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SPECTRA OF ELEMENTS IN THE GROUP RING OF SU(2)

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Abstract. We present a new method for establishing the "gap" property for finitely generated subgroups of SU(2), providing an elementary solution of Ruziewicz problem on S^2 as well as giving many new examples of finitely generated subgroups of SU(2) with an explicit gap. The distribution of the eigenvalues of the elements of the group ring $\mathbf{R}[\mathrm{SU}(2)]$ in the N-th irreducible representation of SU(2) is also studied. Numerical experiments indicate that for a generic (in measure) element of $\mathbf{R}[\mathrm{SU}(2)]$, the "unfolded" consecutive spacings distribution approaches the GOE spacing law of random matrix theory (for N even) and the GSE spacing law (for N odd) as $N \to \infty$; we establish several results in this direction. For certain special "arithmetic" (or *Ramanujan*) elements of $\mathbf{R}[\mathrm{SU}(2)]$ the experiments indicate that the unfolded consecutive spacing distribution follows Poisson statistics; we provide a sharp estimate in that direction.

1. Introduction

The irreducible representations of $G = \mathrm{SU}(2)$ are

$$\pi_N \;=\; \mathrm{sym}^N V, \qquad N \geq 0,$$

where V is the standard two-dimensional representation of G. The dimension of π_N is $N+1$ and it may be realized concretely by the linear action

$$(x,y) \;\to\; (\alpha x + \gamma y\;,\; \beta x + \delta y), \qquad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in G \tag{1}$$

on W_{N+1}, the space of homogeneous polynomials in (x,y) of degree N. The character χ_N of π_N at

$$g \;=\; \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \quad \text{is} \quad \frac{\sin(N+1)\alpha}{\sin\alpha}. \tag{2}$$

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This well known character formula is our main tool for studying the spectra of the elements in the group ring, $\mathbf{R}[G]$. Let z be such an element,

$$z = \sum_{g\in G} z_g g \tag{3}$$

with $z_g \in \mathbf{R}$ and $z_g = 0$ for all but a finite number of $g \in G$. Let $\mathrm{supp}(z) = \{g \in G | z_g \neq 0\}$ and Γ_z the group generated by $\mathrm{supp}(z)$. Clearly Γ_z is a finitely generated subgroup of G.

For $\pi = \pi_N$ as above let $\widehat{z}(\pi)$ be the $(N+1)\times(N+1)$ matrix

$$\widehat{z}(\pi) = \sum_{g\in G} z_g \pi(g) \tag{4}$$

We are interested in the spectrum, $\mathrm{spec}(\widehat{z}(\pi_N))$, of $\widehat{z}(\pi_N)$ (which of course does not depend on the realization of π_N) as $N\to\infty$. This question is of interest in analysis (for example the Ruziewicz problem see [CV], [LPS], [Lub], and "strongly ergodic actions" [Sch]) as well as in the theory of quantization ("quantum chaos" [Sa2], [Zel2]). The connection to the latter comes from Γ_z acting on S^2 (using the double cover $\mathrm{SU}(2)\to\mathrm{SO}(3)$) as symplectic maps and an N-th quantization ("$N = 1/h$") of this classical action being realized by $\widehat{z}(\pi_N)$ [Zel1]. According to general philosophies in quantization theory [BGS] one might expect that if Γ_z acts ergodically on S^2, $\mathrm{spec}(\widehat{z}(\pi_N))$ behaves like the spectrum of a typical member of an appropriate ensemble of random matrices (cf. [M]). With suitable interpretations, this expectation is borne out rather well in the numerical experiments described in Section 5. This appears to be one of the simplest examples of this phenomenon.

We assume further that z is self-adjoint, i.e. $z^* = \sum \bar{z}_g g^{-1} = z$. In this case $\widehat{z}(\pi_N)$ is self-adjoint with respect to a suitable inner product on W_{N+1}. Hence $\mathrm{spec}(\widehat{z}(\pi_N))$ is real and is contained in the interval $[-||z||, ||z||]$, where $||z|| = \sum_{g\in G}|z_g|$. Let $||\widehat{z}(\pi_N)||$ be the norm of the matrix $\widehat{z}(\pi_N)$ with respect to the above inner product. $||\widehat{z}(\pi_N)||$ is equal to the maximum of $|\lambda_j(\widehat{z}(\pi_N))|$ where

$$\lambda_0(\widehat{z}(\pi_N)) \geq \lambda_1(\widehat{z}(\pi_N)) \geq \ldots \geq \lambda_N(\widehat{z}(\pi_N)) \tag{5}$$

are the eigenvalues of $\widehat{z}(\pi_N)$.

A fundamental question for a positive z (i.e. $z_g \geq 0$ for $g \in G$) is whether it has a *gap* by which we mean that

$$\overline{\lim}_{N\to\infty} ||\widehat{z}(\pi_N)|| < ||z||. \tag{6}$$

Constructing any z with a gap is apparently rather difficult and was first achieved by Drinfeld [Dri]. It was used by him to give the final step in the resolution of Ruziewicz problem asserting that the Haar measure S^2 is the unique *finitely* additive rotation invariant measure defined on Lebesgue sets. Drinfeld's method appeals to some sophisticated machinery from the theory of automorphic representations (namely the Jacquet Langlands correspondence and Deligne's solution of the Ramanujan conjectures). One of our aims here is to give a new elementary and perhaps more importantly robust analytic method for establishing that certain z's have a gap. In particular we construct new z's with this property and thus in passing provide an elementary analytic solution of the Ruziewicz problem. The discussion in Section 3 reduces the issue of a gap to estimating the number of elements in Γ_z (ordered suitably with word length) with trace near 2. This estimation can be carried out easily for z's constructed via integers, on the other hand doing so for the case of "generic" z (see below) has thus far eluded us. So let $H(\mathbf{Z})$ denote the ring of Hamilton quaternions $\alpha = x_0 + x_1 i + x_2 j + x_3 k, x_j \in \mathbf{Z}$. Let $\bar{\alpha} = x_0 - x_1 i - x_2 j - x_3 k$ and $N(\alpha) = \alpha\bar{\alpha}$. For $q \geq 3$ a prime number let $\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k$ be a subset of $S = \{\alpha \in H(\mathbf{Z}) | N(\alpha) = q\}$ (it is well known [HW] that the latter has $8(q+1)$ elements) satisfying

(I) $\tilde{g}_{j_1} \neq \varepsilon \tilde{g}_{j_2}$ for $j_1 \neq j_2$ and $\varepsilon \in \{\pm 1, \pm i, \pm j, \pm k\}$ a unit.
(II) $\tilde{g}_{j_1} \neq \varepsilon \overline{\tilde{g}_{j_2}}$ for any j_1, j_2 and ε a unit.

The homomorphism of $H(\mathbf{R})$ into SU(2)

$$\alpha \;\rightarrow\; \frac{1}{N(\alpha)} \begin{bmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{bmatrix}$$

gives us the corresponding elements $g_1, g_2, \ldots, g_k \in G$.

THEOREM 1. *Let $q \geq 3$, $g_1, g_2, \ldots, g_k \in G$ be as above. If $p = 2k - 1 > q^{4/5}$ then $z = g_1 + g_1^{-1} + \ldots + g_k + g_k^{-1}$ has a gap, in fact*

$$\overline{\lim}_{N\to\infty} ||\widehat{z}(\pi_N)|| \;\leq\; p^{1/2} \left(\frac{q^{2/3}}{p^{1/3}} + \frac{p^{1/3}}{q^{2/3}} \right) \;<\; 2k$$

For example if $q = 7$,

$$\tilde{g}_1 = 2 - i + j + k,\ \tilde{g}_2 = 2 - i - j + k,\ \tilde{g}_3 = 2 + i - j + k$$

satisfy the hypotheses and denoting the corresponding z by $\tilde{z}_7$ we have

$$\overline{\lim}_{N\to\infty} ||\widehat{\tilde{z}}_7(\pi_N)|| \;\leq\; 5.83\ (< 6).$$

Remarks:

(1) If $q \equiv 1(\mathrm{mod} 4)$ and we choose a maximal such subset $\tilde{g}_1, \ldots, \tilde{g}_k$ of S above, then $2k - 1 = q$ and we get an element z which we denote z_q which according to the above theorem satisfies

$$\overline{\lim}_{N\to\infty} ||\widehat{z_q}(\pi_N)|| \;\leq\; q^{1/2} \left(q^{1/3} + q^{-1/3} \right) \;<\; q + 1$$

For this element it was shown in [LPS] using automorphic forms and in particular Deligne's proof of the Ramanujan Conjectures that

$$||\widehat{z_q}(\pi_N)|| \;\leq\; 2\sqrt{q} \qquad \mathit{for\ all}\ N \geq 1.$$

This bound is optimal (see Section 4). We call a general element z of the form $g_1 + g_1^{-1} + \ldots + g_k + g_k^{-1}$ (where $k \geq 2$) for which $||\widehat{z}(\pi_N)|| \leq 2\sqrt{2k-1}$ holds when $N \geq 1$, a *Ramanujan element.*

(2) For $z \neq z_q$ as above (i.e. $p < q$) the existing methods from automorphic forms do not, as far as we know, apply to show that z has a gap. In more detail, it is clear that in general z having a gap is a property of Γ_z. The method of Drinfeld [Dri] for establishing a gap ensures that if Γ_z contains an arithmetic group then z has a gap (by an arithmetic subgroup Λ of G we mean a group obtained from a quaternion algebra as follows; let $K/\mathbf{Q}$ be a number field and D/K a quaternion algebra. Assume that K has an archimedean place v such that $D \otimes K_v$ is the Hamiltonian quaternions. In particular $D^{(1)} \otimes K_v$, where the super 1 denotes elements of norm 1, is isomorphic to SU(2). Let $\mathcal{P}$ be a (nonempty) set of places containing the other archimedean places of K and let Γ be the group of elements of $D^{(1)}(K)$ which are integral outside $\mathcal{P}$ and let Γ^v be the projection of Γ in $D^{(1)} \otimes K_v \simeq \mathrm{SU}(2)$. We call $\Lambda \subset \mathrm{SU}(2)$ arithmetic if it is commensurable with a conjugate of such a Γ^v). For $p < q$, Γ_z does not contain an arithmetic group. The element $\tilde{z}_7$ is an example of the above and as numerical computation shows, see Section 5, it is not Ramanujan. We remark that Shalom [Sha] has given (in another context) Γ-s with the analogue of a gap, which are constructed as commutator subgroups of arithmetic groups.

Next we discuss the distribution of the eigenvalues (5) as $N \to \infty$. This requires distinguishing the case N even and N odd, since the corresponding π_N has a different symmetry in each case. As is well known and discussed in Section 2, for N even π_N preserves a symmetric bilinear form on W_{N+1} while for N odd it preserves a skew-symmetric form. It follows that with suitable bases for W_{N+1}, for N even $\widehat{z}(\pi_N)$ lies in $\mathcal{H}_{N+1}$ the (real) linear space of $(N+1) \times (N+1)$ real-symmetric matrices, while for N odd $\widehat{z}(\pi_N)$ lies in the (real) linear space, also denoted $\mathcal{H}_{N+1}$, of $(N+1) \times (N+1)$ matrices H satisfying

$$H^* = H, \; J^t H J = H^t, \; J = \begin{bmatrix} E & 0 & \dots & 0 \\ 0 & E & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E \end{bmatrix}, \; E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{7}$$

In particular in this latter case the eigenvalues of $\widehat{z}(\pi_N)$ are of the form $\lambda_1, \lambda_2, \dots, \lambda_M$ where $M = (N+1)/2$ with each λ_j occuring with multiplicity 2.

While our main interest lies in $\mathrm{spec}(\widehat{z}(\pi_N))$ as $N \to \infty$ we prove in Section 2, see Proposition 1, that if

$$z \;=\; g_1 + g_1^{-1} + \dots + g_k + g_k^{-1} \tag{8}$$

with $(g_1, \dots, g_k) \in G^{(k)}$ chosen at random according to Haar measure, and if k and N *both* go to infinity then local statistics of the eigenvalues of $\widehat{z}(\pi_N)$ follow GOE statistics for N even and GSE statistics for N odd (see Section 2 for the definitions of these ensembles).

Returning to z fixed and $N \to \infty$ we examine the densities of the eigenvalues of $\widehat{z}(\pi_N)$. These are described by the sum of point masses

$$\mu_N(z) \;=\; \frac{1}{N+1}\sum_{j=0}^{N}\delta_{\lambda_j(\widehat{z}(\pi_N))} \tag{9}$$

which is a probability measure supported in $[-||z||, ||z||]$.

It is not difficult to show (see Section 4 and [LPS], as well as [Sa1] and [Ser] for analogous equidistribution results) that there are measures $\nu^{even}(z)$ and $\nu^{odd}(z)$ which are determined entirely by the abstract group Γ_z and its generators (that is supp(z)), such that $\mu_{2N}(z) \to \nu^{even}(z)$ and $\mu_{2N+1}(z) \to \nu^{odd}(z)$, as $N \to \infty$. We call $\nu^{even}(z)$ and $\nu^{odd}(z)$ the *densities of states.* The finer question as to how this equidistribution takes place depends on supp(z) and how it sits in G. We say that $g_1, \dots, g_k \in G$ are *diophantine* if there is a $B = B(g_1, \dots, g_k) > 0$ such that for $m \geq 1$ and R_m a word in $g_1, \dots, g_k$ of length m and $R_m \neq \pm e$, we have

$$||R_m \pm e|| \;\geq\; B^{-m}. \tag{10}$$

Here

$$\left|\left|\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right|\right|^2 \;=\; |a|^2+|b|^2+|c|^2+|d|^2.$$

It is not hard to show (see Proposition 3) that if $g_1, \dots, g_k \in M_2(\overline{\mathbf{Q}})$, that is to say they have algebraic number entries, then $g_1, \dots, g_k$ are diophantine. On the other hand, the topologically generic (i.e. in the sense of Baire category) $g_1, \dots, g_k \in G^{(k)}$ is not diophantine.

THEOREM 2. *Assume that* supp(z) *is diophantine and that* $\nu^{even}(z) = f^{(even)}(x)dx$*,* $\nu^{odd}(z) = f^{(odd)}(x)dx$ *with* $f^{(even)}, f^{(odd)}$ *in* $L^\infty(\mathbf{R})$*, then for* N *large*

$$D(\mu_{N_j}(z), \nu^{even}(z)) \;\ll_z\; \frac{1}{\log N}$$

and

$$D(\mu_N(z), \nu^{odd}(z)) \;\ll_z\; \frac{1}{\log N}$$

(here $D(\nu,\mu)$ is the discrepancy between the measures ν and μ, that is $D(\nu,\mu) \;=\; \sup\{|\nu(I)-\mu(I)| : I = [a,b] \subset \mathbf{R}\}$).

Theorem 2 is false for the topologically generic $z = g_1 + g_1^{-1} + \dots + g_k + g_k^{-1}$, $(g_1, \dots, g_k) \in G^{(k)}$. On the other hand the numerical results of Section 5 suggest that for $k \geq 2$ fixed and z_g chosen with $g \in G^{(k)}$, generic in the measure sense, the discrepancy should behave like those for random matrix models, that is it should be $O((\log N)/N)$, see Figure 1 which displays $\mu_N(z)$ against $\nu(z)$ for such a random z. So for such z's the result in Theorem 2 is probably very far from the truth. For the Ramanujan element z_q (defined in Remark 1) the discrepancy is not small, see Figure 2 for a similar comparison. Note that it has a very large centralizer in $\mathbf{R}[G]$, in fact the z_q's all commute with each other. So the eigenvalues of $\widehat{z}_q(\pi_N)$

should not be expected to behave like any random matrix ensemble. In fact experience with numerics for the spectra of Laplacians for arithmetic hyperbolic surfaces [BGGS], [BSS] suggest that spec($\widehat{z}_q(\pi_N)$) is more likely to behave like random numbers (i.e. "Poisson"). The following lower bound confirms this and is the analogue of the lower bounds for the remainder term in Weyl's law for arithmetic hyperbolic surfaces, see [Hej] and [LS].

THEOREM 3. *Fix $q \geq 3$ and z_q as above, a Ramanujan element. There is a sequence $N_j \to \infty$ such that*

$$D(\mu_{\pi_N}(z_{S_p}), \nu(z)) \gg \frac{1}{N_j^{1/2}(\log N_j)^2}$$

In Section 5 we report on numerical experiments concerning the "unfolded" spacing distribution between the eigenvalues (5) as N gets large. Consider the generic in measure $z = z_g = g_1 + g_1^{-1} + \ldots + g_k + g_k^{-1}$, $g = (g_1, \ldots, g_k) \in G^{(k)}$. For $k = 2$ the centralizer of $z \in \mathbf{R}[G]$ consists of more than just the polynomial ring in z, $\mathbf{R}[z]$. Indeed by conjugating such an element by a suitable member of G we can assume that

$$g_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}, \quad g_2 = \begin{bmatrix} a_1 & b_1 \\ -\bar{b_1} & \bar{a_1} \end{bmatrix}.$$

Put $c = (-b_1/\bar{b_1})^{1/2}$. Then

$$\begin{bmatrix} 0 & c \\ -\bar{c} & 0 \end{bmatrix}$$

commutes with z. That is for $k = 2$ there is a persistent symmetry of order 2 (note that this symmetry is not broken by passing to $z = x(g_1 + g_1^{-1}) + y(g_2 + g_2^{-1})$ with $x, y \in \mathbf{R}$). However for $k \geq 3$ and $g = (g_1, \ldots, g_k)$ generic it appears that the centralizer of z in $\mathbf{R}[G]$ is no bigger than $\mathbf{R}[z]$. In particular the generic such z is "desymmetrized" and we examine the consecutive spacing distribution between the eigenvalues of $\widehat{z}(\pi_N)$. For such a random z with $k = 3$ we found universally that these spacings followed the GOE spacing law for N going to infinity through even values and GSE spacing laws for N odd. Figures 3 and 4 give typical examples of this phenomenon. On the other hand the same question for the Ramanujan element z_5 yielded a Poissonian consecutive spacing distribution (after the obvious symmetries have been taken into account) - see Figures 5 and 6 in Section 5. The element $\tilde{z}_7$ (see the example after Theorem 1) is special in that it is formed out of integral quaternion matrices though $\Gamma_{\tilde{z}_7}$ is not arithmetic. We found that its spacings (again after a symmetry analysis) are GOE and GSE depending on the parity. A typical example of this data is given in Figure 7. For details about the data and computations see Section 5.

Note that for the random $g \in G^{(k)}$, $\Gamma = \langle g_1, \ldots, g_k \rangle$ acts ergodically on S^2. These results point to the conjecture that for $k \geq 3$ fixed and $g \in G^{(k)}$ given and generic (in measure) the eigenvalues of the quantization $\widehat{z}_g(\pi_N)$)

follows GOE spacing statistics as N goes to infinity through even values, while they follow GSE statistics as N goes to infinity through odd values. Besides this conjecture (which if true is no doubt difficult to prove) there are a number of more tractable problems that remain. We mention some of these:

(1) Prove that for $k \geq 2$ and $g \in G^{(k)}$ generic in measure, $z_g = g_1 + g_1^{-1} + \ldots + g_k + g_k^{-1}$ has a gap (see [LPS]).
(2) With our present knowledge it is conceivable that z_g has a gap as long as Γ_{z_g} acts ergodically on S^2 (cf. [LW]).
(3) Consider the parameter space ($k \geq 2$, fixed) $G^{(k)}/\sim$, where z_g is as above and $z_g \sim z_{\tilde{g}}$ if $\delta(g_1, \ldots, g_k)\delta^{-1} = (\tilde{g}_1, \ldots, \tilde{g}_k)$ with $\delta \in G$. What can be said about the Ramanujan elements $z \in G^{(k)}/\sim$? Is this set finite or infinite? One can show it is closed and nowhere dense (cf. [LPS, Theorem 1.4]; in fact its intersection with any 1-dimensional curve is closed and nowhere dense in that curve).
(4) Prove that the generic $(g_1, \ldots, g_k) \in G^{(k)}$ in the sense of measure is diophantine.
(5) One can ask similar questions for $\mathbf{R}[G]$ where G is a more general compact topological group or for $\mathbf{R}[\Gamma]$ relative to a family π_N of unitary finite dimensional representations of Γ arising from geometry or quantization.

2. Properties of π_N

With notation as in Section 1 let $e_0, e_1, \ldots e_N$ be the basis of W_{N+1} given by

$$e_j \;=\; x^j y^{N-j}, \qquad j = 0, 1, \ldots, N. \tag{11}$$

The bilinear form $\langle , \rangle$ on W_{N+1} defined by

$$\langle e_j, e_k \rangle \;=\; \binom{N}{j}^{-1} (-1)^j \delta_{j,N-k} \tag{12}$$

is nondegenerate and is preserved by π_N. That is $\langle \pi_N(g)v, \pi_N(g)w \rangle = \langle v, w \rangle$ for $g \in G$ and $v, w \in W_{N+1}$. The form is symmetric for N even and skew-symmetric for N odd. It follows that π_N is orthogonal for N even and symplectic for N odd. Hence with a suitable basis (over $\mathbf{C}$) we can assume that that π_N preserves

$$I_{N+1} \;=\; \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

for N even and

$$J = \begin{bmatrix} E & 0 & \dots & 0 \\ 0 & E & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for N odd. That is

$$\begin{cases} (\pi_N(g))^t \pi_N(g) = I_{N+1}, & N \textit{ even}; \\ (\pi_N(g))^t \, J \, \pi_N(g) = J, & N \textit{ odd}. \end{cases} \tag{13}$$

Furthermore for $g \in \mathrm{SU}(2)$ we can arrange for $\pi_N(g)$ to be unitary with respect to the standard Hermitian form on W_{N+1}. Thus for N even $\pi_N(g) \in \mathrm{O}(N+1, \mathbf{R})$ while for N odd $\pi_N(g) \in \mathrm{USp}(N+1)$.

If N is even $(\pi_N(g) + \pi_N(g^{-1}))^t = \pi_N(g) + \pi_N(g^{-1})$. It is also selfadjoint hence $\pi_N(g) + \pi_N(g^{-1})$ is a real symmetric matrix. It follows that for any $z \in \mathbf{R}[G]$ which is selfadjoint, $\widehat{z}(\pi_N)$ lies in the real linear space $\mathcal{H}_{N+1}$ of $(N+1) \times (N+1)$ real symmetric matrices.

For N odd, $J^t(\pi_N(g) + \pi_N(g^{-1}))J = (\pi_N(g) + \pi_N(g^{-1}))^t$. Also $\pi_N(g) + \pi_N(g^{-1})$ is selfadjoint so we see that $\widehat{z}(\pi_N)$ lies in the *real* linear space $\mathcal{H}_{N+1}$ of $(N+1) \times (N+1)$ complex matrices satisfying

$$J^t H J = H^t \quad \text{and} \quad H^* = H. \tag{14}$$

It is easily seen that any $H \in \mathcal{H}_{N+1}$, $(N+1 = 2M)$ has its eigenvalues consist of M eigenvalues each of multiplicity 2 (of course some of these M eigenvalues may coincide).

For the generic $(g_1, g_2, \dots, g_k) \in G^k$ (by which we mean here outside a countable union of codimension 1 subsets), $k \geq 2$, $z(g_1, g_2, \dots, g_k) = g_1 + g_1^{-1} + g_2 + g_2^{-1} + \dots + g_k + g_k^{-1}$ has the property that $\widehat{z}(\pi_N)$ has simple spectrum if N is even amd has $M = (N+1)/2$ distinct eigenvalues each of multiplicity 2 if N is odd. To see this it suffices from general principles involving the discriminant of a matrix, to exhibit one such $(g_1, g_2, \dots, g_k)$. For $\alpha \in \mathbf{R}$ and $\mathrm{e}(\alpha) = e^{2\pi i \alpha}$ let $z(\alpha) \in \mathbf{R}[G]$ be given by

$$z(\alpha) = \begin{bmatrix} \mathrm{e}(\alpha) & 0 \\ 0 & \mathrm{e}(-\alpha) \end{bmatrix} + \begin{bmatrix} \mathrm{e}(-\alpha) & 0 \\ 0 & \mathrm{e}(\alpha) \end{bmatrix} + (k-1) \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right).$$

A straightforward calculation yields

$$\mathrm{spec}(\widehat{z(\alpha)}(\pi_N)) =$$

$$\left\{ 2\cos N\alpha \pm (2k-2), 2\cos(N-2)\alpha \pm (2k-2), \dots, 2 + (-1)^{N/2}(2k-2) \right\}$$

if N is even and

$$\begin{aligned} \mathrm{spec}(\widehat{z(\alpha)}(\pi_N)) &= \{2\cos N\alpha, 2\cos N\alpha, \\ & 2\cos(N-2)\alpha, 2\cos(N-2)\alpha, \dots, 2\cos\alpha, 2\cos\alpha\} \end{aligned}$$

if N is odd.

Hence if we choose α generically (e.g. take e(α) to be transcendental) then $z(\alpha)$ will satisfy the claimed property. Note that this $z(\alpha)$ fails to have a gap. The support group $\Gamma_{z(\alpha)}$ is a dihedral group and the densities of states $\nu^{even}(z)$ and $\nu^{odd}(z)$ are easily determined from the explicit description of the spectrum: $\nu^{odd}(z(\alpha)) = dy/\sqrt{4-y^2}$ on $[-2,2]$ while $\nu^{even}(z(\alpha))$ is half of this measure translated by $\pm 2(k-2)$.

While our interest lies in spec($\widehat{z}(\pi_N)$) as $N \to \infty$ for z fixed, we note that if we let k go to infinity as well then the spectrum of the generic $\widehat{z}(g_1, g_2, \dots, g_k)(\pi_N)$ will follow the GOE laws (see [M] for definitions) if N is even and the GSE laws if N is odd.

PROPOSITION 1. *Let $\nu_{N,k}$ be the direct image of $dg_1 dg_2 \dots dg_k$ on $G^{(k)}$ under the map*

$$(g_1, g_2, \dots, g_k) \;\to\; (\frac{1}{\sqrt{k}}(g_1 + g_1^{-1} + \dots + g_k + g_k^{-1}))\hat{}\,(\pi_N)$$

Thus $\nu_{N,k}$ is a probability measure on $\mathcal{H}_{N+1}$. As $k \to \infty$, $\nu_{N,k}$ converges in measure to the standard GOE measure on $\mathcal{H}_{N+1}$ if N is even and to the standard GSE measure if N is odd.

Proof: The measure $\nu_{N,k}$ on the real vector space $\mathcal{H}_{N+1}$ is a sum of i.i.d. random variables. We may therefore appeal to the general vector valued central limit theorem. The distribution of the individual summand is $H(g) = \pi_N(g) + \pi_N(g^{-1}), g \in G$. Thus once we show that $\int_G H(g)dg = 0$ we know that the limit of $\nu_{N,k}$ as $k \to \infty$ is a Gaussian. The issue is to identify this Gaussian and where it is supported.

We begin with the case that N is even. $H(g) = (h_{ij}(g))$ is real symmetric so we can consider the $(N+1)(N+2)/2$ dimensional space $h_{ij}, 1 \le i \le j \le N+1$, i.e. $\mathcal{H}_{N+1}$. We assert that

$$\int_G h_{ij}(g)\, dg \;=\; 0 \tag{15}$$

for $1 \le i \le j \le N+1$ and

$$\int_G h_{ij}(g) h_{rs}(g)\, dg \;=\; \frac{2}{N+1}\,(\delta_{is}\delta_{jr} + \delta_{ir}\delta_{js}) \tag{16}$$

To see this recall that since π_N is irreducible we have from Schur's Lemma [H] (assuming $N \ge 1$) that

$$\int_G 1 \cdot \pi_N(i,j)(g) dg \;=\; 0 \tag{17}$$

and

$$\int_G \pi_N(i,j)(g)\pi_N(m,n)(g^{-1})dg \;=\; \frac{\delta_{in}\delta_{jm}}{N+1} \tag{18}$$

for any $1 \le i, j, m, n \le N+1$.

Thus (15) follows from (17) while (16) from (18) together with the fact that $\pi_N(g)$ is orthogonal and hence

$$\int_G \pi_N(i,j)(g)\pi_N(r,s)(g)dg = \int_G \pi_N(i,j)(g)\pi_N(s,r)(g^{-1})dg = \frac{\delta_{ir}\delta_{js}}{N+1}.$$

The equalities (15) and (16) identify the covariance-matrix for the limiting Gaussian as being:

$$C_N e^{-\frac{N+1}{4}(h_{11}^2+\ldots+h_{N+1,N+1}^2)-\frac{N+1}{2}(\sum_{1\le i<j\le N+1} h_{ij}^2)} \prod_{1\le i\le j\le N+1} dh_{ij} =$$

$$C_N e^{-\frac{N+1}{4}\mathrm{tr}(H^2)} \prod_{i\le j} dh_{ij}$$

This Gaussian is $\mathrm{O}(N+1)$ invariant on $\mathcal{H}_{N+1}$ and is exactly the GOE measure on $\mathcal{H}_{N+1}$, see [M, p. 39].

The case that N is odd is a little more complicated. The linear space over $\mathbf{R}$ of $(N+1)\times(N+1) = 2M \times 2M$ matrices H satisfying (14) takes the form

$$H = \begin{bmatrix} x_{11} & 0 & z_{12} & w_{12} & \cdots & z_{1M} & w_{1M} \\ 0 & x_{11} & -\overline{w_{12}} & \overline{z_{12}} & \cdots & -\overline{w_{1M}} & \overline{z_{1M}} \\ & & x_{22} & 0 & & & \\ & & 0 & x_{22} & & & \\ \vdots & & & & \ddots & & \vdots \\ z_{M1} & w_{M1} & & & & x_{MM} & 0 \\ -\overline{w_{M1}} & \overline{z_{M1}} & & & \cdots & 0 & x_{MM} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1M} \\ A_{21} & A_{22} & \ldots & A_{2M} \\ \vdots & & \ddots & \vdots \\ A_{M1} & A_{M2} & \ldots & A_{MM} \end{bmatrix}$$

where

$$A_{jj} = \begin{bmatrix} x_{jj} & 0 \\ 0 & x_{jj} \end{bmatrix}, \qquad x_{jj} \in \mathbf{R}$$

for $1 \le j \le M$ and

$$A_{ij} = \begin{bmatrix} z_{ij} & w_{ij} \\ -\overline{w_{ij}} & \overline{z_{ij}} \end{bmatrix}, \qquad z_{ij} = x_{ij} + \sqrt{-1}y_{ij},\ w_{ij} = u_{ij} + \sqrt{-1}v_{ij}$$

for $1 \le i < j \le M$.

Hence $\mathcal{H}_{N+1}$ is spanned over the reals by

$$x_{jj}, \qquad j = 1, \ldots, M$$

and

$$x_{ij}, y_{ij}, u_{ij}, v_{ij}, \qquad 1 \le i < j \le M.$$

It is therefore of dimension $M(2M-1)$. Again $\nu_{N,k}$ is a sum of i.i.d. random variables in $\mathcal{H}_{N+1}$, the individual summand being $H(g) = \pi_N(g) + \pi_N(g^{-1}), g \in G$ as before. The equalities (17) and (18) continue to hold since π_N is irreducible so $\int_G H(g)dg = 0$. Thus $\nu_{N,k}$ converges to a Gaussian on $\mathcal{H}_{N+1}$ and we need to identify the covariance matrix. The relation (13) yields

$$
\begin{aligned}
(\pi_N(g))^t \; &= \begin{bmatrix} a_{11}(g) & b_{11}(g) & \dots & a_{1M}(g) & b_{1M}(g) \\ c_{11}(g) & d_{11}(g) & \dots & c_{1M}(g) & d_{1M}(g) \\ \vdots & & \ddots & & \vdots \\ a_{M1}(g) & b_{M1}(g) & \dots & a_{MM}(g) & b_{MM}(g) \\ c_{M1}(g) & d_{M1}(g) & \dots & c_{MM}(g) & d_{MM}(g) \end{bmatrix}^t \qquad (19) \\
&= \begin{bmatrix} d_{11}(g^{-1}) & -c_{11}(g^{-1}) & \dots & d_{1M}(g^{-1}) & -c_{1M}(g^{-1}) \\ -b_{11}(g^{-1}) & a_{11}(g^{-1}) & \dots & -b_{1M}(g^{-1}) & a_{1M}(g^{-1}) \\ \vdots & & \ddots & & \vdots \\ d_{M1}(g^{-1}) & -c_{M1}(g^{-1}) & \dots & d_{MM}(g^{-1}) & -c_{MM}(g^{-1}) \\ -b_{M1}(g^{-1}) & a_{M1}(g^{-1}) & \dots & -b_{MM}(g^{-1}) & a_{MM}(g^{-1}) \end{bmatrix}.
\end{aligned}
$$

¿From (19) one sees easily that the covariance matrix of $H(g) \in \mathcal{H}_{N+1}$ is diagonal. Its diagonal entries may be computed using (17), (18) and (19):

$$
\begin{aligned}
\int_G x_{jj}^2(g)\, dg \;&=\; \int_G (a_{jj}(g) + a_{jj}(g^{-1}))(a_{jj}(g) + a_{jj}(g^{-1}))\, dg \\
&= \frac{2}{N+1} \;+\; 2\int_G a_{jj}^2(g)\, dg \\
&= \frac{2}{N+1} \;+\; 2\int_G a_{jj}(g) d_{jj}(g^{-1})\, dg \\
&= \frac{2}{N+1}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\int_G z_{ij}(g)\overline{z_{ij}}(g)\, dg \;&=\; \frac{2}{N+1}, \\
\int_G z_{ij}^2(g)\, dg \;&=\; \int_G \overline{z}_{ij}^2(g)\, dg \;=\; 0. \\
\int_G w_{ij}(g)\overline{w_{ij}}(g)\, dg \;&=\; \frac{2}{N+1}, \\
\int_G w_{ij}^2(g)\, dg \;&=\; 0.
\end{aligned}
$$

Hence for $1 \le i < j \le M$,

$$
\int_G x_{ij}^2(g)dg = \int_G y_{ij}^2(g)dg = \int_G u_{ij}^2(g)dg = \int_G v_{ij}^2(g)dg = \frac{1}{N+1}
$$

while

$$
\int_G x_{ij}(g) y_{ij}(g)\, dg \;=\; 0
$$

etc.

Thus the Gaussian limit takes the form

$$
C_N\, e^{-\frac{N+1}{2}\left(\sum_{j=1}^M x_{jj}^2\right) - (N+1)\left(\sum_{1\le i<j\le M}(x_{ij}^2 + y_{ij}^2 + u_{ij}^2 + v_{ij}^2)\right)}
$$

$$\times \prod_{j=1}^{M} dx_{jj} \prod_{1\le i<j\le M} dx_{ij}dy_{ij}du_{ij}dv_{ij} \;=$$

$$C_N e^{-\frac{N+1}{4}\operatorname{tr}(H^2)} \prod_{j=1}^{M} dx_{jj} \prod_{1\le i<j\le M} dx_{ij}dy_{ij}du_{ij}dv_{ij}.$$

This is precisely the $\mathrm{USp}(2M)$ invariant Gaussian on $\mathcal{H}_{2M}$, that is the GSE measure, see [M, p. 41].

3. The Gap

In what follows we exhibit z's with a gap. We take z's in $\mathbf{R}[G]$ of the form

$$z = g_1 + g_1^{-1} + g_2 + g_2^{-1} + \ldots + g_k + g_k^{-1} \tag{20}$$

Assume further that $\langle g_1, g_2, \ldots, g_k\rangle = \Gamma_z$ is a free group on these generators and that $k \ge 2$. Hausdorff [Hau] in his work on what later became known as the "Hausdorff-Banach-Tarski" paradox [Wa], exhibits such g's. In fact it is easy to see that the generic $(g_1, g_2, \ldots, g_k) \in G^{(k)}$ (in the sense of being outside a countable union of co-dimension one sets) gives a free Γ_z. Now such a group ($k \ge 2$) being free is Zariski dense in $\mathrm{SL}_2(\mathbf{C})$. The representations π_N extend to $\mathrm{SL}_2(\mathbf{C})$ and are irreducible. Hence if $v \in W_{N+1}$ satisfies $\pi_N(\Gamma)v = \lambda v$ for some $|\lambda| = 1$ then $\pi_N(\mathrm{SL}_2)v = \lambda v$ and hence $v = 0$. It follows that $2k$ and $-2k$ are not in $\operatorname{spec}(\widehat{z}(\pi_N))$, for if $\widehat{z}(\pi_N)v = \pm 2kv$ then $\pi_N(g_j)v = \pm v$ for $1 \le j \le k$ and hence $\pi_N(\Gamma)v = \pm v$ and $v = 0$. Thus for any z with Γ_z free, $\operatorname{spec}(\widehat{z}(\pi_N)) \subset (-2k, 2k)$. In particular for such a z for which we can establish a gap (for N large) we will have

$$\sup_{N\ge 1} ||\widehat{z}(\pi_N)|| < ||z||_1 = 2k \tag{21}$$

Let $p = 2k-1$ (we do not assume that p is prime) and let $U_n(\cos\theta) = \sin(n+1)\theta/\sin\theta$ for $n \ge 0$ be the n-th Chebyshev polynomial of the second kind. In $\mathbf{R}[G]$ we have the following relation which is easily established inductively [LPS]

$$p^{n/2} U_n\left(z/(2\sqrt{p})\right) \;=\; {\sum_{|\omega|\le n}}' \omega \tag{22}$$

where the sum is over all *reduced* words ω in $g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots g_k, g_k^{-1}$ of length $|\omega| = m \le n$ with $m \equiv n (\mathrm{mod} 2)$. The image of this relation under π_N yields

$$p^{n/2} U_n\left(\widehat{z}(\pi_N)/(2\sqrt{p})\right) \;=\; {\sum_{|\omega|\le n}}' \widehat{\omega}(\pi_N). \tag{23}$$

Write the eigenvalues λ_j of $\widehat{z}(\pi_N)$ as

$$\lambda_j \;=\; 2\sqrt{p}\cos(\theta_{j,N}), \qquad j = 0, 1, \ldots, N \tag{24}$$

where

$$\begin{cases} \theta_{j,n} \in [0,\pi], & if\ |\lambda_j| \leq 2\sqrt{p}, \\ \theta_{j,n} = i\xi_{j,N},\ \xi_{j,N} > 0, & if\ \lambda_j > 2\sqrt{p}, \\ \theta_{j,n} = \pi + i\xi_{j,N},\ \xi_{j,N} > 0, & if\ \lambda_j < -2\sqrt{p} \end{cases} \tag{25}$$

We call the $\theta_{j,N}$'s not in $[0,\pi]$ *exceptional*. Indeed since Γ_z is free most of the θ's are not exceptional as $N \to \infty$ (cf. Section 4).

Each $g \in G$ is conjugate to a diagonal matrix

$$\begin{bmatrix} \mathrm{e}(r_g) & 0 \\ 0 & \mathrm{e}(-r_g) \end{bmatrix}$$

with $0 \leq r_g \leq \pi$ which is determined from

$$\mathrm{tr}(g) \ = \ 2\cos r_g \tag{26}$$

Taking the trace of both sides of (23) yields

$$p^{n/2}\sum_{j=0}^{N} \frac{\sin(n+1)\theta_{j,N}}{\sin\theta_{j,N}} \ = \ {\sum_{|\omega|\leq n}}' \frac{\sin(N+1)r_\omega}{\sin r_\omega}. \tag{27}$$

We use (27) to estimate the number of exceptional $\theta_{j,N}$'s. To this end note that if n is even (which we assume henceforth in this Section) then for exceptional $\theta_{j,N}$ we have

$$\frac{\sin(n+1)\theta_{j,N}}{\sin\theta_{j,N}} \ = \ \frac{\sinh(n+1)\xi_{j,N}}{\sinh\xi_{j,N}} \ > \ 0. \tag{28}$$

Hence (27) becomes

$$\sum_{\substack{\xi_{j,N} \\ exceptional}} \frac{\sinh(n+1)\xi_{j,N}}{\sinh\xi_{j,N}} + O(Nn) \ = \ p^{-n/2}{\sum_{0<|\omega|\leq n}}' \frac{\sin(N+1)r_\omega}{\sin r_\omega}. \tag{29}$$

In order to exploit the cancellation in the sum on the right hand side of (29) we sum over N in a range $N \sim N_0$, N_0 large. For technical reasons we do so in a smooth way. Let $f \in C^\infty(1/2,3/2), f \geq 0$ and $f(1) = 1$. Summing over N in (29) we get

$$\sum_{N\sim N_0} f\left(\frac{N+1}{N_0+1}\right) \sum_{\substack{\xi_{j,N} \\ exceptional}} \frac{\sinh(n+1)\xi_{j,N}}{\sinh\xi_{j,N}} \ = \tag{30}$$

$$= \ p^{-n/2}{\sum_{0<|\omega|\leq n}}' \sum_{N\sim N_0} f\left(\frac{N+1}{N_0+1}\right)\frac{\sin(N+1)r_\omega}{\sin r_\omega} \ + \ O(N_0^2 n)$$

Poisson summation applied to the sum on N leads to the following: For any $\delta > 0$, $A \gg 1$ and $r \in [0,\pi]$ we have

$$\sum_{N\sim N_0} f\left(\frac{N+1}{N_0+1}\right)\frac{\sin(N+1)r}{\sin r} \ \ll_{\delta,A} \tag{31}$$

$$\begin{cases} N_0^2, & r \in [0, N_0^{-1+\delta}], \\ \min\left\{N_0^2, \frac{N_0^{-A}}{\sin r}\right\}, & r \in [N_0^{-1+\delta}, \pi]. \end{cases}$$

Since Γ_z is free we have $0 < r_\omega < \pi$ for any reduced word $\omega(|\omega| > 0)$. In fact we assume that there is a constant $B = B(g_1, \dots, g_k)$ such that for $|\omega| = n$,

$$B^{-n} \ \leq \ r_\omega \ \leq \ \pi - B^{-n} \tag{32}$$

We will verify (32) for any of the choices of $g_1, \dots, g_k$ below. Since A is arbitrarily large in (31), we conclude from (30) and (31) that

$$\sum_{N \sim N_0} f\left(\frac{N+1}{N_0+1}\right) \sum_{\substack{\xi_{j,N} \\ exceptional}} \frac{\sinh(n+1)\xi_{j,N}}{\sinh \xi_{j,N}} \ll_{\delta, A} \tag{33}$$

$$N_0^2 n + p^{-n/2} N_0^2 \# \left\{0 < |\omega| < n : r_\omega \in [0, N_0^{-1+\delta}]\right\}.$$

Thus estimating the left hand side above is reduced to estimating the number fo words $\omega', |\omega'| < n$ with small rotation (i.e. size $N_0^{-1+\delta}$). Before continuing we contemplate what to expect as far as the number of such r_ω's. Going back to (27) we note that for N *fixed* if we let $n \to \infty$ and use the fact that $-2k < \lambda_{j,N} < 2k$ for $N \geq 1$, we get

$$\lim_{n \to \infty} p^{-n} \sum_{|\omega| \leq n}' \frac{\sin(N+1) r_\omega}{\sin r_\omega} \ = \ 0. \tag{34}$$

The number of $|\omega| \leq n$ is $(p+1)p^{n-1}$ so (34) asserts that the rotations $r_\omega, |\omega| \leq n$ become equidistributed with respect to $\sin^2 \theta \ d\theta / \pi$, that is with respect to the Weyl measure which is the image of the Haar measure on G onto the maximal torus. We might expect at least generically for $(g_1, \dots, g_k) \in G^{(k)}$ that this continues to hold approximately for small intervals. That is that

$$\# \left\{\omega | r_\omega \in [0, N_0^{-1+\delta}]\right\} \ll \frac{\{\omega : |\omega| \leq n\}}{N_0^{3-3\delta}} \ + \ 1. \tag{35}$$

While we don't know how to establish (35) for the generic $(g_1, \dots, g_k)$, we can prove it for some special choices.

Granting (35) we have (using (28) and (33)) that

$$\sum_{\substack{\xi_{j,N_0} \\ exceptional}} \frac{\sinh(n+1)\xi_{j,N_0}}{\sinh \xi_{j,N_0}} \ \ll \ N_0^2 n + p^{-n/2} \left(\frac{p^n}{N_0^3} + 1\right)$$

and hence for $b > 0$

$$\sum_{e^{\xi_{j,N_0}} \geq p^b} 1 \ \ll \ p^{-nb} \left(N_0^2 n + p^{-n/2} N_0^2 \left(\frac{p^n}{N_0^3} + 1\right)\right)$$

Choose n satisfying $N_0^6 \le p^n \le p^2 N_0^6$ (n even), then

$$\sum_{e^{\xi_{j,N_0}} \ge p^b} 1 \ \ll\ p^{(-b+1/3)n} \ \cdot n \tag{36}$$

If $b > 1/3$ the right hand side of (36) is less than one and we conclude that for N_0 large there are no exceptional ξ_{j,N_0} with $e^{\xi_{j,N_0}} \ge p^b$. That is for N_0 large and any j

$$|\lambda_{j,N_0}| \ \le\ p^{1/2}(p^{1/3} + p^{-1/3}) + \tilde{\varepsilon}, \tag{37}$$

($\tilde{\varepsilon}$ sufficiently small).

So the explicit gap (36) is what we expect to hold for generic $(g_1, g_2, \dots, g_k)$ and we now exhibit g's satisfying (35) (or sufficiently good approximation to it).

Let $\mathbf{H}(\mathbf{Z})$ be the integral Hamilton quaternions, so if $\alpha \in \mathbf{H}(\mathbf{Z})$ then $\alpha = x_0 + x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k}$ with $x_j \in \mathbf{Z}$. We let $\bar{\alpha} = x_0 - x_1\mathrm{i} - x_2\mathrm{j} - x_3\mathrm{k}$ and $N(\alpha) = \bar{\alpha}\alpha \in \mathbf{Z}$. For $q \ge 3$ a prime number we take for $\tilde{g}_1, \tilde{g}_2, \dots \tilde{g}_k$ a subset of the set $\{\alpha | N(\alpha) = q\}$ (as pointed out in the Introduction this set has $8(q+1)$ elements) which satisfies

(I) $\tilde{g}_{j_1} \ne \epsilon \tilde{g}_{j_2}$ if $j_1 \ne j_2$ and $\epsilon \in \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$ is a unit.

(II) $\tilde{g}_{j_1} \ne \epsilon \overline{\tilde{g}_{j_2}}$ for any j_1, j_2 and ϵ a unit.

As is well known (see [LPS] for example) different reduced words $\omega = R(\tilde{g}_1, \overline{\tilde{g}_1}, \dots \overline{\tilde{g}_k})$ in $\tilde{g}_1, \dots, \overline{\tilde{g}_k}$ of length $m \ge 1$ give different quaternions of norm q^m. In particular to each such word corresponds a unique solution of

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 \ = \ q^m, \qquad x_0^2 \ne q^m \tag{38}$$

For any $\alpha = x_0 + x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k} \in \mathbf{H}(\mathbf{Z})$ with $N(\alpha)$ equal to t, set

$$g \ = \ \frac{1}{\sqrt{t}} \begin{bmatrix} x_0 + x_1\mathrm{i} & x_2 + x_3\mathrm{i} \\ -x_2 + x_3\mathrm{i} & x_0 - x_1\mathrm{i} \end{bmatrix} \ \in\ G$$

The image under the homomorphism of $\tilde{g}_1, \dots, \overline{\tilde{g}_k}$ yields a set $g_1, g_1^{-1}, \dots, g_k, g_k^{-1}$ in G. According to (38) above, $\langle g_1, \dots, g_k \rangle$ is a free group. Note that under this correspondence

$$\text{trace}\ \tilde{g} \ = \ 2x_0/\sqrt{t} \tag{39}$$

We proceed to estimate the number fo words ω in $g_1, \dots, g_k^{-1}$ of length m with r_ω close to 0 or π. According to (38) and (39) such a word ω corresponds to a solution

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 \ = \ q^m, \qquad x_0^2 \ne q^m$$

and

$$\sin r_\omega \ = \ \sqrt{q^m - x_0^2}/q^{m/2}$$

From this (32) is clearly satisfied and also

$$\#\left\{\omega : |\omega| = m, r_\omega \in [0, N^{-1+\delta}]\right\} \ll$$

$$\#\left\{x_0^2 + x_1^2 + x_2^2 + x_3^2 = q^m, \sqrt{q^m - x_0^2}/q^{m/2} \le N^{-1+\delta}\right\} \ll$$

$$\#\left\{x_0^2 + x_1^2 + x_2^2 + x_3^2 = q^m; q^{m/2}(1 - N^{2\delta-2}) \le x_0 < q^{m/2}\right\} =$$

$$\sum_{\substack{q^{m/2}(1-N^{2\delta-2}) \\ \le x_0 < q^{m/2}}} r_3(q^m - x_0^2),$$

where $r_3(m)$ is the number of representations of m as a sum of three squares.

From elementary number theory [HW] we have a bound $r_3(m) = O_\varepsilon(m^{1/2+\varepsilon})$ for any $\varepsilon > 0$. Hence the above sum is

$$\ll q^{\varepsilon m} \sum_{q^{m/2}(1-N^{2\delta-2}) \le x_0 < q^{m/2}} \sqrt{q^m - x_0^2} \ll$$

$$\ll q^{\varepsilon m}\left(1 + \int_{q^{m/2}(1-N^{2\delta-2})}^{q^{m/2}} \sqrt{q^m - t^2}\, dt\right) \ll$$

$$\ll q^{\varepsilon m}\left(1 + q^m \int_0^{N^{-1+\delta}} \sin^2\theta\, d\theta\right) \ll$$

$$\ll q^{\varepsilon m}(1 + q^m/N^{3-3\delta}) \tag{40}$$

Summing this for $m \le n$ yields

$$\#\left\{\omega : |\omega| \le n, r_\omega \in [0, N^{-1+\delta}]\right\} \ll q^{\varepsilon n}\left(1 + \frac{q^n}{N^{3-3\delta}}\right) \tag{41}$$

Note that if $p = q$ (which is the largest set of such g_j's that we can choose) then (41) gives the expected bound (35).

Continuing the analysis as above we get

$$\sum_{p^{\xi_{j,N_0}} \ge p^b} 1 \ll_{\varepsilon,\delta} p^{-nb}\left(N_0^2 n + p^{-n/2} N_0^2 q^{\varepsilon n}\left(1 + \frac{q^n}{N_0^{3-3\delta}}\right)\right) \tag{42}$$

Now assume that $p > q^{4/5}$ and choose n even satisfying

$$N_0^3 \le \left(\frac{q}{\sqrt{p}}\right)^n \le \left(\frac{q}{\sqrt{p}}\right)^2 N_0^3$$

For N_0 large enough and $p^b > q^{2/3}/p^{1/3}$ we will have the right hand side of (42) being less than 1. It follows that for $\tilde\varepsilon > 0$ and N_0 large enough

$$|\lambda_j(\widehat{z}(\pi_{N_0}))| \le p^{1/2}\left(\frac{q^{2/3}}{p^{1/3}} + \frac{p^{1/3}}{q^{2/3}}\right) + \tilde\varepsilon$$

and since $p > q^{4/5}$ the latter is less than $p + 1 = 2k$. This completes the proof of the Theorem 1.

□

4. Density of states

Let $z = \sum x_g g \in \mathbf{R}[G]$ and assume that $z = z^*$. Set

$$\mu_N(z) \;=\; \frac{1}{N+1} \sum_{j=0}^{N} \delta_{\lambda_j(\widehat{z}(\pi_N))}$$

PROPOSITION 2. *There are measures $\nu^{even}(z)$ and $\nu^{odd}(z)$ such that*

$$\lim_{\substack{N\to\infty \\ N\ even}} \mu_N(z) \;=\; \nu^{even}(z),$$
$$\lim_{\substack{N\to\infty \\ N\ odd}} \mu_N(z) \;=\; \nu^{odd}(z).$$

(the convergence being in the sense of integration against any continuous function).

Proof: Fix an integer $m \geq 0$. Then

$$z^m \;=\; \sum_{g_1,\cdots,g_m \in \mathrm{supp}(z)} x_{g_1} x_{g_2} \ldots x_{g_m} g_1 g_2 \ldots g_m \tag{43}$$

Hence

$$\mathrm{trace}(\widehat{z^m}(\pi_N)) \;=\; \sum_{g_1,\cdots,g_m} x_{g_1} x_{g_2} \ldots x_{g_m} \chi_{\pi_N}(g_1 g_2 \ldots g_m)$$

or

$$\mu_N(z)(x^m) \;=\; \sum_{g_1,\cdots,g_m} x_{g_1} x_{g_2} \ldots x_{g_m} \chi_{\pi_N}(g_1 g_2 \ldots g_m)/(N+1) \tag{44}$$

Now $\chi_{\pi_N}(g) = sin(N+1)\alpha/(\sin\alpha)$ if g is conjugate to $\begin{bmatrix} \mathrm{e}(\alpha) & 0 \\ 0 & \mathrm{e}(-\alpha) \end{bmatrix}$.
In particular

$$\lim_{\substack{N\to\infty \\ N\ even}} \frac{\chi_{\pi_N}(g)}{N+1} \;=\; \begin{cases} 1, & if\ g = \pm e, \\ 0, & otherwise. \end{cases}$$

$$\lim_{\substack{N\to\infty \\ N\ odd}} \frac{\chi_{\pi_N}(g)}{N+1} \;=\; \begin{cases} 1, & if\ g = e, \\ -1, & if\ g = -e, \\ 0, & otherwise. \end{cases}$$

Hence

$$\lim_{\substack{N\to\infty \\ N\ even}} \mu_{\pi_N}(z)(x^m) := C_m^{even}(z) = \sum_{\substack{g_1,\cdots,g_m \\ g_1\cdots g_m = \pm e}} x_{g_1} \ldots x_{g_m} \tag{45}$$

and

$$\lim_{\substack{N\to\infty \\ N\ odd}} \mu_{\pi_N}(z)(x^m) \;:=\; C_m^{odd}(z) \;= \tag{46}$$

$$\sum_{\substack{g_1,\cdots,g_m \\ g_1\cdots g_m=e}} x_{g_1}\ldots x_{g_m} \quad - \quad \sum_{\substack{g_1,\cdots,g_m \\ g_1\cdots g_m=-e}} x_{g_1}\ldots x_{g_m}$$

Thus for any continuous f, $\mu_N(z)(f) \to \nu^{even}(f)$ as $N \to \infty$ through even values and $\mu_N(z)(f) \to \nu^{odd}(f)$ as $N \to \infty$ through odd values. Here $\nu^{even}(z)$ and $\nu^{odd}(z)$ are the probability measures (whose existence is clear) and whose moments are given by (45) and (46). □

Of special interest are elements of the form

$$z_{g_1,\ldots,g_k} \;=\; g_1 + g_1^{-1} + g_2 + g_2^{-1} + \ldots + g_k + g_k^{-1}.$$

In this case the measures $\nu^{even}(z)$ and $\nu^{odd}(z)$ can be interpreted as the spectral measures for the random walk on Γ_z with respect to the symmetric set of generators $g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_k, g_k^{-1}$ (see Kesten [Kest]). It follows from [Kest] that supp $\nu^{even}(z)$ and supp $\nu^{odd}(z)$ are contained in $[-2\sqrt{k-1}, 2\sqrt{k-1}]$ iff Γ_z is a free group and in this case

$$\nu^{even}(z) = \nu^{odd}(z) = \frac{\sqrt{2k-1-t^2/4}}{2\pi k(1-(t/2k)^2)}\, dt. \tag{47}$$

Moreover Kesten shows that the support of $\nu^{even}(z)$ and $\nu^{odd}(z)$ are contained in $(-2k, 2k)$ iff Γ_z is not amenable. Hence it is only for z with Γ_z not amenable that z can have a gap.

We next examine the speed of convergence in Proposition 2. This will depend on the diophantine properties of $g_1, \ldots, g_k \in G$ (we use this terminology to draw an analogy with diophantine approximation).

DEFINITION 1. *For $k \geq 2$, we say that $g_1, g_2, \ldots, g_k \in G$ are diophantine (or satisfy a diophantine condition) if there is $B = B(g_1, \ldots, g_k) > 0$ such that for any $m \geq 1$ and a word R_m in $g_1, g_2, \ldots, g_k$ of length m with $R_m \neq \pm e$ we have*

$$||R_m \pm e|| \;\geq\; B^{-m}$$

Here

$$\left|\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right|\right|^2 \;=\; |a|^2 + |b|^2 + |c|^2 + |d|^2.$$

Note that for $g \in G$ we have

$$||g \pm e||^2 \;=\; 2|\mathrm{trace}(g) \mp 2|. \tag{48}$$

For example if $\Gamma_{(g_1,\ldots,g_k)}$ is finite then $g_1, \ldots, g_k$ are diophantine, however we are mainly interested in the case that $\Gamma_{(g_1,\cdots,g_k)}$ is free. In this case it follows by a pigeon hole argument similar to Dirichlet's that for any $m \geq 1$ there always is a word $R \neq \pm e$ in $g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}$ of length at most m satisfying

$$||R - e|| \;\leq\; \frac{10}{(2k-1)^{m/6}} \tag{49}$$

(here and elsewhere we assume that $k \geq 2$). This shows that the exponential behavior in the definition of diophantine is the appropriate one.

As was first exploited by Hausdorff [Hau] the relation

$$R_m(g_1, g_1^{-1}, \dots, g_k, g_k^{-1}) \ = \ e$$

where R is a *reduced* word of length $m \geq 1$ is not satisfied identically in $G^{(k)}$. Hence the sets

$$V(R_m) \ := \ \{(g_1, \dots, g_k) | R_m(g) = e\}$$

are of codimension at least one in $G^{(k)}$. It follows that $\cup_{m \geq 1} V(R_m)$ is of zero measure in $G^{(k)}$ and also it is of the first Baire category in $G^{(k)}$. Thus the generic $(g_1, \dots, g_k) \in G^{(k)}$ (in both senses) generates the free group. Now the set of $(g_1, \dots, g_k) \in G^{(k)}$ for which $\langle g_1, \dots, g_k \rangle$ is not free is clearly dense in $G^{(k)}$ so it follows easily that the set of $(g_1, \dots, g_k) \in G^{(k)}$ which are *not* diophantine is of the second (Baire) category in $G^{(k)}$. That is to say the topologically generic $(g_1, \dots, g_k)$ is free but not diophantine. On the other hand we expect that in the measure sense the generic $(g_1, \dots, g_k)$ is diophantine. For elements with algebraic number entries we have:

PROPOSITION 3. *Let* $g_1, \dots, g_k \in G \cap M_2(\overline{\mathbf{Q}})$. *Then* $g_1, \dots, g_k$ *are diophantine.*

Proof. Since the set $\{g_1, \dots, g_k\}$ is finite we have $g_j \in M_2(K)$ where K is a finite extension of $\mathbf{Q}$ of degree d say. We can find a rational integer $N \geq 1$ such that

$$N g_j \ \in \ M_2(\mathcal{O}_K) \tag{50}$$

where $\mathcal{O}_K$ is the ring of integers of K. Choose $M \geq N$ large enough so that for each of the d embeddings σ of K into $\mathbf{C}$ we have

$$||\sigma(N g_j)|| \ \leq \ M, \qquad j = 1, \dots, k. \tag{51}$$

If

$$R \ = \ g_{j_1} g_{j_2} \dots g_{j_n}$$

is a word of length n in the g's then

$$N^n R \ \in \ M_2(\mathcal{O}_K) \tag{52}$$

Set $h = N^n R \pm N^n e = N^n(R \pm e)$, $h \in M_2(\mathcal{O}_K)$ and if $h \neq 0$, say $h_{ij} \neq 0$ then

$$|\sigma(h_{ij})| \leq ||\sigma(N^n R) \pm \sigma(N^n e)|| \leq 2M^n$$

Also

$$h_{ij} \prod_{\sigma \neq Id} \sigma(h_{ij}) \ \in \ \mathbf{Z}$$

and is not zero, so we have

$$|h_{ij}| \ \geq \ \frac{1}{(2M)^{nd}}$$

Hence

$$||R \pm e|| \ \geq \ \frac{1}{((2M)^d N)^n}$$

Choose $B = (2M)^d N$, then $g_1, \ldots, g_k$ is diophantine with this choice of B. $\square$

Our main result about the distribution $\mu_N(z)$ for diophantine z is Theorem 2.

Proof: The proof of Proposition 2 together with the diophantine condition leads to the following. There exists a constant $A = A(z)$ such that for $m \geq 1$,

$$\begin{cases} \mu_N(z)(x^m) = \nu^{even}(z)(x^m) + O(A^m/N), & N \textit{ even}, \\ \mu_N(z)(x^m) = \nu^{odd}(z)(x^m) + O(A^m/N), & N \textit{ odd}, \end{cases} \tag{53}$$

the implied constants in (53) being universal (that is independent of m and N). The passage from (53) to the discrepancy bounds is a straightforward analysis of suitable approximation of characteristic functions of intervals by polynomials; we give the details below.

We choose a large constant K and make a change of variables:

$$\lambda_j \; = \; K \cos\theta_j, \qquad 0 \leq \theta_j \leq \pi.$$

We next define new measures $\widetilde{\mu}_N$ by

$$\widetilde{\mu}_N \; = \; \sum_{j=0}^{N} \frac{1}{2}(\delta_{\theta_j} + \delta_{-\theta_j}). \tag{54}$$

The corresponding limit measures as $N \to \infty$ are denoted by $\widetilde{\nu}^{even}, \widetilde{\nu}^{odd}$; they will both be called $\widetilde{\nu}_\infty$. The definitions insure that all the measures are *even* in θ.

Choosing K large enough, we may assume that the measures $\widetilde{\mu}_N$ and $\widetilde{\nu}_\infty$ are supported in $(-3\pi/4, -\pi/4) \cup (\pi/4, 3\pi/4)$. Given $I \subset (-\pi, \pi)$, we want to estimate $|(\widetilde{\mu}_N - \widetilde{\nu}_\infty)(I)|$. Let χ_I be the characteristic function of the interval I. For every $m \in \mathbf{N}$ there exist trigonometric polynomials $S_m^+(\theta)$ and $S_m^-(\theta)$ of degree m (which depend on I), called *Selberg polynomials*, such that ([Sel], [V])

$$S_m^-(\theta) \leq \chi_I(\theta) \leq S_m^+(\theta), \; \int_{-\pi}^{\pi} (S_m^+(\theta) - S_m^-(\theta))\, d\theta = O(1/m), \tag{55}$$

and the constants are independent of I. The degree m of the polynomials will be chosen later.

By positivity,

$$\int S_m^-(\theta)\, d\widetilde{\mu}_N \leq \widetilde{\mu}_N(\chi_I(\theta)) \leq \int S_m^+(\theta)\, d\widetilde{\mu}_N,$$

and similar inequalities hold for $\widetilde{\nu}_\infty$ in place of $\widetilde{\mu}_N$.

We next remark that since the measures $\widetilde{\mu}_N$ and $\widetilde{\nu}_\infty$ are even in θ,

$$\int S_m^{\pm}(d\widetilde{\mu}_N - d\widetilde{\nu}_\infty) = \int S_{m,even}^{\pm}(d\widetilde{\mu}_N - d\widetilde{\nu}_\infty) \tag{56}$$

where $S_{m,even}^{\pm}$ is the *even part* of $S_m^{\pm}$.

We make another substitution $y = \cos\theta$. We denote the measures $\widetilde{\mu}_N$ (respectively, $\widetilde{\nu}_\infty$) in the new coordinates by corresponding measures by ξ_N (respectively, ξ_∞). To prove Theorem 2, it suffices to estimate the discrepancy $D(\xi_N, \xi_\infty)$.

The functions $S^{\pm}_{m,even}(\theta)$ become polynomials of degree m in y which we denote by $f^{\pm}_m(y)$. It is easy to show (cf. [V]) that there exists a constant $B > 0$, which does not depend on the interval I, such that the coefficients of f_m's grow slower than B^m. That fact and the bounds (53) imply that for a sufficiently small $c > 0$ and for $m = c\ln N$,

$$\int f^{\pm}_m(y)(d\xi_N(y) - d\xi_\infty(y)) \ \ll\ 1/(\ln N).$$

To finish the proof, it suffices to show that for $m = c\ln N$,

$$\int_{\mathrm{supp}(\xi_\infty)} (f^{+}_m(y) - f^{-}_m(y))d\xi_\infty(y) \ \ll\ 1/(\ln N). \tag{57}$$

By assumptions on ν^{even} and ν^{odd} (which remain valid for ξ_∞) it suffices to establish (57) for the measure dy instead of $d\xi_\infty(y)$.

We note that

$$\int_{\mathrm{supp}(\xi_\infty)} (f^{+}_m(y) - f^{-}_m(y))\, dy \ <\ \int_{\mathrm{supp}(\xi_\infty)} \frac{f^{+}_m(y) - f^{-}_m(y)}{\sqrt{1-y^2}}\, dy$$

(the last integral is well-defined since $\mathrm{supp}(\xi_\infty) \subset \{|y| < 1/\sqrt{2}\}$). The last integral by (56) and a change of variables is equal to

$$\int_{-\pi}^{\pi} (S^{+}_m(\theta) - S^{-}_m(\theta))\, d\theta.$$

Accordingly, the bound (55) implies (57) and finishes the proof of Theorem 2.

$\square$

REMARK 1. As pointed out above the generic (in topology) $g_1, \dots, g_k$ is not diophantine and so does not satisfy the assumptions of Theorem 2. In fact in view of Kesten's Theorem above about the support of $\nu(z)$ and Γ_z being free, together with the density of $(g_1, \dots, g_k)$ which are not free, it follows by the usual Baire category argument (cf. [LPS]) that Theorem 2 fails for topologically generic $z(g_1, \dots, g_k)$.

On the other hand the numerical experiments of Section 5 indicate that for the generic in measure $(g_1, \dots, g_k), k \geq 3$, $\mathrm{spec}(\widehat{z}(\pi_N))$ behaves like that of a member of the corresponding random matrix ensemble. If this is so then Theorem 2 for such a generic z is very far from the true upper bound which would be of order $(\log N)/N$ rather than $1/\log N$. For the case of the specific Ramanujan elements z_p of Section 3 we now establish a lower bound for D.

Proof of Theorem 3. A z_p as above satisfies $\mathrm{spec}(\widehat{z}_p(\pi_N)) \subset [-2\sqrt{p}, 2\sqrt{p}]$ for $N \geq 1$. Our aim is to show that $D(\mu_N(z_p), \nu(z_p))$ is large at least on a subsequence $N_j \to \infty$. The measure $\nu(z_p)$ is the Kesten measure defined in (47).

One checks that the discrepancy is invariant under continuous monotone changes of variable in the eigenvalue parameter. Since $\lambda_{j,N} \in [-2\sqrt{p}, 2\sqrt{p}]$ is is convenient to use the variable $\theta_{j,N} \in [0,\pi]$ where

$$2\sqrt{p}\cos\theta_{j,N} \;=\; \lambda_{j,N}, \qquad 0 \le j \le N;\; N \ge 1$$

Set

$$\widetilde{\mu}_N(z_p) \;=\; \frac{1}{N+1}\sum_{j=0}^{n} \delta_{\theta_{j,N}} \tag{58}$$

which is a probability measure on $[0,\pi]$. Let $\widetilde{\nu}_p$ be the corresponding limit of the $\widetilde{\mu}_N(z_p)$ as $N \to \infty$. For $n \ge 3$ let

$$I_{n,N} \;=\; (N+1)\int_0^{\pi}\left(\frac{\sin(n+1)t}{\sin t} - \frac{\sin(n-1)t}{p\sin t}\right) d\widetilde{\mu}_N(t). \tag{59}$$

Note that

$$\int_0^{\pi}\left(\frac{\sin(n+1)t}{\sin t} - \frac{\sin(n-1)t}{p\sin t}\right) d\widetilde{\nu}_p(t) \;=\; 0. \tag{60}$$

Thus an integration by parts in (59) yields

$$(N+1)^{-1}\,|I_{n,N}| \;\le\; 2n^2\, D(\widetilde{\mu}_N, \widetilde{\nu}_p) \tag{61}$$

We will use (61) to give a lower bound for $D(\widetilde{\mu}_N, \widetilde{\nu}_p)$. According to (27) we have

$$I_{n,N} \;=\; p^{-n/2} \sum_{|\omega|=n}{}^{'} \frac{\sin(N+1)r_\omega}{\sin r_\omega}. \tag{62}$$

For the element z_p we saw in Section 3 that the right hand side of (62) may be expressed as

$$p^{-n/2} \sum_{|x|<p^{n/2}} \nu(n,x)\frac{\sin((N+1)\beta_{x,n})}{\sin(\beta_{x,n})} \tag{63}$$

where $\nu(n,x)$ is the multiplicity of reduced words of length n with the trace equal to $x/p^{n/2}$, that is

$$\cos\beta_{x,n} \;=\; \frac{x}{p^{n/2}}, \qquad x \in \mathbf{Z},\; |x| < p^{n/2}. \tag{64}$$

From the case $N = 0$ we have

$$p^{n-1}(1+p) \;=\; \sum_{|x|<p^{n/2}} \nu(n,x) \tag{65}$$

Now rewrite (63) as

$$I_{n,N} \;=\; \sum_{|x|<p^{n/2}} \frac{\nu(x,n)}{\sqrt{p^n - x^2}} \sin((N+1)\beta_{x,n}). \tag{66}$$

Let $f \in \mathcal{S}(\mathbf{R})$ be even, $f \geq 0, f(0) = 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$. Assume further that supp$\hat{f} \subset (-1,1)$. Consider

$$\sum_{N=0}^{\infty} f\left(\frac{N+1}{N_0}\right) |I_{n,N}|^2 = \frac{1}{2}\sum_x \frac{(\nu(x,m))^2}{p^n - x^2} \sum_{m=-\infty}^{\infty} f\left(\frac{m}{N_0}\right)(\sin m\beta_{x,n})^2$$

$$+ \ \frac{1}{2}\sum_{x \neq x'} \frac{(\nu(x,n))\nu(x,n')}{\sqrt{(p^n - x^2)(p^n-(x')^2)}} \sum_{m=-\infty}^{\infty} f\left(\frac{m}{N_0}\right)\sin(m\beta_{x,n})\sin(m\beta_{x',n}) \tag{67}$$

If $x \neq x'$,

$$\frac{1}{p^{n/2}} \leq \left|\frac{x-x'}{p^{n/2}}\right| = |\cos\beta_{x,N} - \cos\beta_{x',N}| \leq |\beta_{x,N} - \beta_{x',N}| \tag{68}$$

while

$$\beta_{x,n} + \beta_{x',N} \ < \ 2\pi - 1/p^{n/2} \tag{69}$$

Thus applying Poisson summation in (67) together with supp$\hat{f} \subset (-1,1)$ we see that for $N_0 > 2p^{n/2}$ the second term in (67) vanishes while the first equals

$$\frac{N_0}{4}\left(\int_{-\infty}^{\infty} f(x)dx\right) \sum_{|x|<p^{n/2}} \frac{(\nu(x,n))^2}{p^n - x^2}.$$

That is we have

$$\sum_{N=0}^{\infty} f\left(\frac{N+1}{N_0}\right)|I_{n,N}|^2 \ = \ \frac{N_0}{4}\sum_{|x|<p^{n/2}} \frac{(\nu(x,n))^2}{p^n - x^2} \tag{70}$$

if $p^{n/2} < N_0/2$. On the other hand from (65) we have

$$p^n \leq \sum_{|x|<p^{n/2}} \nu(x,n) \leq \left(\sum_{|x|<p^{n/2}} \frac{\nu^2(x,n)}{p^n - x^2}\right)^{1/2} \left(\sum_{|x|<p^{n/2}} (p^n - x^2)\right)^{1/2}$$

Hence

$$\sum_{|x|<p^{n/2}} \frac{\nu^2(x,n)}{p^n - x^2} \ \geq \ \frac{p^{n/2}}{4} \tag{71}$$

That is for $p^{n/2} < N_0/2$,

$$\sum_{N=0}^{\infty} f\left(\frac{N+1}{N_0}\right)|I_{n,N}|^2 \ \geq \ \frac{N_0 p^{n/2}}{16}$$

Choose n so that

$$\frac{N_0}{2\sqrt{p}} \ < \ p^{n/2} \ < \ \frac{N_0}{2}.$$

We then have

$$\frac{1}{N_0}\sum_{N=0}^{\infty} f\left(\frac{N+1}{N_0}\right)|I_{n,N}|^2 \ \gg \ N_0$$

and hence from (61) that

$$\frac{1}{N_0}\sum_{N=0}^{\infty} f\left(\frac{N+1}{N_0}\right)|D(\widetilde{\mu}_N,\widetilde{\nu}_p)|^2 \gg \frac{N_0^{-1}}{(\log N_0)^4} \tag{72}$$

In particular this means that there is a subsequence $N_j \to \infty$ such that

$$D(\widetilde{\mu}_{N_j},\widetilde{\nu}_p) \gg \frac{N_j^{-1/2}}{(\log N_j)^2}. \tag{73}$$

This establishes the required lower bound for the discrepancy claimed in Theorem 3.

5. Numerical results

We report on some numerical computations of spec$(\widehat{z}(\pi_N))$ for certain z and for N of size of about 1000. As pointed out in the Introduction, if $z = g_1 + g_1^{-1} + g_2 + g_2^{-1}$ and is in generic position there is a nontrivial $h \in G$ commuting with z (as elements of $\mathbf{R}[G]$). This impacts the statistics of the distributions of the eigenvalues of $\widehat{z}(\pi_N)$. To avoid this symmetry we consider in this Section elements of the form $g_1+g_1^{-1}+g_2+g_2^{-1}+g_3+g_3^{-1}$. For such a z with (g_1,g_2,g_3) generic it appears[1] that centralizing subalgebra of z in $\mathbf{R}[G]$ is $\mathbf{R}[z]$. The numerics below also confirm that the generic such g is "desymmetrized". The precise z's which we consider are

(I) z chosen at random (i.e. g_1, g_2, g_3 chosen at random).

(II) z is the Ramanujan element z_5, that is

$$g_1 = A = \frac{1}{\sqrt{5}}\begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix}, \quad g_2 = B = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix},$$

and

$$g_3 = C = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2i \\ 2i & 1 \end{bmatrix}.$$

(III) z the element $\tilde{z}$ defined after Theorem 1 with $q=7$,

$$\widetilde{A} = \frac{1}{\sqrt{7}}\begin{bmatrix} 2-i & 1+i \\ -1+i & 2+i \end{bmatrix}, \quad \widetilde{B} = \frac{1}{\sqrt{7}}\begin{bmatrix} 2-i & -1+i \\ 1+i & 2+i \end{bmatrix},$$

and

$$\widetilde{C} = \frac{1}{\sqrt{7}}\begin{bmatrix} 2+i & -1+i \\ 1+i & 2-i \end{bmatrix}.$$

In all these cases Γ_z is a free group on 3 generators so that the density of states is given by (47) with $k=3$.

We first describe the method of computation. A direct computation of the matrix of $\widehat{z}(\pi_N)$ using the basis (11) (or variants thereof using orthogonal polynomials) turn out to be too time consuming and inaccurate when N is

[1]This question will be addressed in more detail in a separate paper.

of size 1000. What is much better is to use matrix exponentiation as follows. The Lie algebra su(2) of SU(2) is spanned by the vectors

$$X_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \; X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; X_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{74}$$

For $X \in \mathrm{su}(2)$ and π a representation of G (in particular π_N) we have the relation

$$\pi(\exp X) \;=\; \exp(d\pi(X)) \tag{75}$$

The matrix of $d\pi_N(X_j), j = 1,2,3$ may be computed explicitly from the action (1) and the basis of W_{N+1} normalized according to (11). For $j = 0,1,\ldots,N$ let

$$\widehat{e}_j \;=\; \frac{x^j y^{N-j}}{\sqrt{j!(N-j)!}} \tag{76}$$

Then

$$\begin{cases} d\pi_N(X_1)(\widehat{e}_j) &= i(2j-N)\widehat{e}_j, \\ d\pi_N(X_2)(\widehat{e}_j) &= -\sqrt{j(N-j+1)}\widehat{e}_{j-1} + \sqrt{(N-j)(j+1)}\widehat{e}_{j+1}, \\ d\pi_N(X_3)(\widehat{e}_j) &= i\sqrt{j(N-j+1)}\widehat{e}_{j-1} + i\sqrt{(N-j)(j+1)}\widehat{e}_{j+1}. \end{cases} \tag{77}$$

Thus $d\pi_N(X_j)$ is skew-Hermitian with $d\pi_N(X_1)$ diagonal and $d\pi_N(X_2), d\pi_N(X_3)$ tridiagonal. The exponentials of these matrices were computed using MATLAB which appeals to the Pade approximation with scaling and squaring algorithm (see [GV, method 11.3.1]). A nice discussion of the subtleties of computing such exponentials of matrices is given in [MV]. Once the exponential is computed we use (75) to obtain $\widehat{z}(\pi_N)$ in this basis. The eigenvalues of this matrix were then computed, again using MATLAB.

We turn to an analysis of the results and their interpretation. The first data concerns the density of states. For the random $z = g_1 + g_1^{-1} + g_2 + g_2^{-1} + g_3 + g_3^{-1}$ (that is to say we chose $X^{(j)} = \xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3$ at random for $j = 1,2,3$ and then used these to get $g_j = \exp(X^{(j)})$) the distributions $\mu_N(z)$ were computed for various N's. The graphs all looked alike and in Figure 1 the case $N = 1004$ with a histogram of $\mu_{1004}(z)$ is drawn against the density (47). The fit is excellent and is consistent with the discrepancy being $O((\log N)/N)$.

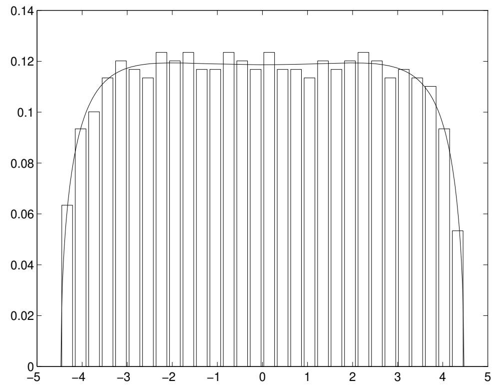

Figure 1. Density for a random z, $N = 1004, k = 3$, vs. Kesten's measure.

On the other hand the same experiment was carried out for the Ramanujan element z_5 (note that A, B, C in z_5 are very conveniently expressed as $\exp(\xi X_1), \exp(\xi X_2), \exp(\xi X_3)$ with $\cos\xi = 3/5$) and the result is displayed in Figure 2. The discrepancy is much larger and is consistent with being of size $1/\sqrt{N}$ as proven in Theorem 2.

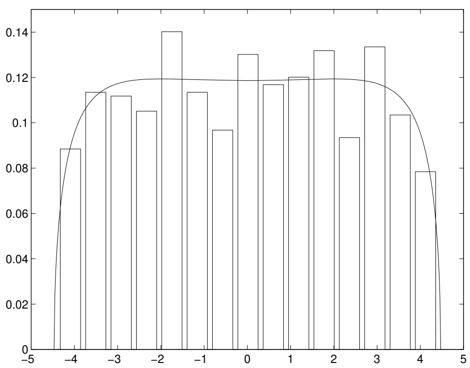

Figure 2. Density for A, B, C of (II), $N = 1004$, vs. Kesten's measure.

The second set of data concerns the consecutive spacings distribution between the eigenvalues of $\widehat{z}(\pi_N)$. As is apparent from Figure 1 the density of eigenvalues is essentially constant between -3 and 3. To use more of the data set one straightens or unfolds the eigenvalues by the change of variables

$$F(x) \;=\; \int_{-2\sqrt{5}}^{x} \frac{6\sqrt{20-t^2}}{2\pi(36-t^2)}\,dt$$

which renders the density uniform. The histogram of the distribution for the resulting consecutive spacings between these straightened numbers (renormalized to have mean 1) was then computed.

For a random z and 10 values of N in the range $[700, 1100]$ the results were all alike. In Figure 3 the case $N = 1099$ is displayed against the Gaussian Symplectic Ensemble prediction [M] (actually we used the Wigner surmise which is a very good approximation to the latter). Note that in the case of N odd we know that the spectrum of $\widehat{z}(\pi_N)$ comes in doublets and we took one eigenvalue out of each of these. The fit is very good (theoretically we might expect $1/\sqrt{N}$ discrepancy).

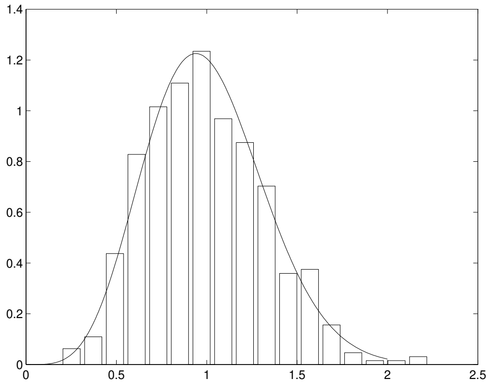

Figure 3. Spacings for a random z, $N = 1099$, vs. GSE.

A similar calculation for N even gave GOE statistics. A typical case being the histogram in Figure 4 for $N = 1004$ which is compared with the GOE Wigner surmise.

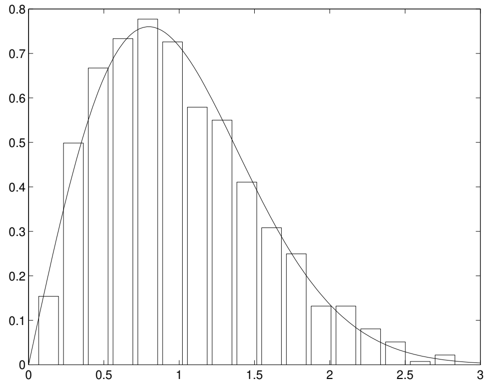

Figure 4. Spacings for a random z, $N = 1004$, vs. GOE.

For the case of z_5 the spacings behave very differently. Firstly there is added multiplicity to the eigenvalues. This may be accounted for by symmetry. The subgroup H of G which centralizes z_5 is isomorphic to the symmetric group S_4. A convenient geometric way of seeing this is to represent a rotation R in SO(3) as $\xi = (\xi_1, \xi_2, \xi_3)$, where ξ is the axis of rotation of R and whose length is the angle of rotation. The direction is chosen so that the rotation is at most π about the oriented axis. For A, B, C in (II) above, the ξ representations are $\xi_A = (0, 0, \tau), \xi_B = (0, \tau, 0)$ and $\xi_C = (\tau, 0, 0)$ with $\cos\tau = 3/5$. Now if g_0 corresponds to ξ and g is in SO(3), then $g^{-1}g_0g$ corresponds to $g\xi$. Hence since the vectors $\xi_A, \xi_{A^{-1}}, \xi_B, \xi_{B^{-1}}, \xi_C, \xi_{C^{-1}}$ are the vertices of a regular octahedron, we see that H , the rotational symmetry group, is the octahedral group S_4.

We decompose the action of H on W_{N+1} into irreducible subspaces on each of which z acts as a scalar. The irreducible representations and character table of S_4 are well known. $|S_4| = 24$ and there are five conjugacy classes. The identity element, 6 transpositions (ab), 3 elements of order 2 $(ab)(cd)$, 8 elements (abc) of order 3 and 6 elements of order 4 $(abcd)$. There are two one dimensional irreducible representations χ_0 and ε, one two dimensional representation θ and two three dimensional representations ψ and $\varepsilon\psi$. The character table together with the character $\chi_N, N = 2l$ of SO(3) (using the double cover SU(2) $\to$ SO(3)) is given in Table 1.

Table 1.

	1	(ab)	$(ab)(cd)$	(abc)	$(abcd)$
χ_0	1	1	1	1	1
ε	1	-1	1	1	-1
θ	2	0	2	-1	0
ψ	3	1	-1	0	-1
$\varepsilon\psi$	3	-1	-1	0	1
χ_{2l}	$2l+1$	$(-1)^l$	$(-1)^l$	$\varepsilon_3(l)$	$\varepsilon_4(l)$

where

$$\varepsilon_3(l) = \begin{cases} 1, & if\ l \equiv 0(\mathrm{mod}3), \\ 0, & if\ l \equiv 1(\mathrm{mod}3), \\ -1, & if\ l \equiv 2(\mathrm{mod}3), \end{cases}$$

and

$$\varepsilon_4(l) = \begin{cases} 1, & if\ l \equiv 0,1(\mathrm{mod}4), \\ -1, & if\ l \equiv 2,3(\mathrm{mod}4). \end{cases}$$

Using the scalar product $\langle \chi_{2l}, \chi_\sigma \rangle$ we can count the multiplicity m_σ with which σ appears in W_{2l+1}. We get

$$\begin{aligned} m_0 &= (2l+1+9(-1)^l + 8\varepsilon_3(l) + 6\varepsilon_4(l))/24 \\ m_\varepsilon &= (2l+1-3(-1)^l + 8\varepsilon_3(l) - 6\varepsilon_4(l))/24 \\ m_\theta &= (2l+1+3(-1)^l - 4\varepsilon_3(l))/12 \\ m_\psi &= (2l+1+(-1)^l - 2\varepsilon_4(l))/8 \\ m_{\varepsilon\psi} &= (2l+1-3(-1)^l + 2\varepsilon_4(l))/8 \end{aligned}$$

Thus if there are no degeneracies in the spectrum of $\widehat{z}_5(\pi_{2l+1})$ beyond what is forced by the symmetry group H then we would have m_0+m_ε simple eigenvalues, m_θ eigenvalues of multiplicity two, and $m_\psi + m_{\varepsilon\psi}$ eigenvalues of multiplicity three. We checked this for 10 different l's and found exactly those multiplicities in each case. Note that the multiplicity two part is "pure" and the consecutive spacing distribution for it was computed. The results were all more or less the same. Figure 5 gives the pure spectrum for three values of N of order 1000 and compares the distribution with $e^{-x}dx$ (i.e. "Poissonian"). Several values of N were taken because the pure part of the spectrum for N of that order is rather small (less than 100).

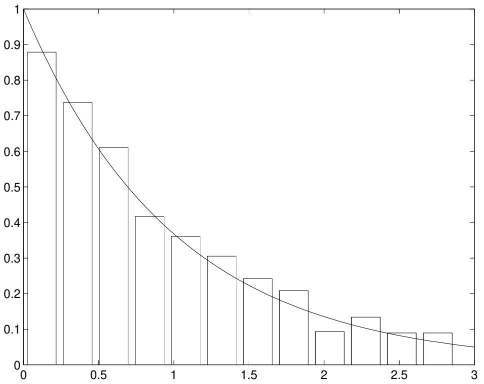

Figure 5. Spacings for A, B, C of (II), $N \approx 1000$, vs. $e^{-x}dx$ ("pure" part).

Combining all the spectrum for $\widehat{z}_5(\pi_{2l+1})$ (taking one eigenvalue out of each multiple set) gives a similar distribution, see Figure 6. The reason is that each "pure" piece is Poissonian and the superposition of independent Poissonians is again Poissonian.

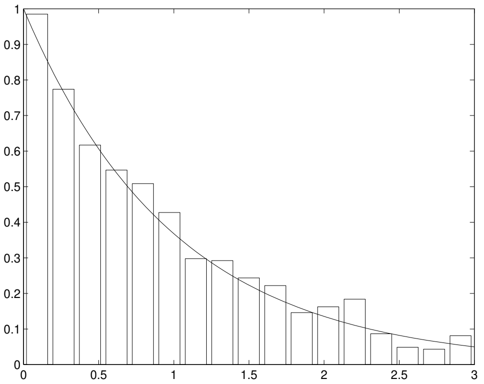

Figure 6. Spacings for A, B, C of (II), $N = 1004$, vs. $e^{-x}dx$ ("combined").

A similar analysis for the element $\tilde{z}$ (cf. (III) above) was carried out. Recall that in Theorem 1 we showed that $||\tilde{z}(\pi_N)|| \leq 5.83 < 6$. It turns out that $\tilde{z}$ is not a Ramanujan element since for example $||\tilde{z}(\pi_{80})|| = 4.5842\ldots > 4.4721\ldots = 2\sqrt{5}$. Again there are degeneracies in the spectrum due to symmetry. This time the ξ vectors corresponding to $\widetilde{A}, \widetilde{B}, \widetilde{C}$ are

$$\sqrt{\frac{\alpha}{3}}(-1,1,1), \sqrt{\frac{\alpha}{3}}(-1,-1,1), \sqrt{\frac{\alpha}{3}}(1,-1,1)$$

where $\cos\alpha = 1/7$. The vectors $\xi_{\widetilde{A}}, \xi_{\widetilde{A}^{-1}}, \xi_{\widetilde{B}}, \xi_{\widetilde{B}^{-1}}, \xi_{\widetilde{C}}, \xi_{\widetilde{C}^{-1}}$ form the three diagonals of the cube with barycenter at the origin. The rotation symmetry group is S_3 and an analysis as above with its irreducibles (there is one 2-dimensional and two 1-dimensional representations) yields a "pure" part in W_{2l+1} of multiplicity two. The dimension of the part is $2(2l+1-\varepsilon_3(l))/3$ and the rest of the space corresponds to the 1-dimensional pieces. Again this accounted for all the degeneracy in the spectrum for the l's we checked. The consecutive spacing distribution for the pure multiplicity two part is given in Figure 7 when $N = 912$. Again it was typical of what was found for other N's. The approximation by GOE is quite good. As for the multiplicity one part of the spectrum, it is not GOE. The reason is that it consists of eigenvalues from both of the 1-dimensional representations which are being superimposed.

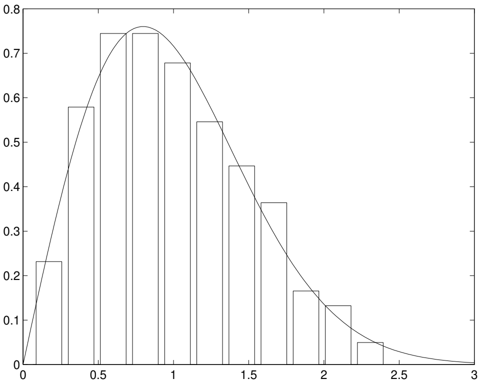

Figure 7. Spacings for $\widetilde{A}, \widetilde{B}, \widetilde{C}$ of (III), $N = 912$, vs. GOE ("pure" part).

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