly, it turned out that for an eigenfunction φ of a Shrödinger operator, the smoothness of Γ_1 is one higher than that of φ itself. For example, if the potential of H is in $C^\alpha, 0 < \alpha < 1$ then according to the classical theorem of Schauder, φ is in $C^{2,\alpha}$ but Γ_1 is $C^{3,\alpha}$, as shown in [Ho-Ho-Nad1]. For $n = 2$, Bers's theorem 2.2 implies that Γ_0 is discrete. In dimension n, Γ_0 has locally finite $(n-2)$dimensional Hausdorff measure, [Ho-Ho-Nad2, Har-Ho-Ho-Nad].

For an eigenfunction on a compact manifold (or a domain) it is interesting to estimate the $(n-1)$-dimensional measure of the whole nodal set Γ. Indeed, such an estimate should depend on the eigenvalue λ and one can expect that there exists $C > 0$ such that

$$C^{-1}\sqrt{\lambda} \leq \mathrm{meas}_{n-1}\Gamma \leq C\sqrt{\lambda}. \tag{2}$$

This estimate was proved by Donnelly and Fefferman [Don-F3] (see also [Don-F1], [Nad3]) for n-dimensional real-analytic manifolds; we lack such knowledge in the smooth case. For $n = 2$ there is such estimate from below [Bru], but the best known estimate from above ([Don-F2])

$$\mathrm{meas}_1\Gamma \ \leq \ C\lambda^{3/4}$$

[1] The limiting distribution of $x_j = \nu_j/j$ is supported in $[0, \mu]$ for some $0 < \mu < 1$, and has a $(1 - (x/mu))^{-1/2}$ singularity near $x = \mu$.

is probably not optimal.

We know much less about topological structure of nodal sets than about their metric structure. The problems start already when we consider the second Dirichlet eigenfunction of a planar domain. According to Theorem 2.1, the second eigenfunction has exactly two nodal domains. Melas [Mel] proved that for a convex planar domain, the nodal line is not closed, i.e. it is a segment connecting two boundary points of the domain. For multiply-connected domains this is no longer true, and a nodal line can be a closed curve ([Ho-Ho-Nad3, Ho-Ho-Nad4]); for examples in higher dimensions, see [Fou]. It remains a very interesting open problem whether Melas' Theorem holds for arbitrary simply-connected planar domains. For special class of long domains the geometry of nodal sets wa studied in [G-Jer, Jer1].

2.1. **Eigenvalue multiplicities.** Let M be a compact surface with a smooth metric g, and let $0 < \lambda_1(g) \leq \lambda_2(g) < \dots$ be the eigenvalues in (1). We denote by $m_k(g)$ the multiplicity of $\lambda_k(g)$, and we let

$$m_k(M) = \sup_g \ m_k(g),$$

where the supremum is taken over all Riemannian metrics g on M, and so depends only on the topology of M.

Theorems 2.1 and 2.2 discussed above allow to get considerable information about the numbers $m_k(M)$. The following theorem holds:

Theorem 2.3. *The number $m_k(M)$ is finite for all k and M. Moreover, for an orientable surface M_γ of genus $\gamma \geq 2$*

$$m_k(M_\gamma) \leq 2k + 4\gamma - 1,$$

while $m_k(S^2) \leq 2k - 1$ for $k \geq 2$ and $m_k(\mathbf{T}^2) \leq 2k + 4$ for $k \geq 1$.

For small values of k, $m_1(S^2) = 3$, $m_2(S^2) = 3$, $m_1(\mathbf{T}^2) = 6$, $m_1(\mathbf{RP}^2) = 5$, $m_1(\textit{Klein bottle}) = 5$ and $m_1(\mathbf{T}^2 \# \mathbf{RP}^2) = 7$.

Various parts of Theorem 2.3 were proved in [Bess, Che, CV4, Ho-Ho-Nad5, Nad2, Sev]. For other inequalities for non-orientable surfaces, see [Nad2, Sev]. Analogous estimates hold for multiplicities in the Dirichlet problem for planar domains, see [Nad1, Ho-Mi-Nad]. When k and the genus γ increase, it becomes very difficult to determine exact values for $m_k(M)$. For large k and γ, the estimate in Theorem 2.3 is linear in k and γ. However, in all the known examples multiplicity grows like $\sqrt{k}$ and $\sqrt{\gamma}$. Finding the correct rate of growth of $m_k(M)$ in k and γ is an outstanding unsolved problem.

Let M be an n-dimensional manifold, and let G be a discrete subgroup of isometries of M. Let $A_1, A_2, \dots, A_k \subset H^1(M)$ be symmetry subspaces corresponding to the irreducible representations of G. Then A_i are invariant under Δ, and we can denote by $\lambda_1^i \leq \lambda_2^i \leq \dots$ the eigenvalues of Δ on A_i. Let the dimension of the i-th irreducible representation be l_i, and let the multiplicity of λ_1^i equal to $N_i l_i$. It is a challenging problem to get an estimate for N_i depending on G and on the topology of M and M/G. In general, it is even unknown whether there exists such an estimate for N_i. The first step in that direction was made in [H-Ho-Ho-Nad]. For Dihedral groups $\mathbf{D}_{2n}$ acting on a two-dimensional disk, it was shown that all $N_i = 1$. There were also established universal inequalities between $\lambda_1^i, 1 \leq i \leq k$. Of course, the cases when $N_i = 1$ are especially interesting. They are equivariant analogues of the fact that λ_1 is always simple for the Dirichlet problem.

3. Eigenfunctions and their critical points

Describing geometry of an eigenfunction itself is a more general problem than describing geometry of its nodal set. This is a fundamental question in Quantum Mechanics, especially when the eigenfunction is the ground state of a quantum-mechanical system. However, very little is known even about a simpler problem of geometry of eigenfunctions of a membrane.

Let φ be the first Dirichlet eigenfunction of a bounded convex set in $\mathbf{R}^n$. Then any set bounded by a level sets of φ is itself convex [Br-Li], see also [Kaw]. In [Jer-Nad] critical points of Neumann eigenfunctions corresponding to the first nonzero eigenvalue of a bounded planar domain are studied. It is proved, in particular, that in convex domains with two symmetry axes such an eigenfunction has exactly two critical points: supremum and infimum, and that both points lie on the boundary. In the last result it is essential to assume *both* the convexity and the presence of two symmetry axes; otherwise it is easy to construct a counterexample. For instance, a narrow circular sector is a convex domain with a single symmetry axis. However, gradient of the first eigenfunction vanishes everywhere on the circular part of the boundary.

However, the following conjecture seems plausible: all critical points of a first eigenfunction of the Neumann problem in a convex planar domain lie on the boundary of the domain. In a general multiply-connected set the last conjecture is not true; the corresponding counterexamples are constructed in [Bur-W], see also [Ban-Bur, Bas-Bur].

In [Y1, # 76], [Y2, # 43] and [Y3] S. T. Yau considered questions about the number and the distribution of critical points of eigenfunctions of the Laplacian on Riemannian manifolds. In [Jak-Nad1] the first and the second authors provide an example of a metric on the two-dimensional torus, and a sequence of eigenfunctions such that the corresponding eigenvalues go to infinity but the number of critical points remains bounded. This answers in the negative the question raised in [Y1]; however, the metric is rather special, and it is possible that for a generic metric the number of critical points increases with the growth of the eigenvalue. The main idea of the construction in [Jak-Nad1] is to consider a sequence of eigenfunctions $f_n(x,y) = \sin(nx+y)$ (on $\mathbf{T}^2$ with the flat metric) whose critical points lie on a union of two line segments, and then change a metric in such a way that instead of two critical "ridges" we shall have a bounded number of critical points.

4. Limits of eigenfunctions

Many questions about eigenfunctions in the limit $\lambda \to \infty$ are motivated by the correspondence principle in quantum mechanics. It asserts that certain features of the classical system manifest themselves in the semiclassical (as Planck's constant $h \to 0$) limit of a quantization of the classical system. When the classical system is the geodesic flow G^t on a Riemannian manifold M, its quantization is the Laplacian $-h^2\Delta$ on M. The the semiclassical limit for this system is the limit $\lambda_j \to \infty$. Accordingly, properties of high energy eigenfunctions should reflect the properties of the geodesic flow.

One question that has attracted a lot of attention concerns limits of eigenfunctions. To an eigenfunction φ_j one can associate its "probability density" $d\mu_j = |\varphi_j(x)|^2 dx$ on M (where dx is the normalized Riemannian volume), and a distribution $d\omega_j$ on S^*M, projecting to $d\mu_j$. These measures are regarded as functionals on the spaces $\mathcal{D}(\mathcal{M})$ and $\mathcal{D}(\mathcal{S}^*\mathcal{M})$. Given "observables" $f \in \mathcal{D}(\mathcal{M})$ and $a \in \mathcal{D}(\mathcal{S}^*\mathcal{M})$ (regarded as a homogeneous function of order 0 on T^*M), we choose an auxiliary pseudo-differential operator $A = Op(a)$ of order zero with principal symbol a, and let

$$\begin{aligned} \langle f, d\mu_j \rangle &= \int_M f(x)|\varphi_j(x)|^2 \, dx \\ \langle a, d\omega_j \rangle &= \int_M (A\varphi_j)(x)\overline{\varphi_j(x)}dx := \langle A\varphi_j, \varphi_j \rangle. \end{aligned} \tag{3}$$

The choice of $Op(a)$ is not canonical, but any two choices differ by an operator of order -1, and and that does not affect the asymptotic behavior of the (diagonal) matrix elements $\langle A\varphi_j, \varphi_j \rangle$. When the correspondence $a \to Op(a)$ is Weyl quantization, the distribution $d\omega_j$ is called *Wigner measure* corresponding to φ_j; sometimes related objects called *Husimi* and *Bargman* measures are considered (cf. [Oz2, Oz3, Vor, P-U1, P-U2]). A basic problem is to study the set $\mathcal{L}$ of weak* limit points of μ_j-s and the set $\mathcal{U}$ of weak* limit points of ω_j-s; it follows from Egorov's theorem ([Eg]) that any limit measure in $\mathcal{U}$ is invariant under the geodesic flow. The sets $\mathcal{L}$ and $\mathcal{U}$ turn out to be quite different for manifolds with integrable and ergodic geodesic flows.

5. Ergodic geodesic flows

The following beautiful theorem describes limits on manifolds with ergodic geodesic flows; it shows that for a subsequence φ_{j_k} of eigenfunctions of the full density, $d\mu_{j_k} \to dx$ and $d\omega_{j_k} \to d\omega$, the Liouville measure on S^*M. Various versions of Theorem 5.1 were proved in [Shn1, Shn2, Z1, Z8, CV3, H-M-R] by Shnirelman, Zelditch, Colin de Verdiere and others. We let $N(\lambda) = \#\{\lambda_j < \lambda\}$ and $f \in \mathcal{D}(\mathcal{M})$

Theorem 5.1. *Let M be a compact manifold with ergodic geodesic flow. Then*

$$\begin{aligned} \sum_{\lambda_j \le \lambda} \left| \int_M f|\varphi_j|^2 - \int_M f \right|^2 &= o(N(\lambda)), \\ \sum_{\lambda_j \le \lambda} \left| \int_{S^*M} \langle A\varphi_j, \varphi_j \rangle - \int_{S^*M} a d\omega \right|^2 &= o(N(\lambda)). \end{aligned} \tag{4}$$

The proof uses Egorov's theorem and the ergodicity of the geodesic flow on S^*M.

One can say that on manifolds with ergodic geodesic flows, *almost all*[2] eigenfunctions become "uniformly distributed." This is sometimes called *Quantum Ergodicity*. Theorem 5.1 was extended by Zelditch to hyperbolic surfaces of finite area [Z4]. An analogous theorem was established for ergodic billiards in [Ger-L] and [Z-Z]. Partial converse results to Theorem 5.1 were obtained in [Su] and [Z9]. Other results relating high energy asymptotics of the matrix elements $\langle A\varphi_i, \varphi_j \rangle$ to classical dynamics can be found in [Hel-K3, Z10, Z11].

The rate of growth of the sum in (4) (which we denote by $S(\lambda)$) is called the *rate of quantum ergodicity*. For negatively curved manifolds, an upper bound for that rate was improved from $o(N(\lambda))$ to $O(N(\lambda)/\log \lambda)$ in [Z6]; Lower bounds for $S(\lambda)$ were considered in [Z7]. The rate of quantum ergodicity for hyperbolic surfaces and billiards was also studied in [A-T, B-S-S1]; the results for arithmetic hyperbolic manifolds are discussed in Section 5.1.

[2] A subsequence of the full density.

A related question is studying the asymptotics as $\lambda_j \to \infty$ of the integrals $\int \varphi_i^2 \varphi_j$ or, more generally,

$$I_j = \int P(\varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_k})\varphi_j, \tag{5}$$

where P is a polynomial in eigenfunctions $\varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_k}$. Sarnak [Sar1] proved that on n-dimensional hyperbolic manifold (of curvature -1),

$$I_j = O\left(e^{-\pi\sqrt{\lambda_j}/2}(\lambda_j+1)^B\right), \tag{6}$$

where B depends only on n and the degree of P. He also proved that (6) holds on average for non-compact hyperbolic manifolds of finite volume when φ_j is a (generalized) eigenfunction corresponding to the continuous spectrum of Δ; Good ([Go1, Go2]) proved a similar statement when $n = 2$ and P is a square of a holomorphic cusp form. The *exponential* rate of decay is optimal in (6); it is of major importance in many applications. The *polynomial* rate in (6) was not optimal and was improved by Petridis in [P1, P2]. The (essentially) sharp results on the polynomial rate *on average over the spectrum* were obtained in [Jut] for a class of such problems on $M = PSL_2(\mathbf{Z})\backslash\mathbf{H}$, and in [B-Rez] for compact hyperbolic surfaces.

A natural question is whether on manifolds with ergodic geodesic flows there exist *exceptional* sequences φ_{j_l} of eigenfunctions such that $d\mu_{j_l} \to d\nu \neq dx$ or $d\omega_{j_l} \to d\alpha \neq d\omega$; they are called *strong scars*. By Theorem 5.1 they must have density zero. WKB construction of "quasi-modes" concentrating on *stable* closed geodesics (discussed in Section 6, cf. [B-L]) suggests that high energy eigenfunctions may localize on *unstable* closed geodesics. Other natural candidates for such localization include continuous families of closed geodesics (bouncing balls) in ergodic billiards. Such localization was investigated in [A-E-F-K-M-M, A-F1, A-F2, A-S1, A-S2, A-S4, A-F, B-S, B-S-S2, B-H-S-T, Bal-Be, Be1, Be2, Be3, Be4, B-H-O, Bog, F-G-P, Gutz, Hel1, Hel2, Hel-K1, Hel-K2, M-K, Oz1, Oz3, Pr2, Tan] and other papers. However, it seems that on negatively curved manifolds the rate of concentration is not strong enough to produce strong scars; understanding that rate remains a very interesting problem.

Rudnick and Sarnak ([R-S]) conjecture that on negatively curved manifolds strong scars do not exist (this is sometimes called *quantum unique ergodicity*). Evidence supporting this conjecture can be found in [A-S2, A-S4, B-G-G-S, Hej-R]. Also, many results supporting that conjecture have been established for arithmetic hyperbolic manifolds and are described below.

Despite the general absence of strong scars, one does generically observe the phenomenon of eigenfunction mass concentrated along periodic, unstable geodesics at least when averaging over large families of eigenfunctions for which the Laplace eigenvalues λ_j satisfy $\lambda_j - \lambda \ll \frac{1}{\lambda}$. This property is, however, not asymptotic (as $\lambda \to \infty$) and depends on, among other things, the Liapunov exponents of the orbit. One way of understanding how much eigenfunction mass (at least on average) stays near a given orbit involves the computation of its contribution to the semi-classical trace formula (see [P-U1, P-U2]).

Random Wave conjectures in Quantum Chaos [Be1] suggest that high-energy eigenfunctions behave like random superpositions of waves when the classical system is ergodic. Accordingly ([Kah], [S-Z]), eigenfunctions for such systems are expected to have local Gaussian limit distributions, and that their $||\varphi_\lambda||_\infty$ grows $\sqrt{\log \lambda}$. These conjectures were considered in [A-S2, A-S3, A-S4, Be1, B-Rob, B-B-L, Han, Hej-R, Hel2, Hel-K3, Hort-S, Leb, Leb-Sh, Leb-V, M-K, N-V, Oz-T, Pr1, Shu, Sr-St, T-Vor, Z9, Z10].

Random Wave conjectures also relate the statistics of nodal sets of Wigner distributions (associated to high energy eigenfunctions) to the statistics of zeros of random polynomials. Those statistics were studied in [A-S3, Ber2, Bl-Di, Bl-S-Z1, Bl-S-Z2, B-B-L, Han, Leb-Sh, Leb-V, N-V, Oz-T, Pr1, Shu, Sh-Z].

It was shown in [N-V, Sh-Z, Z11] that quantum ergodicity implies uniform distribution of the zeros of holomorphic sections of quantizing line bundles at least in an "almost-everywhere" sense. More precisely, let M is a compact, complex manifold and (L, h) is a positive Hermitian holomorphic line bundle over M. Shiffman and Zelditch [Sh-Z] study the distribution of zeros of holomorphic sections $s \in H^0(M, L^N)$ as $N \to \infty$. In the special case where $M = CP^k$ and L is the canonical bundle, these sections are just homogeneous, holomorphic polynomials of degree N on $\mathbb{C}^{N+1}$ (see [N-V]). The curvature $\omega = c_1(h)$ of h is a Kahler form and together with h they give a Hermitian inner product on the vector space $H^0(M, L^N)$. This inner product allows one to define the notion of a 'random' holomorphic section rigorously: Form the probability space $S = \prod_{N=1}^{\infty} SH^0(M, L^N)$ where $SH^0(M, L^N)$ is the unit sphere on $H^0(M, L^N)$ and put the product Haar measure $d\mu$ on S. Define $Z_{s_N} := \{z \in M; s_N(z) = 0\}$ and by a slight abuse of notation, denote its Poincaré - Lelong dual $\partial\overline{\partial} \log |s_N|$ (see [GH]) also by Z_{s_N}. Then, in [Sh-Z] it is proved that for any $\phi \in \Omega^{k-1,k-1}(M)$, and almost all $\mathbf{s} = \{s_N\} \in S$,

$$\lim_{N\to\infty} \frac{1}{N} \int_M Z_{s_N} \wedge \phi = \int_M \omega \wedge \phi. \tag{7}$$

Thus, the sections $\mathbf{s} = \{s_N\}$ are, at least on average, equidistributed with respect to the Kahler form ω on M. In [Bl-S-Z1, Bl-S-Z2] more complicated correlation statistics are computed for the zeros.

In the case of real manifolds, the computation of the asymptotics of the density of states for the eigenfunction zeros is a difficult problem which may or may not have a limiting distribution. However, in the case of the round sphere, $\mathbb{S}^n$, Neuheisel [Neu] has proved nodal equidistribution results for random spherical harmonics that are analogous to (7).

5.1. **Arithmetic hyperbolic manifolds.** An *arithmetic hyperbolic manifold* is a quotient of a hyperbolic space by an arithmetic subgroup Γ of an isometry group G. Γ is called *arithmetic* if it has a large *commensurator* $\mathrm{COM}(\Gamma)$, i.e. the set of all $\delta \in G$ for which the group $\Gamma(\delta) := \delta^{-1}\Gamma\delta \cap \Gamma$ has finite index in both Γ and $\delta^{-1}\Gamma\delta$. Margulis showed in [Mar] that for semisimple G, Γ is arithmetic iff $\mathrm{COM}(\Gamma)$ is dense in $G(\mathbf{R})$; otherwise $\mathrm{COM}(\Gamma)/\Gamma$ is finite. An element $\delta \in \mathrm{COM}(\Gamma)$ gives rise to a Hecke operators T_δ acting on $L^2(M)$ as follows. Let $\Gamma = \cup_{i=1}^k \Gamma(\delta)\beta_i$ for $\beta_i \in G$. Then

$$T_\delta f(x) = \sum_{i=1}^{k} f(\delta\beta_j x) \tag{8}$$

Accordingly, on arithmetic hyperbolic manifolds there exist infinitely many Hecke operators; they commute with the Laplacian and with each other. Techniques from analytic number theory can be used to study joint eigenfunctions of Δ and the Hecke operators;[3] it is often convenient to choose such *test* functions as well. A lot of progress in studying questions related to quantum ergodicity has been made for eigenfunctions of Laplacians on arithmetic hyperbolic manifolds; we give a short survey of those results below. For more comprehensive surveys of the results on high-energy spectra and eigenfunctions of Laplacians for arithmetic hyperbolic manifolds and related results in Number Theory, we refer the reader to [Sar2, Sar3, I-S2].

The Quantum Unique Ergodicity conjecture has been established in [L-S2] and [Jak2] for (generalized) eigenfunctions corresponding to the *continuous* spectrum of Δ on $PSL_2(\mathbf{Z})\backslash\mathbf{H}$. Analogous results for arithmetic hyperbolic 3-manifolds were reduced in [Ko2] to certain L-function estimates; those estimates were established in [P-S]. Rudnick and Sarnak showed in [R-S] that on certain arithmetic hyperbolic manifolds, an exceptional measure $d\nu$ cannot have singular support contained in a finite union of closed geodesics (or totally geodesic surfaces). S. Wolpert showed in [Wol] that on $\Gamma_0(m)\backslash PSL_2(\mathbf{R})$, any such measure with compact singular support is null on each countable union of closed geodesics and geodesic laminations for finite index subgroups. E. Lindenstrauss has considered in [Lin] limiting measures on $\Gamma\backslash G/K$, where G is a semi-simple Lie group[4] of rank ≥ 2, K is its compact subgroup and Γ is an irreducible lattice. He conjectures that for joint eigenfunctions of *all* G-invariant differential operators, an analogue of QUE conjecture holds. He deduced his conjecture from a conjecture of Margulis in rigidity theory, and extended the results in [R-S] and [Wol] to the case of uniform irreducible congruence lattices in $PSL_2(\mathbf{R})^n$.

A new identity proved by Watson [W] allows to relate the rate of growth of the integral $\int_{PSL_2(\mathbf{Z})\backslash\mathbf{H}} \varphi_1\varphi_2\varphi_3$ (where φ_j-s are eigenfunctions in $L^2(PSL_2(\mathbf{Z})\backslash\mathbf{H})$) to a special value of a certain L-function. Using this identity, quantum unique ergodicity has recently been established ([I-S2, Sar4]) for "CM-forms" on $\Gamma(N)\backslash\mathbf{H}$ constructed by Maass and Hecke (see [Maa, He1, He2]). It has also been shown that improving Phragmen-Lindelöf "convexity" bound for a certain L-function implies quantum unique ergodicity for square-integrable eigenfunctions of Δ on $PSL_2(\mathbf{Z})\backslash\mathbf{H}$ (see [I-S2]). It has been recently shown ([W]) that on $PSL_2(\mathbf{Z})\backslash\mathbf{H}$, $\int\varphi_\lambda^3 \to 0$; Also, for certain special ("dihedral") Hecke eigenforms on an arithmetic hyperbolic manifold M, $\int|\varphi_\lambda|^4 \to 3/\mathrm{vol}(M)$ These are exactly the values predicted by the Random Wave conjectures!

For $M = PSL_2(\mathbf{Z})\backslash\mathbf{H}$, an upper bound for the rate of quantum ergodicity was improved in [L-S2, Jak3] from $o(\lambda)$ to $O_\varepsilon(\lambda^{1/2+\varepsilon})$ for any $\varepsilon > 0$; here $N(\lambda) = c\lambda$. This established what seems to be an optimal estimate for the rate of quantum ergodicity. Recently, the range of summation for holomorphic Hecke eigenforms in [L-S1] has been considerably shortened by Luo and Sarnak [L-S3].

It is known (see Section 7) that on an n-dimensional manifold, $||\varphi||_\infty = O(\lambda^{(n-1)/4})$ for $\Delta\varphi + \lambda\varphi = 0, ||\varphi||_2 = 1$. This bound was improved in [I-S1] for arithmetic hyperbolic surfaces, and in [Ko1] for compact arithmetic hyperbolic 3-manifolds. It was conjectured in [Sar2] that on negatively curved surfaces $||\varphi||_\infty = O(\lambda^\varepsilon)$ for any

[3]The existence of L^2 eigenfunctions of Δ for non-compact hyperbolic surfaces of finite area depends very strongly on whether surface is arithmetic or not; it is conjectured that L^2 eigenfunctions generally do not exist for non-arithmetic surfaces. We will not discuss this question here.

[4]without compact factors and with finite center

$\varepsilon > 0$. In [R-S] the authors construct a sequence φ_j of eigenfunctions ("theta-lifts") on a hyperbolic 3-manifold with $||\varphi_j||_\infty > C\lambda_j^{1/4}$.

While the presence of Hecke operators changes the behavior of the eigenvalues of Δ ([L-S1, B-L-S, St]), it doesn't seem to affect the behavior of the eigenfunctions ([Hej-R, A-T, A-B-S-T]). So, one might hope that the results obtained for joint eigenfunctions of Δ and the Hecke operators on arithmetic hyperbolic manifolds gives a better understanding of the eigenfunctions on other manifolds of negative curvature.

6. Integrable geodesic flows

The Hamiltonian for geodesic flow on a Riemannian manifold (M^n, g) is the function

$$H_0(x,\xi) = \frac{1}{2}|\xi|_g^2,$$

where, $|\xi|$ denotes the induced length of the covector $\xi \in T_x^*M$. The function H_0 is completely integrable if there exist $n-1$ distinct functions $H_i; i = 1, ..., n-1$, such that:

$$\{H_i, H_j\} = 0; \quad \forall i, j = 0, ..., n-1$$

and $dH_0, ..., dH_{n-1}$ are linearly independent on an open dense subset of T^*M. Integrability is very exceptional, but there are many important examples including geodesic flow on surfaces of revolution, hyperellipsoids, families of left-invariant metrics on Lie groups, families of nilmanifolds and Liouville metrics on tori[5] among many others (see [But], [Heck], [T1]). The Laplacian $\hat{H}_0 := \Delta$ is called *quantum completely integrable* (QCI) if there exist $n-1$ pseudodifferential operators $\hat{H}_1, ..., \hat{H}_{n-1}$ with principal symbols $H_1, ..., H_{n-1}$ above that satisfy:

$$[\hat{H}_i, \hat{H}_j] = 0; \quad \forall i, j = 0, ..., n-1.$$

It is a curious empirical fact that the known classically integrable geodesic flows all have QCI quantizations. However, at present no rigorous results have been proved along these lines. One can define a joint flow on T^*M as follows: For each $t = (t_0, ..., t_{n-1}) \in \mathbb{R}^n$ and $(x,\xi) \in T^*M$ we put

$$\Phi_t(x,\xi) = \phi_{0,t_0} \circ ... \circ \phi_{n-1,t_{n-1}}(x,\xi),$$

where $\phi_{i,t}$ denotes the flow of the Hamilton vector field of H_i for time t. This way, the cotangent space T^*M splits up into a union of orbits for the joint flow. These are either tori, cylinders or copies of $\mathbb{R}^n$. By a celebrated result of Arnold [Arn2], on a dense subset of T^*M the joint orbits are Lagrangian tori (which have dimension $= n$). The singular orbits (ie. compact orbits of dimension $< n$) can be easily shown to be ω- limit sets for the joint flow on a level set $\Sigma_b = \{(x,\xi) \in T^*M; H_0 = b_0, ..., H_{n-1}(x,\xi) = b_{n-1}\}$. Furthermore, in [T-Z1] it is proved that singular orbits always exist and that given such orbits $\gamma_1, ..., \gamma_l \subset \Sigma_b$ there exist a subsequence $(\psi_{\lambda_j})_{j=1}^\infty$ of (joint) eigenfunctions of $\hat{H}_0, ..., \hat{H}_{n-1}$ such that for some $\alpha_i \geq 0$ with $\sum_i \alpha_i = 1$,

$$\lim_{\lambda\to\infty} \langle Op(q)\psi_\lambda, \psi_\lambda\rangle = \sum_{i=1}^{l} \alpha_i \int_{\gamma_i} q. \tag{9}$$

[5]cf. [Bl-K-S, K-M-S]

Thus, in sharp contrast with ergodic flows, in the QCI case eigenfunctions *always* concentrate along singular orbits.

In addition, under suitable non-degeneracy and non-resonance assumptions on the singular orbits, Toth [T4] proves the following sharp rate of convergence for the eigenfunction mass in (9):

$$\langle \mathrm{Op}(q)\psi_\lambda, \psi_\lambda\rangle = \sum_i \alpha_i \int_{\gamma_i} q_i + O(|\log\lambda|^{-1/2}). \tag{10}$$

The proof of (10) hinges on a normal form for the commuting operators in a neighbourhood of any singular orbit. This "quantum Birkhoff normal form" is derived from a classical mechanical theorem of H. Ito [Ito] in the real-analytic case and the C^∞ analogue is obtained from Eliasson's normal forms; see [VuNg]. More precisely, a slight extension of Eliasson's result implies that under generic non-resonance conditions, there exists a neighbourhood, Ω, of the compact singular orbit, and a canonical map $\kappa : \mathbb{T}^k \times D \to \Omega$ with the property that, for all $i = 1, ..., n$,

$$H_i \circ \kappa = \sum_{j=1}^{H} F_{ij}\cdot I_j^h + \sum_{j=H+1}^{H+L+1} F_{ij}\cdot I_j^{ch} + \sum_{j=H+L+1}^{n-k} F_{ij}\cdot I_j^e + \sum_{j=n-k+1}^{n} \tag{11}$$

Here $I^h := (I_1^h, ..., I_H^h), I^{ch} := (I_{H+1}^{ch}, ..., I_{H+L+1}^{ch})$ and $I^e := (I_{H+L+2}^e, ..., I_{n-k}^e)$ denote the standard generators of the Cartan algebra on the Poincaré cross-section of the singular orbit, $I := (I_1, ..., I_k)$ the regular action coordinates and $D \subset \mathbb{R}^{n-k}$ a small ball around the origin. The F_{ij} Poisson-commute with all the action functions. At the operator level, there is a quantum version of this result [VuNg]:

Let $(b_0, ..., b_{n-1}) \in \mathbb{R}^n$ be a singular value of the moment map Φ and $\mathbb{R}^n \cdot v_0$ be a dimension k Eliasson non-degenerate orbit of the joint flow. Then, there exists a microlocally elliptic Fourier integral operator, F, and a microlocally invertible $n\times n$ matrix of pseudodifferential operators, $\mathcal{M}_{ij}$, and vector-valued C^∞ symbols $\epsilon^h(\hbar) \sim (\sum_{j=1}^\infty \epsilon_{1,j}^h \hbar^j, ..., \sum_{j=1}^\infty \epsilon_{H,j}^h \hbar^j), \epsilon^{ch}(\hbar) \sim (\sum_{j=1}^\infty \epsilon_{H+1,j}^{ch}\hbar^j, ..., \sum_{j=1}^\infty \epsilon_{L+H+1,j}^{ch}\hbar^j), \epsilon^e(\hbar) \sim (\sum_{j=1}^\infty \epsilon_{L+H+1,j}^{e}\hbar^j, ..., \sum_{j=1}^\infty \epsilon_{n-k,j}^{e}\hbar^j)$, and $\epsilon(\hbar) \sim (\sum_{j=1}^\infty \epsilon_{n-k+1,j}\hbar^j, ..., \sum_{j=1}^\infty \epsilon_{n,j}\hbar^j)$, such that

$$\begin{aligned} &F^{-1}(\hat{H}_1 - b_0, ..., \hat{H}_n - b_{n-1})\,F =_\Omega \\ &\mathcal{M}\cdot(\,\hat{I}^h - \epsilon^h(\hbar), \hat{I}^{ch} - \epsilon^{ch}(\hbar), \hat{I}^e - \epsilon^e(\hbar),\, \hbar D_\theta - \epsilon(\hbar)\,) + \mathcal{O}(\hbar^\infty). \end{aligned} \tag{12}$$

Here, $=_\Omega$ denotes microlocal equivalence on Ω and $\hat{I}_j^h = \hbar(D_{y_j}y_j + y_jD_{y_j})$, $\hat{I}_j^e = \hbar^2 D_{y_j}^2 + y_j^2$, $\hat{I}^{ch} = \hbar[(y_jD_{y_j} + y_{j+1}D_{y_{j+1}}) + \sqrt{-1}(y_jD_{j+1} - y_{j+1}D_{y_j})]$.

Since the singular orbit $\mathbb{R}^n\cdot v_0$ is an ω-limit set for the joint flow on $\mathcal{P}^{-1}(c)$ an application of the semiclassical Egorov Theorem implies that for (joint) eigenfunctions with (joint) eigenvalues sufficiently close (as $\hbar \to 0$) to $(b_0, ..., b_{n-1})$ the eigenfunction mass is concentrated in an arbitrarily small neighbourhood of $\mathbb{R}^n\cdot v_0$. The proof of (10) then reduces to a careful analysis of the explicit microlocal masses of the model (joint) eigenfunctions of $\hat{I}^h, \hat{I}^{ch}, \hat{I}^e$.

There are several important examples of degenerate integrable metrics where the results in (9) and (10) do not directly apply. These include:

6.1. **Spheres.** In [Jak-Z] it was shown that on $M = S^n$ with the standard metric, the set $\mathcal{U}$ contains all probability measures on S^*M invariant under the geodesic flow and under time reversal. If we consider real eigenfunctions, it is easy to show that only such measures can arise as limit points, so on S^n the set $\mathcal{U}$ is as large as it can possibly be.

In contrast, it was shown in [Z5, Van] that if we choose a basis of the space of spherical harmonics of degree N *at random* for all N, then for almost all such choices the resulting basis of eigenfunctions on S^n will be *quantum uniquely ergodic*.

On S^n one can define certain operators (similarly to (8)) acting on $L^2(S^n)$. It was shown in [Jak-Z] that for a fairly general choice of g_j-s, joint eigenfunctions of Δ and T cannot localize on a *single* closed geodesic. Vanderkam [Van] has improved upper bounds for the rate of growth of the L^∞ norms of joint eigenfunctions of Δ and *arithmetic* Hecke operators (using the techniques introduced in [I-S1]). He and Bourgain provided examples of a basis of $L^2(S^n)$ consisting of spherical harmonics whose L^∞ norms grow logarithmically; it is unknown whether there exists such a basis with *uniformly bounded* L^∞ norms.

6.2. **Tori.** We next discuss limits of eigenfunctions on flat tori $M = \mathbf{T}^n$. For $n = 2$, it is easy to see that for an eigenfunction $\varphi(x, y) = e^{2\pi i(mx+ny)}$ the corresponding measure $d\omega_\varphi$ on $S^*(\mathbf{T}^2)$ is *equal* to the delta-measure on the Liouville torus corresponding to geodesics in the direction $\theta = \arctan(n/m)$; this provides perhaps the easiest example of the localization discussed at the beginning of this section. The measure $d\mu_\varphi$ is just the Lebesgue measure $dxdy$ on $\mathbf{T}^2$.

However, on *arithmetic* flat tori $\mathbf{R}^n/\mathbf{Z}^n$ eigenvalues can have arbitrarily large multiplicities for $n \geq 2$, corresponding to more complicated measures $d\mu$.[6] Their limit set $\mathcal{L}$ was studied in [Jak1]; it is was shown that on $\mathbf{T}^2$ all limit measures in $\mathcal{L}$ have densities that are trigonometric polynomials.[7] Bourgain showed that for any n, any measure in $\mathcal{L}(\mathbf{T}^n)$ is absolutely continuous with respect to the Lebesgue measure; one can prove [Jak1] that the limiting densities on $\mathbf{T}^3$ have absolutely convergent Fourier series, while on $\mathbf{T}^4$ they are in $L^2(\mathbf{T}^4)$. The key ingredient in the proofs is a geometric lemma about co-dimension one simplices whose vertices are lattice points on spheres; its proof follows from the finiteness of the number of solutions of two simultaneous Pell's equations. That lemma relates the L^p properties of the densities of weak* limits on $\mathbf{T}^n$ (for $n \geq 4$) to possible generalizations (or lack thereof) to higher dimensions ($n-2$ and $n-3$) of the result of Cooke and Zygmund ([C1, C2, Zyg]) on the uniform bound on $||\varphi||_4/||\varphi||_2$ for the eigenfunctions φ on $\mathbf{T}^2$. It is unknown if a similar bound for $||\varphi||_p/||\varphi||_2$ exists for $p > 2, n \geq 4$; Bourgain ([Bo2]) has provided upper bounds on the rate of growth of for that ratio for $n \geq 4$ as $\lambda \to \infty$.

Let $g(x) = |\varphi(x)|^2$ (where φ is an eigenfunction on $\mathbf{T}^n$, and let $\hat{g}$ denote the corresponding Fourier transform. The first author conjectured in [Jak1] that for $n \geq 2$,

$$||\widehat{g}||_{l^n} \; < \; C(n)||\varphi||_2, \tag{13}$$

where $C(n)$ doesn't depend on the eigenvalue of φ.

Theorem 6.1. *The inequality* (13) *holds for* $n = 2, 3, 4$.

[6] For an eigenfunction φ with the eigenvalue λ, the density of $d\mu_\varphi$ is a trigonometric polynomial whose frequencies are chords connecting lattice points on the sphere of radius $\sqrt{\lambda}$ in $\mathbf{R}^n$.

[7] Their nonzero frequencies lie on (at most) *two* different circles centered at the origin.

For $n=2$, this is equivalent to the bound proved by Cooke and Zygmund; for $n=3$, the proof was given in in [Jak1]. The proof for $n=4$ is given below. Theorem 6.1 implies that any limiting density $h(x)=\sum_\tau c_\tau e^{2\pi i(x,\tau)}$ on $\mathbf{T}^n$ satisfies

$$\sum_{\tau\in\mathbf{Z}^n}|c_\tau|^{n-2} < \infty$$

for $4\le n\le 6$.

Proof of Theorem 6.1 for $n=4$. Let $\Delta\varphi_\lambda+\lambda\varphi_\lambda=0$ on $\mathbf{T}^n$, and let $S=S(n,\lambda)$ denote the sphere of radius $\sqrt{\lambda}$ centered at the origin in $\mathbf{R}^n$. Then

$$\varphi(x)=\sum_{\xi\in\mathbf{Z}^n\cap S}a(\xi)e^{2\pi i(x,\xi)},$$

where $a(\xi)\in\mathbf{C}, a(-\xi)=\overline{a(\xi)}, \sum_\xi|a(\xi)|^2=1$ (we consider real eigenfunctions with $||\varphi||_2=1$). Accordingly,

$$g(x)=1+\sum_{0\neq\tau\in\mathbf{Z}^n}b(\tau)e^{2\pi i(x,\tau)},\ b(\tau)=\sum_{\xi-\eta=\tau}a(\xi)\overline{a(\eta)},\qquad \xi,\eta\in\mathbf{Z}^n\cap S. \tag{14}$$

Without loss of generality, we may assume that $a(\xi)$-s are *real and positive*, and so $a(-\tau)=\overline{a(\tau)}=a(\tau)$. We would like to prove that

$$I:=\sum_\tau|b(\tau)|^n<P_n\left(\sum_{\xi\in\mathbf{Z}^n\cap S}a(\xi)^2\right)\equiv P_n(1), \tag{15}$$

where $P_n(y)$ is a fixed polynomial of degree n depending only on the dimension (and not on the radius of S).

Substituting the expression for b_τ into (15) we see that for $n=4$,

$$I=\sum_{0\neq\tau\in\mathbf{Z}^n}\left(\sum_{\xi_j-\eta_j=\tau}\prod_{j=1}^4 a(\xi_j)a(\eta_j)\right), \tag{16}$$

where $\xi_j,\eta_j\in\mathbf{Z}^n\cap S$.

Every term in (16) is a product of 8 factors, corresponding to eight lattice points on S; we shall separate them into two groups of *four* lattice points, and estimate the corresponding product trivially. The idea is to first change the order of summation, and then to divide 8 lattice points into groups of four in such a way that any 4-tuple $\{\zeta_1,\zeta_2,\zeta_3,\zeta_4\}$ appears only for a *bounded* number of 'frequencies' τ. We shall group the terms $\{\xi_1,\eta_1,\dots,\xi_4,\eta_4\}$ and $\{-\xi_1,\dots,-\eta_4\}$ together since the corresponding products are equal. Finally, we may assume that no two of ξ_j,η_j-s coincide or are diametrically opposite, since the contribution to (16) from those cases is easily estimated as in [Jak1].

Assume first that ξ_j-s *do not* lie in the same two-plane, and so form the vertices of a *fixed* tetrahedron $\mathcal{T}$. Estimate the corresponding term in (16) by

$$\frac{1}{2}\left[(a(\xi_1)a(\xi_2)a(\xi_3)a(\xi_4))^2+(a(\eta_1)a(\eta_2)a(\eta_3)a(\eta_4))^2\right].$$

There are at most *two* ways of putting $\mathcal{T}$ on S (since the latter is convex), so the estimate above is used for at most *two* vectors τ in (16), and those terms give a bounded contribution to that sum.

Now suppose that all ξ_j-s lie in *the same* two-dimensional affine subspace $\alpha\subset\mathbf{R}^4$, but one of $-\eta_j$s (say, $-\eta_4$) is *not* in α. Since the simplex $(\eta_1,\eta_2,\eta_3,\eta_4)$ is a parallel translate of $(\xi_1, \xi_2, \xi_3, \xi_4)$, the simplex $(\eta_1, \eta_2, \eta_3, -\xi_4)$ is a parallel translate of $(\xi_1, \xi_2, \xi_3, -\eta_4)$, and the latter two simplices are *not* planar! We are thus reduced to the previous case and can estimate the corresponding term in (16) by

$$\frac{1}{2}\left[(a(\xi_1)a(\xi_2)a(\xi_3)a(\eta_4))^2 + (a(\eta_1)a(\eta_2)a(\eta_3)a(\xi_4))^2\right].$$

Finally, assume that $-\eta_j \in \alpha$ for *all* $1 \le j \le 4$. Then $\xi_j \eta_j$ all lie on the same *circle* $C_1 = S \cap \alpha$. We can assume that $\eta_1 \notin C_1$ (otherwise, C_1 is a great circle, *alpha* contains the origin, $\tau = \xi_j - \eta_j \in \alpha$ and it is easy to see that two of ξ-s have to coincide, contradicting our assumption). Let β be the affine subspace containing both α and η_1. Then $-\xi_j \in \beta, \eta_j \in \beta$ for all j, and also $0 \in \beta$. Accordingly, $\tau = \xi_j - \eta_j \in \beta$. Let $C_2 = S \cap \beta$ be the two-sphere containing all $\pm\xi_j, \pm\eta_j$, and let $\mathcal{T}'$ be the triangle (ξ_1, ξ_2, ξ_3). There are at most *two* ways of putting $\mathcal{T}'$ on C_2 (since the latter is convex), so the for given ξ_j-s our assumptions can be satisfied for at most *two* vectors τ, hence the corresponding term in (16) can be estimated by

$$\frac{1}{2}\left[(a(\xi_1)a(\xi_2)a(\xi_3)a(\xi_4))^2 + (a(\eta_1)a(\eta_2)a(\eta_3)a(\eta_4))^2\right],$$

giving a bounded contribution to (16). We have considered all possible cases, and the proof is finished. $\square$

Finally, we mention an interesting result of Mockenhaupt ([Mo]), who constructed sequences φ_j of eigenfunctions on $\mathbf{T}^2$ (e.g. those with eigenvalues $\lambda_j = q^j$, where $q = 4m + 1$ is a fixed prime) such that $||\varphi_j||_\infty \to \infty$ but for all $p < \infty$, $||\varphi_j||_p < C(p) < \infty$. It is unknown whether there exists $p > 4$ such that $||\varphi||_p < C(p) < \infty$ holds for *all* eigenfunctions φ on $\mathbf{T}^2$.

7. L^p NORMS

The behavior of L^p norms of eigenfunctions of the Laplacian on Riemannian manifolds has been a subject of active investigations. The rate of growth of L^∞ norms was studied in [Hor, Dui-G, Vol, Iv]; in [Ber1, Don1] for negatively curved manifolds (see also [I-S1, R-S, Sar3, A-B-S-T]); and by Colin de Verdiére ([CV1, CV2]) and others for manifolds with integrable geodesic flows (see [Bo1, T2, T3, T-Z2, Van, Zyg]). Seeger and Sogge ([Se-So, So]) gave an improved upper bound for the rate of growth of L^p norms of eigenfunctions. See also [Gr1, Gr2, Sm-So] for L^p norm results on manifolds with boundary. The rate of growth for arithmetic hyperbolic manifolds was discussed in Section 5.1, and that for flat tori was discussed in Section 6.2.

There are also recent results on recurrence properties for the geodesic flow and L^∞ bounds for eigenfunctions: For each $x \in M$ put

$$L_x = \{\xi \in S^*_x M;\ \exp(t\xi) \cdot x = x \text{ for some } t \in \mathbb{R}\}. \tag{17}$$

Then, $x \in M$ is called recurrent if $L_x = S^*_x M$. In recent work, Sogge and Zelditch [S-Z] have shown that the universal Hörmander L^∞ eigenfunction bound $\mathcal{O}(\lambda^{\frac{n-1}{4}})$ is necessarily attained at recurrent points for the geodesic flow. The existence of recurrent points is however not sufficient to guarantee the existence of subsequences of normalized eigenfunctions with maximal sup-norm blow-up. There are however some very recent results of Toth and Zelditch [T-Z2] which provide a partial converse to the above result of Sogge and Zelditch in the quantum completely integrable case: Let Δ be QCI and assume that the geodesic flow is non-degenerate with Bott-Morse type singularities and assume that there exists a recurrent point $x \in M$ for the geodesic flow which is also a blow-down singularity. This means that S_x^*M is contained in a single leaf of the moment map, Φ; that is $S_x^*M \subset \Phi^{-1}(b)$. Then it is proved in [T-Z2] that there exists a subsequence of Laplace eigenfunctions with the property that for any $\epsilon > 0$,

$$\|\phi_\lambda\|_{L^\infty} \gg_\epsilon \lambda^{\frac{n-1}{4}-\epsilon}. \tag{18}$$

Moreover, it is shown that the maximizing eigenfunctions achieve this blow-up near the recurrent point. Thus, in such cases one does get arbitrarily close to the universal Hörmander bound. Examples here include the triaxial ellipsoid where the four umbilic points are recurrent.

Another natural question related to eigenfunction bounds is the following: What effect does extreme eigenfunction growth have on the geometry of the underlying manifold? In [T-Z1] it is shown that if Δ is QCI and the spectrum is of bounded multiplicity then:

$$\sup_{\lambda \in Spec(\Delta)} \|\phi_\lambda\|_{L^\infty} = \mathcal{O}(1) \tag{19}$$

implies that

$$(M, g) \text{ is a flat torus.}$$

Whether such an implication holds in general is not known at present. Under appropriate non-degeneracy and non-resonance conditions, it is proved in [T-Z2] that the result in (19) can be weakened somewhat: The condition that the L^2-normalized joint eigenfunctions, ϕ_λ, satisfy

$$\|\phi_\lambda\|_{L^\infty} = \mathcal{O}(\lambda^{1/8-\epsilon})$$

for some $\epsilon > 0$ already implies that (M, g) is flat torus. Finally, there are also recent related results of Donnelly [Don2] on the minimal blow-up of eigenfunctions in the case where the metric is not necessarily integrable but does admit an isometric $\mathbb{S}^1$-action. In such a case, he shows that there exists a subsequence of eigenfunctions that are $\Omega(\lambda^{1/8})$ in sup norm as $\lambda \to \infty$.

8. Quasi-symmetry properties of eigenfunctions

In view of the predictions of the random wave conjectures of quantum chaos (discussed in Section 5) it seems natural to investigate the relationship between positive and negative parts of real eigenfunctions on Riemannian manifolds. In particular, it seems natural to consider the ratio $||\varphi_+||_p/||\varphi_-||_p$ of the L^p norms of the positive part φ_+ and and the negative part φ_- of an eigenfunction φ. It was proved in [Nad4] that for $p = \infty$ this ratio is bounded away from zero and infinity by constants that depend only on the manifold (and *do not* depend on the eigenvalue). In the paper [Jak-Nad2] the first and the second authors extended this result to arbitrary $1 \le p < \infty$.[8] The exact value of C for $p = \infty$ was considered in [Kr]. Similar questions on S^n were considered in [Ar]. It is unclear whether $||\varphi_+||_p/||\varphi_-||_p \to 1$ as $\lambda \to \infty$ for $1 < p < \infty$ on a given manifold. One can show

[8]For $p = 1$ the ratio is equal to 1 since $\int \varphi = 0$.

([Jak-Nad2]) that zonal spherical harmonics provide an example of a sequence of eigenfunctions with $||\varphi_+||_\infty/||\varphi_-||_\infty > C > 1$.

The second author has also considered in [Nad4] the domains $M_+ = \{x \in M : \varphi(x) > 0\}$ and $M_- = \{x \in M : \varphi(x) < 0\}$ for real-analytic two-dimensional M, and proved that the ratio $\mathrm{Vol}(M_+)/\mathrm{Vol}(M_-)$ is is bounded away from zero and infinity by constants depending only on M. It would be very interesting to extend this result to a more general situation. Blum, Gnutzman and Smilansky in [Blu-Gn-Sm] obtained a random wave theory prediction for the rate of decay of the *variance* $(\mathrm{Vol}(M_+) - \mathrm{Vol}(M_-))^2$ as $\lambda \to \infty$ and compared it with numerical results for ergodic planar billiards.

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