QUANTUM UNIQUE ERGODICITY FOR EISENSTEIN SERIES ON $PSL_2(\mathbb{Z})\backslash PSL_2(\mathbb{R})$

DMITRY JAKOBSON

ABSTRACT. In this paper we prove microlocal version of the equidistribution theorem for Wigner distributions associated to Eisenstein series on $PSL_2(\mathbb{Z})\backslash PSL_2(\mathbb{R})$. This generalizes a recent result of W. Luo and P. Sarnak who prove equidistribution for $PSL_2(\mathbb{Z})\backslash\mathbb{H}$. The averaged versions of these results have been proven by Zelditch for an arbitrary finite-volume surface, but our proof depends essentially on the presence of Hecke operators and works only for congruence subgroups of $SL_2(\mathbb{R})$. In the proof the key estimates come from applying Meurman's and Good's results on L-functions associated to holomorphic and Maass cusp forms. One also has to use classical transformation formulas for generalized hypergeometric functions of a unit argument.

RÉSUMÉ. Nous donnons la preuve d'une version microlocale d'un resultat de W. Luo et
P. Sarnak concernant la répartition asymptotique des fonctions de Wigner associées aux series d'Eisenstein sur $PSL_2(\mathbb{Z})\backslash PSL_2(\mathbb{R})$. La preuve utilise les opérateurs de Hecke, donc il est valable ceulement pour les sous-groupes de congruence de $SL_2(\mathbb{R})$.

0. Introduction. Let Γ denote $PSL_2(\mathbb{Z}) \subset G = PSL_2(\mathbb{R})$ and let $X = \Gamma\backslash\mathbb{H}$ be the usual Riemann surface of constant negative curvature (with the metric $\frac{dx\,dy}{y^2}$) in the upper half-plane. We denote by φ_j an orthonormal basis of L^2-eigenfunctions of a Laplacian $\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2})$ with the eigenvalues $\lambda_j = \frac{1}{4} + r_j^2$ on X (discrete spectrum), and by $E(z,s)$ the Eisenstein series, which correspond to the continuous spectrum (eigenvalue $\frac{1}{4} + t^2$) when $s = \frac{1}{2} + it$.

We write Iwasawa decomposition of an element g of $SL_2(\mathbb{R})$ (cf. [K]) as

$$g = n(x)a(y)\mathbf{k}(\theta), \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ a(y) = \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix},$$

$$\mathbf{k}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{0.1}$$

Then $\mathbb{H}$ can be identified with G/K (where $K = SO(2)/\pm I$) and $\Gamma\backslash G$ can be considered as the unit tangent bundle S^*X of $X = \Gamma\backslash\mathbb{H}$. We denote by $z = x + iy$ points in $\mathbb{H}$.

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Let A be a pseudo-differential operator of order zero on $\Gamma\backslash G$, σ_A be its principal symbol. Zelditch has shown in [Z2] that one can find a subsequence j_k of j-s of density one such that

$$\lim_{k\to\infty}(A\varphi_{j_k},\varphi_{j_k}) \;=\; \frac{1}{vol(S^*X)}\int_{S^*X}\sigma_A\,d\omega, \tag{0.2}$$

where $d\omega = \frac{dx\,dy\,d\theta}{2\pi y^2}$ is a Liouville measure[1] on $\Gamma\backslash G$ and (,) is a scalar product on S^*X. This is a generalization to finite-area surfaces of a well-known result for compact surfaces due to Shnirelman, Zelditch, and Colin-de-Verdiere (cf. [CdV], [Sn1], [Sn2] and [Z1]). Actually, Zelditch showed that Maass cusp forms φ_j and Eisenstein series $E(z,\frac{1}{2}+it)$ become "on average" equidistributed in S^*X (cf. [Z2] for the precise definitions).

While results in [Z2] are valid not only for $\Gamma = SL_2(\mathbb{Z})$ but also for an arbitrary finite-area surface $\Gamma\backslash G$, the question of individual equidistribution for cusp forms and Eisenstein series remained open. One can reformulate this questions for $\Gamma\backslash\mathbb{H}$. Namely, it is a consequence of (0.2) that if Ω is a Jordan set in X, then

$$\lim_{k\to\infty}\int_\Omega|\varphi_{j_k}|^2 \;=\; \frac{area(\Omega)}{area(X)},$$

where *area* means hyperbolic area in $\mathbb{H}$ and j_k is a subsequence of j of density one. The question then arises whether the above is true for *all* j. This question hasn't been settled, but an analogue of the above statement for Eisenstein series was proved by W. Luo and P. Sarnak in [L-S] (for congruence subgroups of $SL_2(\mathbb{R})$). Namely, it is shown that for arbitrary Jordan Ω_1 and Ω_2 in X,

$$\lim_{t\to\infty}\frac{\int_{\Omega_1}|E(z,\frac{1}{2}+it)|^2\,\frac{dx\,dy}{y^2}}{\int_{\Omega_2}|E(z,\frac{1}{2}+it)|^2\,\frac{dx\,dy}{y^2}} \;=\; \frac{area(\Omega_1)}{area(\Omega_2)}$$

$(area(\Omega_2)\neq 0)$, which follows from

$$\int_X F(z)|E(z,1/2+it)|^2\frac{dx\,dy}{y^2} \;\sim\; \frac{48}{\pi}\int_X F(z)\,\frac{dx\,dy}{y^2}\;\ln t, \tag{0.3}$$

where $F(z)$ is a continuous function of a compact support and $E(z,s)$ is an Eisenstein series.

The above result is proved without considering S^*X, so it becomes natural to generalize (0.3) by proving an individual $PSL_2(\mathbb{Z})$-version of Zelditch's "average" results in [Z2], which we do in this paper.

To formulate an analogue of (0.3) for S^*X one has to "lift" the distribution $|E(z,\frac{1}{2}+it)|^2$ to S^*X. One can formulate it as follows: given a pseudo-differential operator A of order zero with a principal symbol σ_A, we want to find such a distribution $d\mu_t$ on S^*X that $(AE(z,\frac{1}{2}+it),E(z,\frac{1}{2}+it)) = \int_{S^*X}\sigma_A d\mu_t$.

[1] In this paper functions on $PSL_2(\mathbb{R})$ are identified with functions on $SL_2(\mathbb{R})$ that are invariant under the action of $SL_2(\mathbb{Z})$, and "integration" means integration over $SL_2(\mathbb{R})$ in coordinates (0.1).

The answer was pointed out by Zelditch in [Z2] (it first appeared in [Sa] in a different context). It is given (in the notation of [Sa] and in the coordinates (0.1)) by

$$(0.4)\quad d\mu_t \ = \ E(\cdot, 1/2+it)\bar{E}^{\infty}(\cdot, 1/2+it)d\omega,\ E^{\infty}(z,\theta,s) \ \sim \ \sum_k E_{2k}(z,s)e^{2ik\theta}$$

Here $E(z,s) = E_0(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y(\gamma z)^s$ is the usual Eisenstein series, and $E_{2k}(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y(\gamma z)^s(\epsilon_\gamma(z))^{2k}$, where $\epsilon_\gamma(z) = \frac{(cz+d)}{|cz+d|}$ with $\gamma = \begin{pmatrix} * & * \\ c & d\end{pmatrix}$. In the preceding formulas $\Gamma = PSL_2(\mathbb{Z})$ and $\Gamma_\infty = \{\begin{pmatrix} 1 & m \\ 0 & 1\end{pmatrix}|m \in \mathbb{Z}\}$. The distribution $d\mu_t$ can be called a Wigner distribution or a Wigner function (cf. [Sn2]) since the usual Fourier-Wigner transform satisfies the defining property of $d\mu_t$ ([Fo], Proposition 2.5).

As remarked by Zelditch, $d\mu_t$, though useful for asymptotic computations, is not a positive distribution. To make it positive, one uses the technique of Friedrichs symmetrization. A new distribution $d\mu_t^F$ is defined (cf. [Z1], [Z2]) by

$$(0.5)\qquad (\sigma, d\mu_t^F) \ = \ (\sigma^F, d\mu_t)$$

for $\sigma \in C_c(\Gamma\backslash G)$, where σ^F is a Friedrichs symmetrization of σ (cf. [Z1], [Z2]). The expression $d\mu_t^F$ is now a positive distribution and is asymptotically equivalent (cf. §4) to $d\mu_t$ ([Z2], Proposition 3.8).

While first defined for compactly supported symbols, $d\mu_t$ and $d\mu_t^F$ can be proved to lie in $\mathcal{S}'(\Gamma\backslash G)$, where $\mathcal{S}$ is a Schwartz space ([Z2], Proposition 3.6).

The analogue of W. Luo and P. Sarnak's result for $\Gamma\backslash G$ can now be given in terms of measure $d\mu_t^F$:

Theorem 1. *Let Ω_1 and Ω_2 be arbitrary Jordan sets in $\Gamma\backslash G$. Then*

$$\lim_{t\to\infty}\frac{\int_{\Omega_1} d\mu_t^F}{\int_{\Omega_2} d\mu_t^F} \ = \ \frac{vol(\Omega_1)}{vol(\Omega_2)}$$

($vol(\Omega_2) \neq 0$).

The proof of Theorem 1 follows the general outline of W. Luo and P. Sarnak's proof of (0.3). Namely, one establishes the asymptotic estimates of $(f, d\mu_t)$ (as $t \to \infty$) for functions f constituting orthonormal basis of $L^2(\Gamma\backslash G)$: holomorphic cusp forms, "shifted" Maass cusp forms, and incomplete Eisenstein series. In section 1 we find Fourier expansions of all relevant objects (analogous calculations can be found in [Z2] and in Fay's article [Fa]). In section 2 we consider holomorphic and Maass cusp forms, and in section 3 we turn to incomplete Eisenstein series. In section 4 we use the estimates obtained in sections 2 and 3 to prove an analogue of W. Luo and P. Sarnak's result (0.3) (Proposition 4.1), from which Theorem 1 follows. In the Appendix we prove a technical lemma needed in section 3.

The arithmetical considerations in the proof of Theorem 1 are similar to those in [L-S], but in order to reduce the problem to some known estimates of L-functions on a critical lineone has to deal with generalized hypergeometric series and their transformation formulas. The basic reference for this is Bailey's book [Ba].

1. Fourier expansions. The author is indebted to Professors Zelditch and Eskin for helpful discussions. There exists a unique operator Ω (called Casimir) on functions on $G = SL_2(\mathbb{R})$ commuting with the action of G. In the coordinates (0.1) it is given by

$$y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\right)+y\frac{\partial^2}{\partial x\partial\theta}=\Delta+y\frac{\partial^2}{\partial x\partial\theta}$$

This operator acts as $\Delta_{2k} = \Delta - 2iky\frac{\partial}{\partial x}$ on functions "of weight $2k$" ($f(g) = f(x,y)e^{-2ik\theta}$), with $g \in G$ as in (0.1). In particular, when $k = 0$ Casimir becomes the usual Laplacian. Functions on $\Gamma\backslash G$ can be identified with functions on G satisfying $f(\gamma g) = f(g), \forall g \in G, \gamma \in \Gamma$. The space of all eigenfunctions of Ω on $\Gamma\backslash G$ is a direct sum of spaces of functions of weight $2k, k \in \mathbb{Z}$ (cf. [K]).

There are two operators E_+ and E_- (called "raising" and "lowering") which map eigenfunctions of Casimir on $\Gamma\backslash G$ of weight $2k$ into those of weight $2k+2$ and $2k-2$, respectively. These operators are given by

$$E_+ \;=\; e^{-2i\theta}(2iy\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}+i\frac{\partial}{\partial\theta}),\quad E_- \;=\; e^{2i\theta}(2iy\frac{\partial}{\partial x}-2y\frac{\partial}{\partial y}+i\frac{\partial}{\partial\theta})$$

in coordinates (0.1) (cf. [L]; note that in Lang's book the angle in Iwasawa decomposition is equal to minus that in (0.1), hence the difference in formulas).
To every function on $\Gamma\backslash G$ of weight $2k$ there corresponds a function on $\Gamma\backslash\mathbb{H}$ satisfying $f(\gamma z) = \left(\frac{(cz+d)}{|cz+d|}\right)^{2k} f(z)$ for $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ (we shall say that such functions are also of weight $2k$) and vice versa. Namely, if $f(g)$ a function on $\Gamma\backslash G$ of weight $2k$, the expression $f_1(z) = f(g)e^{2ik\theta}$ in the coordinates (0.1) gives a well-defined function on $\Gamma\backslash\mathbb{H}$ of weight $2k$ (cf. [K]). Conversely, suppose that a function f depends on g in the coordinates (0.1) as

$$f(g) = f_1(x,y)f_2(\theta) = f_1(z)f_2(\theta),\quad z = x+iy.$$

and that f_1 is of weight $2k$. Since the left multiplication of g by $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ can be written in the coordinates (0.1) as $z \to \left(\frac{(az+b)}{|cz+d|}\right)$ and $\theta \to \theta + arg(cz+d)$, to get a well-defined function on $\Gamma\backslash G$ we must let

$$f_2(\theta) = e^{-2ik\theta}.$$

The action of E_+ and E_- on functions of weight $2k$ on G corresponds to the action on functions of weight $2k$ on $\mathbb{H}$ of "Maass" operators K_{2k} and Λ_{2k} given by

$$K_{2k} \;=\; iy\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+k,\quad \Lambda_{2k} \;=\; iy\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}-k \tag{1.1}$$

(our notation is that of [Ro] and different from [Fa], [Z2]). In fact, for F and f as above,

$$E_+F(z,\theta) = 2e^{-2i(k+1)}K_{2k}f(z),\;\; E_-F(z,\theta) = -2e^{-2i(k-1)}\Lambda_{2k}f(z) \tag{1.2}$$

([Ro], p. 318). It is also shown in [Ro] that K_{2k} maps eigenfunctions of Δ_{2k} into those of Δ_{2k+2}, while Λ_{2k} maps them into eigenfunctions of Δ_{2k-2} (as well as numerous relations between K_k-s, Λ_k-s, and Δ_k-s).

Now, "shifted" Maass cusp forms on $PSL_2(\mathbb{Z})\backslash PSL_2(\mathbb{R})$ are none other than images of Maass cusp forms φ_j (cf. Introduction) on $PSL_2(\mathbb{Z})\backslash\mathbb{H}$ under repeated applications of E_+ and E_-. In view of the above relations, to find the Fourier expansions of shifted Maass cusp forms it suffices to understand the action of K_{2k} and Λ_{2k} on eigenfunctions of Δ_{2k} (if we know the Fourier expansion of φ_j-s). Moreover, $E_{2k}(z,s), k>0$ is a function of weight $-2k$ (which can be seen from the definition $E_{2k}(z,s)=\sum_{\sigma\in\Gamma_\infty\backslash\Gamma} y(\sigma z)^s(\epsilon_\sigma(z))^{2k}$ and an identity $\epsilon_\sigma(\gamma z)=\epsilon_{\sigma\gamma}(z)/\epsilon_\gamma(z)$); it is also an eigenfunction of Δ_{-2k} (cf. [K]) and is a multiple of $\Lambda_{-(2k-2)}...\Lambda_{-2}\Lambda_0 E_0(z,s)$. Similarly, $E_{-2k}(z,s), k>0$ is a multiple of $K_{2k-2}...K_2K_0E_0(z,s)$.

Before proceeding to calculate Fourier expansions, we state several facts about the eigenfunctions of Δ_{2k} on $\Gamma\backslash\mathbb{H}$. Such functions are periodic in x with period 1, so one can expand them into Fourier series $\sum_{n=-\infty}^{\infty} c_n(y)\mathbf{e}(nx)$, where $\mathbf{e}(nx)=e^{2\pi i n x}$. Moreover, by separation of variables the equation $\Delta_{2k}f(z,s)+s(1-s)f(z,s)=0$ reduces to the second-order ODE for the coefficients $c(n,y)$. Solving that equation yields

$$c(n,y) \;=\; a(n)W_{k\cdot sgn(n),s-1/2}(4\pi|n|y),$$

where $W_{\lambda,\mu}(y)$ is a Whittaker function satisfying (cf. [G-R] 9.220, p.1059)

$$\frac{d^2W}{dy^2}+\left[-\frac{1}{4}+\frac{\lambda}{y}+\frac{1/4-\mu^2}{y^2}\right]W \;=\; 0.$$

So, the problem reduces to finding $a(n)$-s.

To do this, we first write the Fourier expansion of $E(z,s)$ as a function of x (where $z=x+iy$). It is

$$E(z,s) \;=\; y^s+\phi(s)y^{1-s}+2\frac{\sqrt{y}}{\xi(2s)}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)K_{s-\frac{1}{2}}(2\pi n y)\cos(2\pi n x),$$

where K is the K-Bessel function, $\phi(s)=\frac{\xi(2s-1)}{\xi(2s)}$, $\xi(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, and where $\sigma_\nu(n)=\sum_{d|n} d^\nu$.

We now rewrite the above formula in terms of Whittaker function W_0, also substituting $s=1/2+it$. Functions K and W are connected by the formula ([G-R] 9.235.2, p. 1062) $W_{0,\mu}(y)=\sqrt{(y/\pi)}K_\mu(y/2)$. Also, it is convenient for later calculations to write $\cos 2\pi n x$ as $(\mathbf{e}(nx)+\mathbf{e}(-nx))/2$, so that the above formula becomes

$$E(z,1/2+it) \;=\; y^{1/2+it}+\phi(1/2+it)y^{1/2-it}+$$

$$\frac{1}{2\xi(1+2it)}\sum_{n\neq 0}|n|^{-\frac{1}{2}+it}\sigma_{-2it}(|n|)W_{0,it}(4\pi|n|y)\mathbf{e}(nx) \tag{1.3}$$

The (even) Maass cusp forms φ_j are assumed to be Hecke eigenforms, and their Fourier expansion is given by

$$\varphi_j(z) = \sum_{n\neq 0} \frac{c_j(|n|)}{\sqrt{|n|}} W_{0,ir_j}(4\pi|n|y)\mathbf{e}(nx), \tag{1.4}$$

where $\Delta\varphi_j + (1/4+r_j^2)\varphi_j = 0$, and $c_j(n)$-s are Hecke eigenvalues (henceforth simply denoted by $c(n)$ for fixed j). Recall that $c(1)$ is normalized to be 1 and $c(n)$-s satisfy $c(mn) = c(m)c(n)$ for $(m,n) = 1$ and $c(p^n)c(p) = c(p^{n+1}) + p^{n-1}c(p^{n-1})$ for p prime and $n \geq 1$. The odd Maass cusp forms have a similar expansion (each term is multiplied by $sgn(n)$.)

We are now ready to make the computations. The basic question is how operators K_{2k} and Λ_{2k} act on $W_{k\cdot sgn(n),it}(4\pi|n|y)\mathbf{e}(nx)$, which is a typical term in a Fourier expansion of an eigenfunction of Δ_{2k} with an eigenvalue $1/4+t^2$. The answer is given in the Lemma below[2] (compare [Fa], (72)).

Lemma 1.1. *With the above notation,*

$$K_{2k}(W_{k\cdot sgn(n),it}(4\pi|n|y)\mathbf{e}(nx)) = \mathbf{e}(nx)\begin{cases} -W_{k+1,it}(4\pi|n|y), & n>0, \\ \left[\frac{1}{4}+t^2+k(k+1)\right] W_{-(k+1),it}(4\pi|n|y), & n<0 \end{cases}$$

and

$$-\Lambda_{2k}(W_{k\cdot sgn(n),it}(4\pi|n|y)\mathbf{e}(nx)) = \mathbf{e}(nx)\begin{cases} -W_{k-1,it}(4\pi|n|y), & n<0, \\ \left[\frac{1}{4}+t^2+k(k-1)\right] W_{-(k-1),it}(4\pi|n|y), & n>0 \end{cases}$$

We shall postpone the proof of the Lemma until the end of section 1, and use it now to derive Various Fourier expansions. We first compute $K_{2k-2}...K_2K_0E(z,1/2+it)$. Clearly, $K_{2k}y^{1/2+it} = (1/2+it+k)y^{1/2+it}$. Using this and Lemma 1.1 we find that

$$\begin{aligned} K_{2k-2}...K_2K_0E(z,1/2+it) &= d_k(t)y^{1/2+it} + d_k(-t)\phi(1/2+it)y^{1/2-it} + \\ \frac{1}{2\xi(1+2it)}\Bigg[(-1)^k \sum_{n>0} \frac{|n|^{it}\sigma_{-2it}(|n|)}{\sqrt{|n|}} W_{k,it}(4\pi|n|y)\mathbf{e}(nx) &+ b_k(t)\sum_{n<0}\frac{|n|^{it}\sigma_{-2it}(|n|)}{\sqrt{|n|}} W_{-k,it}(4\pi|n|y)\mathbf{e}(nx)\Bigg], \end{aligned} \tag{1.5}$$

where

$$d_k(t) = \left(\frac{1}{2}+it\right)\left(\frac{3}{2}+it\right)...\left(\frac{2k-1}{2}+it\right),$$

$$b_k(t) = \left(\frac{1}{4}+t^2\right)\left(\frac{1}{4}+t^2+1\cdot 2\right)...\left(\frac{1}{4}+t^2+(k-1)k\right)$$

We notice that $b_k(t) = d_k(t)\cdot d_k(-t)$.

On the other hand, we know (cf. [Sa], [K]) that the zeroth term of the Fourier expansion of $E_{-2k}(z,1/2+it)$ is equal to

$$y^{1/2+it} + (-1)^k \frac{\Gamma^2(\frac{1}{2}+it)}{\Gamma(\frac{1}{2}+it-k)\Gamma(\frac{1}{2}+it+k)}\phi(1/2+it)y^{1/2-it}.$$

[2] Since the action of E_- is conjugate to the action of $-2\Lambda_{2k}$ by (1.2), we give the answer for $-\Lambda_{2k}$.

Accordingly, we need to divide (1.5) by $d_k(t)$ (the coefficient of $y^{1/2+it}$ in (1.5)) to get the Fourier expansion of $E_{-2k}(z,1/2+it)$. We can then check that we get the same answer for the coefficient of $y^{1/2-it}$ from (1.5) as above, since $d_k(t) = \Gamma(1/2+it+k)/\Gamma(1/2+it)$ and $d_k(-t) = (-1)^k\Gamma(1/2+it)/\Gamma(1/2+it-k)$.

Finally, we get the desired Fourier expansion of $E_{-2k}(z,1/2+it)$ (where $k>0$):

$$\begin{aligned}(1.6)\qquad E_{-2k}(z,1/2+it) \;=\;& y^{1/2+it} + \frac{(-1)^k\Gamma^2(\frac{1}{2}+it)}{\Gamma(\frac{1}{2}-k+it)\Gamma(\frac{1}{2}+k+it)}\phi(1/2+it)y^{1/2-it}+\\ & \frac{(-1)^k\Gamma(1/2+it)}{2\Gamma(\frac{1}{2}+k+it)\xi(1+2it)}\sum_{n>0}\frac{|n|^{it}\sigma_{-2it}(|n|)}{\sqrt{|n|}}W_{k,it}(4\pi|n|y)\mathrm{e}(nx)+\\ & \frac{(-1)^k\Gamma(1/2+it)}{2\Gamma(\frac{1}{2}-k+it)\xi(1+2it)}\sum_{n<0}\frac{|n|^{it}\sigma_{-2it}(|n|)}{\sqrt{|n|}}W_{-k,it}(4\pi|n|y)\mathrm{e}(nx)\end{aligned}$$

An analogous calculation yields the expansion of $E_{2k}(z,1/2+it)$ (where $k>0$):

$$\begin{aligned}(1.7)\qquad E_{2k}(z,1/2+it) \;=\;& y^{1/2+it} + \frac{(-1)^k\Gamma^2(\frac{1}{2}+it)}{\Gamma(\frac{1}{2}-k+it)\Gamma(\frac{1}{2}+k+it)}\phi(1/2+it)y^{1/2-it}+\\ & \frac{(-1)^k\Gamma(1/2+it)}{2\Gamma(\frac{1}{2}+k+it)\xi(1+2it)}\sum_{n<0}\frac{|n|^{it}\sigma_{-2it}(|n|)}{\sqrt{|n|}}W_{k,it}(4\pi|n|y)\mathrm{e}(nx)+\\ & \frac{(-1)^k\Gamma(1/2+it)}{2\Gamma(\frac{1}{2}-k+it)\xi(1+2it)}\sum_{n>0}\frac{|n|^{it}\sigma_{-2it}(|n|)}{\sqrt{|n|}}W_{-k,it}(4\pi|n|y)\mathrm{e}(nx)\end{aligned}$$

We now turn to shifted Maass forms. Let us denote by $\varphi_{j,k}$ (where $k>0$) the image of φ_j under $K_{2k-2}...K_2K_0$ normalized to have the same norm as φ_j. The expression $K_{2k-2}...K_2K_0\varphi_j$ can be computed using (1.4) and Lemma 1.1. The result is an analogue of (1.5) with r_j substituted for t. It turns out that the correct normalization for $\varphi_{j,k}$ is obtained (in the above notation) by division by $d_k(r_j)$ (cf. [Z1] and [Z2], p. 25). Accordingly, the Fourier expansion of (an even) shifted Maass cusp form is given by

$$\begin{aligned}(1.8)\qquad \varphi_{j,k}(z) \;=\;& \frac{(-1)^k\Gamma(1/2+ir_j)}{\Gamma(\frac{1}{2}+k+ir_j)}\sum_{n>0}\frac{c_j(|n|)}{\sqrt{|n|}}W_{k,ir_j}(4\pi|n|y)\mathrm{e}(nx)+\\ & \frac{(-1)^k\Gamma(1/2+ir_j)}{\Gamma(\frac{1}{2}-k+ir_j)}\sum_{n<0}\frac{c_j(|n|)}{\sqrt{|n|}}W_{-k,ir_j}(4\pi|n|y)\mathrm{e}(nx).\end{aligned}$$

The odd Maass cusp form has (up to a change of sign) the same expansion with the sign of the second sum changed.

Similarly, denoting by $\varphi_{j,-k}$ (where $k>0$) the expression $\Lambda_{-(2k-2)}...\Lambda_{-2}\Lambda_0\varphi_j$ and normalizing correctly (cf. [Z1] and [Z2], p. 25), we get the analogue of (1.8) for $\varphi_{j,-k}$ which will be omitted here.

Finally, we give the

Proof of Lemma 1.1. We first remark that K_{2k} acts on $W_{k\cdot sgn(n),it}(4\pi|n|y)\mathrm{e}(nx)$ as $y\frac{\partial}{\partial y}-2\pi ny+k$. The first formula in Lemma 1.1 now becomes an easy consequence of two recursion relations for Whittaker functions. The first property ([G-R] 9.234.3, p. 1062) is

$$(1.9\text{a})\qquad z\frac{\partial}{\partial z}\left[W_{\lambda,\mu}(z)\right] \;=\; \left[\lambda-\frac{z}{2}\right]W_{\lambda,\mu}(z)-\left[\mu^2-\left(\lambda-\frac{1}{2}\right)^2\right]W_{\lambda-1,\mu}(z)$$

and the second one ([A-S] 13.4.31, p. 507) is

$$W_{\lambda,\mu}(z) \;=\; [2-2\lambda+z]\,W_{\lambda-1,\mu}(z)+\left[\mu^2-\left(\lambda-\frac{3}{2}\right)^2\right]W_{\lambda-2,\mu}(z) \tag{1.9b}$$

After substituting $z=4\pi|n|y, \lambda=k\cdot sgn(n), \mu=it$ into (1.9a) we get

$$y\frac{\partial}{\partial y}\left[W_{k\cdot sgn(n),it}(4\pi|n|y)\right] \;=\; [k\cdot sgn(n)-2\pi|n|y]\,W_{k\cdot sgn(n),it}(4\pi|n|y)+ \tag{1.10}$$

$$\left[\frac{1}{4}+t^2+k^2-k\cdot sgn(n)\right]W_{k\cdot sgn(n)-1,it}(4\pi|n|y).$$

Direct application of (1.10) proves the first formula in Lemma 1.1 for $n<0$, while for $n>0$ we need to substitute $z=4\pi|n|y, \lambda=k+1$ and $\mu=it$ into (1.9b) and apply the resulting formula. The proof of the second formula in Lemma 1.1 is analogous.

2. Holomorphic and Maass cusp forms. We now turn to studying asymptotics of integrals of holomorphic and Maass cusp forms against $d\mu_t$ (cf. (0.4). We start with holomorphic cusp forms. A "classical" holomorphic cusp form on $\mathbb{H}/SL_2(\mathbb{Z})$ of weight $2k$ is a function $F(z)$ satisfying $F(\gamma z)=(cz+d)^{2k}F(z)$ for $\gamma=\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in\Gamma$. Also, we assume it to be an eigenfunction of all Hecke operators, hence its Fourier expansion is

$$F(z) \;=\; \sum_{n>0}c(n)\mathbf{e}(nz) \;=\; \sum_{n>0}c(n)e^{-2\pi ny}\mathbf{e}(nx),$$

where $c(n)$-s satisfy the same relations as in (1.4). One can check that if we define $f(x,y,\theta)=e^{-2ik\theta}y^kF(x,y)$ (in the coordinates (0.1)), we will get a well-defined eigenfunction of Casimir in $L^2(\Gamma\backslash G)$ with an eigenvalue $k(1-k)$. It is this function that we want to integrate against $d\mu_t$.

We want to prove the following

Proposition 2.1. *With the above notation,*

$$|(f,d\mu_t)| \;\ll_\varepsilon\; |t|^{-\frac{1}{6}+\varepsilon}$$

Proof of Proposition. The Fourier expansion of $f(z,\theta)$ is

$$f(z,\theta) \;=\; e^{-2ik\theta}y^k\sum_{n>0}c(n)e^{-2\pi ny}\mathbf{e}(nx) \tag{2.1}$$

It follows from (0.4) that after integrating θ out in the expression

$$(f,d\mu_t) \;=\; \int_{\Gamma\backslash G}f(\cdot)E(\cdot,1/2-it)E^{\infty}(\cdot,s)d\omega$$

we get

$$\int_{\Gamma\backslash\mathbb{H}} f(z,0)E(z,\frac{1}{2}-it)E_{2k}(z,s)\frac{dxdy}{y^2} \;=\; \int_{\Gamma\backslash\mathbb{H}} f(z,0)E(z,\frac{1}{2}-it)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y(\gamma z)^s(\epsilon_\gamma(z))^{2k}\frac{dxdy}{y^2}$$

We change summation and integration signs and rewrite the last expression as

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\Gamma\backslash\mathbb{H}} f(z,0)E(z,\frac{1}{2}-it)y(\gamma z)^s(\epsilon_\gamma(z))^{2k}\,\frac{dxdy}{y^2} =$$

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma(\Gamma\backslash\mathbb{H})} f(w,0)E(w,\frac{1}{2}-it)y(w)^s\,\frac{dxdy}{y^2}$$

(in the last equality we made a change of variables $w=\gamma z$ and used the transformation properties of f).

The last expression is equal to

$$\int_{\bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma(\Gamma\backslash\mathbb{H})} f(w,0)E(w,\frac{1}{2}-it)y(w)^s\,\frac{dxdy}{y^2} =$$

$$\int_{\Gamma_\infty\backslash\mathbb{H}} f(w,0)E(w,\frac{1}{2}-it)y(w)^s\,\frac{dxdy}{y^2}$$

By analytic continuation the above equality holds for $s=\frac{1}{2}+it$. After substituting (2.1) for $f(z,0)$, we find that

$$(f,d\mu_t) \;=\; \int_0^\infty\int_0^1 y^{1/2+it}y^k\sum_{n>0}c(n)e^{-2\pi ny}\mathbf{e}(nx)E(z,1/2-it)\frac{dxdy}{y^2}$$

We substitute (1.3) for $E(z,1/2-it)$ (with sign of t changed) into the above expression, we change summation and integration signs, and after integrating out x (only terms with $n<0$ in (1.3) give nonzero contribution), we get

$$(f,d\mu_t) \;=\; \frac{1}{2\xi(1-2it)}\sum_{n=1}^{\infty}\int_0^\infty y^{k+1/2+it}\frac{c(n)|n|^{-it}\sigma_{2it}(|n|)}{\sqrt{|n|}}e^{-2\pi ny}W_{0,-it}(4\pi|n|y)\frac{dy}{y^2},$$

which after change of variables $u=4\pi|n|y$ gives us the final answer:

(2.2)
$$(f,d\mu_t) \;=\; \frac{1}{2\xi(1-2it)(4\pi)^{k-1/2+it}}\left(\sum_{n=1}^{\infty}\frac{c(n)n^{-it}\sigma_{2it}(n)}{n^{k+it}}\right)\int_0^\infty W_{0,-it}(u)e^{-u/2}u^{k-1/2+it}\frac{du}{u}$$

The integral in (2.2) is equal to ([G-R] 7.621.11, p. 861)

$$\frac{\Gamma(k+2it)\Gamma(k)}{\Gamma(k+\frac{1}{2}+it)}$$

The infinite sum in brackets is an Euler product of degree four which decomposes into a product of two Euler products of degree two. It can be evaluated (cf. [L-S]) and is equal to

$$\frac{L(F,k+2it)L(F,k)}{\zeta(2k+2it)}$$

where $L(\psi, s)$ is an L-function associated to a holomorphic cusp form ψ. This function can be meromorphically continued to the whole plane and satisfies a functional equation (the critical line is $\mathrm{Re}(s) = k$).

To prove that $(f, d\mu_t) \to 0$ as $t \to \infty$ we need an estimate for $L(F, s)$ on a critical line of the form

$$|L(F, k + it)| \ll_\varepsilon |t|^{\alpha+\varepsilon}, \quad \alpha < \frac{1}{2}$$

The trivial "convexity bound" for L (coming from applying Hadamard's theorem) gives $\alpha = \frac{1}{2}$, which is not sufficient to prove the decay of $(f, d\mu_t)$. Therefore, it is necessary to "break convexity," which was done by Good in [Go].

Good proves that $L(F, s)$ satisfies

$$|L(F, k + it)| \ll_\varepsilon |t|^{\frac{1}{3}+\varepsilon} \tag{2.3}$$

Now, substituting for ξ in (2.2), we write

$$(f, d\mu_t) = \frac{\Gamma(k + 2it)\Gamma(k)L(F, k + 2it)L(F, k)}{4^{k+it}\pi^{k-1+2it}\Gamma(\frac{1}{2} - it)\zeta(1 - 2it)\Gamma(k + \frac{1}{2} + it)\zeta(2k + 2it)}$$

Finally, we use Stirling's formula, inequality (2.3), and the estimate (cf. [Ti])

$$\ln|t|^{-1} \ll |\zeta(1 + it)| \ll \ln|t| \tag{2.4}$$

to finish the proof of Proposition 2.1.

We now turn to Maass cusp forms. Given $\varphi_{j,k}(z)$ as in (1.8), define $\varphi_{j,k}(z, \theta) = \varphi_{j,k}(z)e^{-2ik\theta}$ to get a well-defined eigenfunction of Casimir on $\Gamma\backslash G$ with an eigenvalue $1/4 + r_j^2$. We want to estimate asymptotics of $(\varphi_{j,k}, d\mu_t)$. We will make calculations only for even Maass cusp forms $\varphi_{j,k}$ with k positive. The calculations for k negative and for odd Maass cusp forms are analogous (since their Fourier expansions are very similar).

We will prove the following

Proposition 2.2. *Notation as above,*

$$|(\varphi_{j,k}, d\mu_t)| \ll_{j,k,\varepsilon} |t|^{-\frac{1}{6}+\varepsilon}$$

Proof of Proposition. We proceed as we did for holomorphic cusp forms. We first integrate out θ (again, only $E_{2k}(z, 1/2+it)e^{2ik\theta}$ in $E^\infty(g, 1/2+it)$ – cf. (0.4) – will give a nonzero contribution). Then we "unfold" $E_{2k}(z, 1/2 + it)$ and the Fourier expansions (1.8) and (1.3) into the formula to get

$$(\varphi_{j,k}, d\mu_t) = \int_0^\infty \int_0^1 y^{\frac{1}{2}+it}(-1)^k\Gamma(\frac{1}{2} + ir_j)$$

$$\sum_{n=1}^\infty \frac{c_j(n)}{\sqrt{n}} \left(\frac{W_{k,ir_j}(4\pi ny)\mathrm{e}(nx)}{\Gamma(\frac{1}{2} + k + ir_j)} + \frac{W_{-k,ir_j}(4\pi ny)\mathrm{e}(-nx)}{\Gamma(\frac{1}{2} - k + ir_j)} \right)$$

$$\left[y^{1/2-it}+\phi(1/2-it)y^{1/2+it}+\frac{1}{\xi(1-2it)}\sum_{n=1}^{\infty}\frac{\sigma_{2it}(n)}{n^{1/2+it}}W_{0,-it}(4\pi ny)\left(\frac{\mathrm{e}(nx)+\mathrm{e}(-nx)}{2}\right)\right]\frac{dxdy}{y^2}$$

After changing summation and integration signs and integrating x out, we get

$$(\varphi_{j,k},d\mu_t)=\frac{(-1)^k\Gamma(\frac{1}{2}+ir_j)}{2\xi(1-2it)}\sum_{n=1}^{\infty}\frac{c_j(n)\sigma_{2it}(n)}{n^{1+it}}$$
$$\left[\frac{\int_0^\infty W_{k,ir_j}(4\pi ny)W_{0,-it}(4\pi ny)y^{-\frac{1}{2}+it}\frac{dy}{y}}{\Gamma(\frac{1}{2}+k+ir_j)}+\frac{\int_0^\infty W_{-k,ir_j}(4\pi ny)W_{0,-it}(4\pi ny)y^{-\frac{1}{2}+it}\frac{dy}{y}}{\Gamma(\frac{1}{2}-k+ir_j)}\right]$$

After making a change of variable $u=4\pi ny$ in both integrals, we obtain the following expression for $(\varphi_{j,k},d\mu_t)$:

$$\frac{(-1)^k\Gamma(\frac{1}{2}+ir_j)}{2(4\pi)^{it-\frac{1}{2}}\xi(1-2it)}\sum_{n=1}^{\infty}\frac{c_j(n)\sigma_{2it}(n)}{n^{1/2+2it}}\tag{2.5}$$
$$\left[\frac{\int_0^\infty W_{k,ir_j}(u)W_{0,-it}(u)u^{it-\frac{1}{2}}\frac{du}{u}}{\Gamma(\frac{1}{2}+k+ir_j)}+\frac{\int_0^\infty W_{-k,ir_j}(u)W_{0,-it}(u)u^{it-\frac{1}{2}}\frac{du}{u}}{\Gamma(\frac{1}{2}-k+ir_j)}\right]$$

The infinite sum in (2.5) is an Euler product of degree four. As for holomorphic cusp forms, it is a product of two Euler products of degree 2. It can be evaluated (cf. [L-S]) and is equal to

$$\frac{L(\varphi_j,1/2+2it)L(\varphi_j,1/2)}{\zeta(1+2it)},$$

where $L(\varphi_j,s)$ is an L-function associated to Maass cusp form φ_j. This function can be meromorphically continued to the whole plane and satisfies a functional equation (with the critical line $\mathrm{Re}(s)=1/2$).

We now turn to the integrals in (2.5). Denote the first integral in brackets by $I_1^k(t)$, and the second one by $I_2^k(t)$. These integrals can be evaluated explicitly ([G-R] 7.611.7, p.858)† in terms of generalized hypergeometric series ${}_3F_2$. $I_1^k(t)$ is equal to

$$\frac{\Gamma(\frac{1}{2}+ir_j)\Gamma(\frac{1}{2}+ir_j+2it)\Gamma(-2ir_j)}{\Gamma(\frac{1}{2}-k-ir_j)\Gamma(1+ir_j+it)}{}_3F_2(\frac{1}{2}+ir_j,\frac{1}{2}+ir_j+2it,\frac{1}{2}-k+ir_j;1+2ir_j,1+ir_j+it;1)+$$
$$\frac{\Gamma(\frac{1}{2}-ir_j)\Gamma(\frac{1}{2}-ir_j+2it)\Gamma(2ir_j)}{\Gamma(\frac{1}{2}-k+ir_j)\Gamma(1-ir_j+it)}{}_3F_2(\frac{1}{2}-ir_j,\frac{1}{2}-ir_j+2it,\frac{1}{2}-k-ir_j;1-2ir_j,1-ir_j+it;1)\tag{2.6}$$

and $I_2^k(t)$ is equal to

$$\frac{\Gamma(\frac{1}{2}+ir_j)\Gamma(\frac{1}{2}-ir_j)\Gamma(2it)}{\Gamma(\frac{1}{2}+it)\Gamma(k+1)}{}_3F_2(\frac{1}{2}+ir_j,\frac{1}{2}-ir_j,\frac{1}{2}-it;1-2it,k+1;1)+\tag{2.7}$$
$$\frac{\Gamma(\frac{1}{2}+ir_j+2it)\Gamma(\frac{1}{2}-ir_j+2it)\Gamma(-2it)}{\Gamma(\frac{1}{2}-it)\Gamma(k+1+2it)}{}_3F_2(\frac{1}{2}+ir_j+2it,\frac{1}{2}-ir_j+2it,\frac{1}{2}+it;1+2it,k+1+2it;1)$$

†One can make two different substitutions in [G-R] 7.611.7. The correct substitution makes functions ${}_3F_2$ convergent.

The generalized hypergeometric function ${}_pF_q$ is a generalization of a Gauss hypergeometric function $F = {}_2F_1$. It is defined by

$$(2.8)\quad {}_pF_q(a_1,...,a_p;b_1,...,b_q;z) \;=\; {}_pF_q\left(\begin{matrix} a_1,...,a_p \\ b_1,...,b_q \end{matrix}; z\right) \;=\; 1+\sum_{n=1}^{\infty} \frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{z^n}{n!},$$

where $(x)_n = x(x+1)...(x+n-1)$. If $p \le q+1$, the series (2.7) converges for $|z| < 1$. If $p = q+1$, the series (2.8) converges for $z = 1$ whenever $\mathrm{Re}(b_1 + ... + b_q - a_1 - ... - a_{q+1}) > 0$. When the argument z in (2.8) is equal to one, it is usually omitted, which we will do throughout the rest of the paper. A good reference for generalized hypergeometric functions is Bailey's book [Ba].

We need to study asymptotics of $I_1^k(t)$ and $I_2^k(t)$ as $t \to \infty$. It is easier to begin with $I_2^k(t)$ (formula (2.7)). Asymptotics of Γ-functions can be found using Stirling's formula, so we need to get asymptotics of ${}_3F_2$-s. By far, the easiest case is the first ${}_3F_2$ in (2.7):

$$(*)\qquad {}_3F_2(\frac{1}{2}+ir_j, \frac{1}{2}-ir_j, \frac{1}{2}-it; 1-2it, k+1).$$

Namely, it is easy to show that for every n we have $|(1/2-it)_n/(1-2it)_n| < 1$. It means that the series (2.8) for (*) is majorized by

$$1 \;+\; \sum_{n=1}^{\infty} \frac{(\frac{1}{2}+ir_j)_n(\frac{1}{2}-ir_j)_n}{(k+1)_n}\frac{z^n}{n!} \;=\; {}_2F_1(\frac{1}{2}+ir_j, \frac{1}{2}-ir_j; k+1; 1),$$

which is absolutely convergent. Hence limit as t goes to infinity can be taken term by term in (*), yielding

$$1 \;+\; \sum_{n=1}^{\infty} \frac{(\frac{1}{2}+ir_j)_n(\frac{1}{2}-ir_j)_n}{(k+1)_n}\frac{z^n}{2^n n!} \;=\; {}_2F_1(\frac{1}{2}+ir_j, \frac{1}{2}-ir_j; k+1; \frac{1}{2})$$

It is more difficult to evaluate asymptotics of the second term in (2.7) as it stands. Fortunately, one can use transformation formulas for ${}_3F_2$-s to transform the second term into something analogous to (*). We use the following formula (cf. [Ba] §3.5, p. 18):

$$(2.9)\quad {}_3F_2(a,b,c;e,f) = \frac{\Gamma(e+f-a-b-c)\Gamma(f)}{\Gamma(f-a)\Gamma(e+f-b-c)}{}_3F_2(a, e-b, e-c; e, e+f-b-c)$$

After substituting $a = \frac{1}{2}+it, b = \frac{1}{2}+ir_j+2it, c = \frac{1}{2}-ir_j+2it, e = 1+2it$ and $f = k+1+2it$ into (2.8) and cancellations, the second term in (2.7) becomes

$$\frac{\Gamma(\frac{1}{2}+ir_j+2it)\Gamma(\frac{1}{2}-ir_j+2it)\Gamma(-2it)\Gamma(\frac{1}{2}+k-it)}{\Gamma(\frac{1}{2}-it)\Gamma(\frac{1}{2}+k+it)\Gamma(k+1)}{}_3F_2(\frac{1}{2}+ir_j, \frac{1}{2}-ir_j, \frac{1}{2}+it; 1+2it, k+1)$$

Now, ${}_3F_2$ is again asymptotic to a constant $({}_2F_1(\frac{1}{2}+ir_j, \frac{1}{2}-ir_j; k+1; \frac{1}{2}))$, and Stirling's formula shows that the second term in (2.7) is exponentially smaller than the first one, so we need only to consider the first term when estimating asymptotics for $I_2^k(t)$.

We now turn to asymptotics of $I_1^k(t)$ (formula (2.6)). Two terms in (2.6) have the same t-dependence and differ only by the sign of r_j, so their t-asymptotics are the same. Accordingly, we shall deal only with the first term. Our strategy will be again to use transformation formulas to get an analogue of (*). We use the following formula ([Ba] 3.2.2, p. 15)

$$(2.10)\quad {}_3F_2\begin{pmatrix} a,b,c \\ e,f \end{pmatrix} = \frac{\Gamma(1-a)\Gamma(e)\Gamma(f)\Gamma(c-b)}{\Gamma(e-b)\Gamma(f-b)\Gamma(1+b-a)\Gamma(c)}\,{}_3F_2\begin{pmatrix} b, b-e+1, b-f+1 \\ 1+b-c, 1+b-a \end{pmatrix} + \frac{\Gamma(1-a)\Gamma(e)\Gamma(f)\Gamma(b-c)}{\Gamma(e-c)\Gamma(f-c)\Gamma(1+c-a)\Gamma(b)}\,{}_3F_2\begin{pmatrix} c, c-e+1, c-f+1 \\ 1+c-b, 1+c-a \end{pmatrix}$$

After substituting $a=\frac{1}{2}-k+ir_j, b=\frac{1}{2}+ir_j, c=\frac{1}{2}+ir_j+2it, e=1+2ir_j$ and $f=1+ir_j+it$ into (2.9) and cancellations, the first term in (2.6) becomes

$$\frac{\Gamma(k+\frac{1}{2}-ir_j)\Gamma(1+2ir_j)\Gamma(2it)\Gamma(-2ir_j)}{\Gamma(\frac{1}{2}+it)\Gamma(\frac{1}{2}-k-ir_j)\Gamma(k+1)}\,{}_3F_2\begin{pmatrix} \frac{1}{2}+ir_j, \frac{1}{2}-ir_j, \frac{1}{2}-it \\ 1-2it, k+1 \end{pmatrix} + \frac{\Gamma(\frac{1}{2}+ir_j+2it)\Gamma(\frac{1}{2}+k-ir_j)\Gamma(-2it)\Gamma(1+2ir_j)\Gamma(-2ir_j)}{\Gamma(\frac{1}{2}+ir_j-2it)\Gamma(\frac{1}{2}-k-ir_j)\Gamma(\frac{1}{2}-it)\Gamma(k+1+2it)}\,{}_3F_2\begin{pmatrix} \frac{1}{2}+ir_j+2it, \frac{1}{2}-ir_j+2it, \frac{1}{2}+it \\ 1+2it, k+1+2it \end{pmatrix}$$

Now, using formula (2.9) the second term in the above formula can be written as

$$\frac{\Gamma(\frac{1}{2}+ir_j+2it)\Gamma(\frac{1}{2}+k-it)\Gamma(-2it)\Gamma(\frac{1}{2}+k-ir_j)\Gamma(1+2ir_j)\Gamma(-2ir_j)}{\Gamma(\frac{1}{2}+ir_j-2it)\Gamma(\frac{1}{2}+k+it)\Gamma(\frac{1}{2}-it)\Gamma(\frac{1}{2}-k-ir_j)\Gamma(k+1)}\,{}_3F_2\begin{pmatrix} \frac{1}{2}+ir_j, \frac{1}{2}-ir_j, \frac{1}{2}+it \\ 1+2it, k+1 \end{pmatrix}$$

Now we utilize the above computations to estimate $I_1^k(t)$ and $I_2^k(t)$:

$$(2.11)\qquad |I_1^k(t)| \ll_{j,k} \left|\frac{\Gamma(2it)}{\Gamma(\frac{1}{2}+it)}\right| + \left|\frac{\Gamma(-2it)}{\Gamma(\frac{1}{2}-it)}\right| \quad ; \quad |I_2^k(t)| \ll_{j,k} \left|\frac{\Gamma(2it)}{\Gamma(\frac{1}{2}+it)}\right|.$$

Note that we are using

$$\lim_{t\to\infty} {}_3F_2(\frac{1}{2}+ir_j, \frac{1}{2}-ir_j, \frac{1}{2}\pm it; 1\pm 2it, k+1; 1) \;=\; {}_2F_1(\frac{1}{2}+ir_j, \frac{1}{2}-ir_j; k+1; \frac{1}{2})$$

established above.

Recall now that $(\varphi_{j,k}, d\mu_t)$ is equal to (cf. (2.5)):

$$\frac{(-1)^k\Gamma(\frac{1}{2}+ir_j)}{2(4\pi)^{it-\frac{1}{2}}\xi(1-2it)}\frac{L(\varphi_j,\frac{1}{2}+2it)L(\varphi_j,\frac{1}{2})}{\zeta(1+2it)}\left[\frac{I_1^k(t)}{\Gamma(\frac{1}{2}+k+ir_j)}+\frac{I_2^k(t)}{\Gamma(\frac{1}{2}-k+ir_j)}\right]$$

(where $L(\varphi_j, s)$ is an L-function associated to holomorphic Maass cusp form φ_j).

As for holomorphic cusp forms, to show that $(\varphi_{j,k}, d\mu_t) \to 0$ as $t \to \infty$ we need an estimate for $L(\varphi_j, s)$ on a critical line of the form

$$|L(\varphi_j, \frac{1}{2} + it)| \ \ll_{j,\varepsilon} \ |t|^{\alpha+\varepsilon}, \quad \alpha < \frac{1}{2}$$

Analogously to the holomorphic case, the trivial "convexity bound" for L (coming from applying Hadamard's theorem) gives $\alpha = \frac{1}{2}$, which is not sufficient to prove the decay of $(\varphi_{j,k}, d\mu_t)$. Therefore, it is again necessary to "break convexity," which for Maass cusp forms was done by Meurman in [Me].

Meurman proved that $L(\varphi_j, s)$ satisfies

$$|L(\varphi_j, 1/2 + it)| \ \ll_{j,\varepsilon} \ |t|^{\frac{1}{3}+\varepsilon} \tag{2.12}$$

We can now substitute definition of $\xi(1-2it)$ into the last formula for $(\varphi_{j,k}, d\mu_t)$, and then use formulas (2.11), (2.12), inequalities (2.4) and Stirling's formula to finish the proof of Proposition 2.2.[3]

3. Incomplete Eisenstein series. We now turn to the consideration of incomplete Eisenstein series[4]. Let $\psi(y) \in C_0^\infty((0,\infty))$ and let

$$L_\psi(s) \ = \ \int_0^\infty \psi(y) y^{-s} \frac{dy}{y}$$

be the Mellin transform of the function ψ.

Then incomplete Eisenstein series of weight $-2k$ is equal to (cf. [K] and [Z2])

$$F_\psi(z,\theta) \ = \ \frac{1}{2\pi i} e^{2ik\theta} \int_{\mathrm{Re}(s)=s_1>1} L_\psi(s) E_{2k}(z,s) ds, \tag{3.1}$$

where z and θ are as in (0.1). We want to find the asymptotics of $(F_\psi, d\mu_t)$. The answer for $k=0$ was given in [L-S]. It was shown that

$$(F_\psi, d\mu_t) \ \sim \ \frac{48}{\pi} \int_{S^*X} F\, d\omega \cdot \ln t$$

as t goes to infinity. We will show that for $k \neq 0$, the expression $(F_\psi, d\mu_t)$ is asymptotically less than that for $k=0$. This will be achieved by shifting the contour of integration in (3.1) from $\mathrm{Re}(s) = s_1 > 1$ to $\mathrm{Re}(s) = 1/2$, evaluating residue, and estimating the integral along $\mathrm{Re}(s) = 1/2$. Transformation formulas for generalized hypergeometric functions will play a role in the proof.

We will give the proof only for incomplete Eisenstein series of weight $-2k$. The calculations for weight $2k$ are analogous.[5]

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[3] The Lindelöf hypothesis $|L(\varphi_j, 1/2+it)| \ \ll_{j,\varepsilon} \ |t|^\varepsilon$ will imply the correct decay rate $|(\varphi_{j,k}, d\mu_t)| \ll_{j,k,\varepsilon} |t|^{-\frac{1}{2}+\varepsilon}$ in Proposition 2.2.

[4] Incomplete Eisenstein series are called incomplete theta-series in [K] and [Z2].

[5] The formulas below are *not* valid for $k=0$ since generalized hypergeometric functions in expressions for integrals $I_3^k(s,t)$ and $I_4^k(s,t)$ (see (3.4) below) fail to converge. However, using a different formula in [G-R] for $I_3^k(s,t)$ and $I_4^k(s,t)$ when $k=0$ will recover W. Luo and P. Sarnak's results in [L-S].

We substitute the definition (0.4) for $d\mu_t$. After integrating out θ, we see that

$$(F_\psi, d\mu_t) = \int_{\Gamma\backslash\mathbb{H}} F_\psi(z,0)E(z,1/2-it)E_{-2k}(z,1/2+it)\frac{dxdy}{y^2}$$

We then substitute (3.1) into the above formula, change the order of integration, and after "unfolding" $E_{2k}(z,s)$ get

$$\begin{aligned}(F_\psi, d\mu_t) &= \int_{\mathrm{Re}(s)=s_1>1}\int_0^\infty\int_0^1 L_\psi(s)y^s E(z,1/2-it)E_{-2k}(z,1/2+it)\,\frac{dxdy}{y^2}ds = \\ &\int_{\mathrm{Re}(s)=s_1>1}\int_0^\infty\int_0^1 y^s L_\psi(s)\Big[y^{1/2-it}+\phi(1/2-it)y^{1/2+it}+\frac{1}{\xi(1-2it)}\\ &\qquad\sum_{n=1}^\infty \frac{\sigma_{2it}(n)}{n^{1/2+it}}W_{0,-it}(4\pi ny)\left(\frac{\mathrm{e}(nx)+\mathrm{e}(-nx)}{2}\right)\Big]\\ &\qquad\Big[y^{1/2+it}+\frac{(-1)^k\Gamma^2(\frac12+it)}{\Gamma(\frac12-k+it)\Gamma(\frac12+k+it)}\phi(1/2+it)y^{1/2-it}+\\ &\frac{(-1)^k\Gamma(1/2+it)}{2\xi(1+2it)}\sum_{n=1}^\infty\frac{n^{it}\sigma_{-2it}(n)}{\sqrt{n}}\left(\frac{W_{k,it}(4\pi ny)\mathrm{e}(nx)}{\Gamma(\frac12+k+it)}+\frac{W_{-k,it}(4\pi ny)\mathrm{e}(-nx)}{\Gamma(\frac12-k+it)}\right)\Big]\;\frac{dxdy}{y^2}ds\end{aligned}$$

(the last equality is obtained by substituting Fourier expansions (1.3) and (1.6) into the formula).

After integrating x out, we get the following:

$$(F_\psi, d\mu_t) = 2\int_{\mathrm{Re}(s)=s_1>1}\int_0^\infty y^s L_\psi(s)\,\frac{dy}{y}ds + \left(\begin{matrix}\textit{rapidly de-}\\ \textit{creasing in } t\end{matrix}\right) + \tag{3.2}$$
$$\frac{(-1)^k\Gamma(1/2+it)}{4|\xi(1+2it)|^2}\int_{\mathrm{Re}(s)=s_1}\int_0^\infty y^s L_\psi(s)\sum_{n=1}^\infty\frac{|\sigma_{2it}(n)|^2}{n}W_{0,-it}(4\pi ny)\left(\frac{W_{k,it}(4\pi ny)}{\Gamma(\frac12+k+it)}+\frac{W_{-k,it}(4\pi ny)}{\Gamma(\frac12-k+it)}\right)\,\frac{dy}{y^2}ds$$

We use the inversion formula for the Mellin transform

$$\psi(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_1} L_\psi(s)y^s\,ds$$

to conclude that the first term in (3.2) is equal to

$$2\int_0^\infty \psi(y)\,\frac{dy}{y} \tag{3.3}$$

After changing summation and integration and changing variable in the infinite sum in (3.2), we find that the second term is equal to

$$(-1)^k\frac{\Gamma(1/2+it)}{4|\xi(1+2it)|^2}\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_1}L_\psi(s)\left(\sum_{n=1}^\infty\frac{|\sigma_{2it}(n)|^2}{n^s(4\pi)^{s-1}}\right) \tag{3.4}$$
$$\left[\frac{1}{\Gamma(\frac12+k+it)}\int_0^\infty W_{k,it}(u)W_{0,-it}(u)u^s\,\frac{du}{u^2}+\frac{1}{\Gamma(\frac12-k+it)}\int_0^\infty W_{-k,it}(u)W_{0,-it}(u)u^s\,\frac{du}{u^2}\right]\,ds$$

The infinite sum in the above expression was first computed by Ramanujan and found to be (cf. also [L-S])

$$\sum_{n=1}^{\infty} \frac{|\sigma_{2it}(n)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s-2it)\zeta(s+2it)}{\zeta(2s)} \tag{3.5}$$

Let us denote the first and the second integrals in (3.4) by $I_3^k(s,t)$ and $I_4^k(s,t)$, respectively. They can be evaluated (cf. [G-R] 7.611.7, p. 858). $I_3^k(s,t)$ is equal to

$$\begin{aligned}(3.6a)\ &\frac{\Gamma(s)\Gamma(s+2it)\Gamma(-2it)}{\Gamma(\frac{1}{2}-k-it)\Gamma(s+\frac{1}{2}+it)}{}_3F_2(s,s+2it,\frac{1}{2}-k+it;1+2it,s+\frac{1}{2}+it)+\\ &\frac{\Gamma(s)\Gamma(s-2it)\Gamma(2it)}{\Gamma(\frac{1}{2}-k+it)\Gamma(s+\frac{1}{2}-it)}{}_3F_2(s,s-2it,\frac{1}{2}-k-it;1-2it,s+\frac{1}{2}-it),\end{aligned}$$

and $I_4^k(s,t)$ is equal to

$$\begin{aligned}(3.7a)\ &\frac{\Gamma(s)\Gamma(s+2it)\Gamma(-2it)}{\Gamma(\frac{1}{2}-it)\Gamma(\frac{1}{2}+k+s+it)}{}_3F_2(s,s+2it,\frac{1}{2}+it;1+2it,\frac{1}{2}+k+s+it)+\\ &\frac{\Gamma(s)\Gamma(s-2it)\Gamma(2it)}{\Gamma(\frac{1}{2}+it)\Gamma(\frac{1}{2}+k+s-it)}{}_3F_2(s,s-2it,\frac{1}{2}-it;1-2it,\frac{1}{2}+k+s-it)\end{aligned}$$

It is convenient for later use to transform (3.6) and (3.7); this will also yield meromorphic continuation of $I_3^k(s,t)$ and $I_4^k(s,t)$ to the whole complex plane.

For $I_3^k(s,t)$ we use the formula (2.9), substituting $a=s$, $b=s+2it$, $c=\frac{1}{2}-k+it$, $e=1+2it$, $f=1/2+s+it$ for the top integral in (3.6a) and $a=s$, $b=s-2it$, $c=\frac{1}{2}-k-it$, $e=1-2it$, $f=1/2+s-it$ for the bottom one. We get the following expression:

$$\begin{aligned}(3.6b)\ &\frac{\Gamma(s)\Gamma(s+2it)\Gamma(-2it)\Gamma(k+1-s)}{\Gamma(\frac{1}{2}-k-it)\Gamma(\frac{1}{2}+it)\Gamma(k+1)}{}_3F_2(s,1-s,\frac{1}{2}+k+it;1+2it,k+1)+\\ &\frac{\Gamma(s)\Gamma(s-2it)\Gamma(2it)\Gamma(k+1-s)}{\Gamma(\frac{1}{2}-k+it)\Gamma(\frac{1}{2}-it)\Gamma(k+1)}{}_3F_2(s,1-s,\frac{1}{2}+k-it;1-2it,k+1)\end{aligned}$$

For $I_4^k(s,t)$ we use the formula (cf. [Ba] §3.5, p. 18)

$$\begin{aligned}{}_3F_2(1+a+b+c-e-f,b,c;1+b+c-e,1+b+c-f) &= \frac{\Gamma(f-a)\Gamma(2+b+c-e-f)}{\Gamma(1-a)\Gamma(1+b+c-e)}\\ {}_3F_2(1+a+b+c-e-f,1-f+b,1-f+c;2+b+c-e-f,1+b+c-f)&\end{aligned}$$

After making the appropriate substitutions, the expression (3.7a) becomes

$$\begin{aligned}(3.7b)\ &\frac{\Gamma(s)\Gamma(s+2it)\Gamma(-2it)\Gamma(k+1-s)}{\Gamma(\frac{1}{2}+k+it)\Gamma(\frac{1}{2}-it)\Gamma(k+1)}{}_3F_2(s,1-s,\frac{1}{2}+it;1+2it,k+1)+\\ &\frac{\Gamma(s)\Gamma(s-2it)\Gamma(2it)\Gamma(k+1-s)}{\Gamma(\frac{1}{2}+k-it)\Gamma(\frac{1}{2}+it)\Gamma(k+1)}{}_3F_2(s,1-s,\frac{1}{2}-it;1-2it,k+1)\end{aligned}$$

Observe that in the above expressions for $I_3^k(s,t)$ and $I_4^k(s,t)$ all generalized hypergeometric functions ${}_3F_2$ converge for arbitrary s,t. Also, presence of s and $1-s$ as the arguments makes it possible to estimate the expressions uniformly in s along the line $\mathrm{Re}(s) = 1/2$, which suggests shifting there the contour of integration in (3.4).

As we shift the line of integration from $\mathrm{Re}(s) = s_1 > 1$ to $\mathrm{Re}(s) = 1/2$, we pass through the poles at $s = 1$ and $s = 1 \pm 2it$ coming from the poles of ζ-functions in (3.5). We will prove below the following

Proposition 3.1. *Let F_ψ be an incomplete Eisenstein series of weight $-2k$. Then*

$$|(F_\psi, d\mu_t)| \; < C(k,\psi)$$

as t goes to infinity.

The expression $(F_\psi, d\mu_t)$ is the sum of several terms: the integral along $\mathrm{Re}(s) = 1/2$; residues at $s = 1$ and $s = 1 \pm 2it$; the "top" and the "bottom" contributions; the constant term (3.3). We shall prove the following Lemma (which has an analogue in [L-S]):

Lemma 3.2. *Notation as above, when the line of integration in (3.4) is shifted to* $\mathrm{Re}(s) = 1/2$, *the resulting expression is* $\ll_{k,\varepsilon,\psi} |t|^{-\frac{1}{6}+\varepsilon}$

The proof is given at the end of section 3.

We next turn to evaluating the residue at $s = 1$. We prove the following

Lemma 3.3. *The residue in (3.4) at $s = 1$ is $O(1)$.*

Proof of Lemma 3.3. After substituting (3.5) into (3.4), we get the following expression (the residue of which at $s = 1$ we have to compute):

$$\zeta^2(s)(-1)^k \frac{\Gamma(1/2+it)}{4|\xi(1+2it)|^2} L_\psi(s) \frac{\zeta(s-2it)\zeta(s+2it)}{\zeta(2s)}$$

$$\left(\frac{I_3^k(s,t)}{\Gamma(\frac{1}{2}+k+it)} + \frac{I_4^k(s,t)}{\Gamma(\frac{1}{2}-k+it)} \right)$$

The above expression has a double pole at $s = 1$ coming from $\zeta^2(s)$ (as in the corresponding calculation in [L-S]), but we also get a single zero coming from the expression in brackets in the above formula.

Namely, after substituting (3.6b) and (3.7b) for I_3^k and I_4^k, we find that the expression in brackets in the above formula at $s = 1$ is equal to

$$\begin{aligned}(3.8)\quad & \frac{1}{\Gamma(\frac{1}{2}+k+it)} \left(\frac{\Gamma(1+2it)\Gamma(-2it)}{\Gamma(\frac{1}{2}-k-it)\Gamma(\frac{1}{2}+it)} + \frac{\Gamma(1-2it)\Gamma(+2it)}{\Gamma(\frac{1}{2}-k+it)\Gamma(\frac{1}{2}-it)} \right) + \\ & \frac{1}{\Gamma(\frac{1}{2}-k+it)} \left(\frac{\Gamma(1+2it)\Gamma(-2it)}{\Gamma(\frac{1}{2}+k+it)\Gamma(\frac{1}{2}-it)} + \frac{\Gamma(1-2it)\Gamma(+2it)}{\Gamma(\frac{1}{2}+k-it)\Gamma(\frac{1}{2}+it)} \right)\end{aligned}$$

and the resulting expression can be shown to be zero by the usual transformation formulas of Γ-function.

This results in a single pole at $s = 1$. The residue again can be computed using the formulas (3.6b) and (3.7b) and the definition (2.8) of a generalized hypergeometric function. The resulting expression can be estimated using the inequality (A.3) of the Appendix, and the Lemma can be proved by calculations analogous to those in the Appendix.

We now turn to estimating integral (3.4) along the line $\mathrm{Re}(s) = 1/2$. We start by substituting (3.5), (3.6b) and (3.7b) into (3.4) and letting $s = \frac{1}{2} + iu$. The resulting expression is

$$\frac{\Gamma(1/2+it)}{4|\xi(1+2it)|^2}\frac{(-1)^k}{2\pi i}\int_{-\infty}^{\infty} L_\psi(\frac{1}{2}+iu)\frac{\zeta^2(\frac{1}{2}+iu)\zeta(\frac{1}{2}+iu-2it)\zeta(\frac{1}{2}+iu+2it)}{\zeta(1+2iu)}$$
$$\frac{\Gamma(\frac{1}{2}+iu)\Gamma(k+\frac{1}{2}-iu)}{\Gamma(k+1)}$$
$$\Bigg[\frac{1}{\Gamma(\frac{1}{2}+k+it)}\left(\frac{\Gamma(\frac{1}{2}+iu+2it)\Gamma(-2it)}{\Gamma(\frac{1}{2}-k-it)\Gamma(\frac{1}{2}+it)}{}_3F_2(\frac{1}{2}+iu,\frac{1}{2}-iu,\frac{1}{2}+k+it;1+2it,k+1)+\right.$$
$$\left.\frac{\Gamma(\frac{1}{2}+iu-2it)\Gamma(2it)}{\Gamma(\frac{1}{2}-k+it)\Gamma(\frac{1}{2}-it)}{}_3F_2(\frac{1}{2}+iu,\frac{1}{2}-iu,\frac{1}{2}+k-it;1+2it,k+1)\right)+$$
$$\frac{1}{\Gamma(\frac{1}{2}-k+it)}\left(\frac{\Gamma(\frac{1}{2}+iu+2it)\Gamma(-2it)}{\Gamma(\frac{1}{2}+k+it)\Gamma(\frac{1}{2}-it)}{}_3F_2(\frac{1}{2}+iu,\frac{1}{2}-iu,\frac{1}{2}+it;1+2it,k+1)+\right.$$
$$\left.\frac{\Gamma(\frac{1}{2}+iu-2it)\Gamma(2it)}{\Gamma(\frac{1}{2}+k-it)\Gamma(\frac{1}{2}+it)}{}_3F_2(\frac{1}{2}+iu,\frac{1}{2}-iu,\frac{1}{2}-it;1-2it,k+1)\right)\Bigg]\,du$$

We want first to estimate ${}_3F_2\left(\frac{1}{2}+iu,\ \frac{1}{2}-iu,\ \frac{1}{2}+it;\ 1+2it,\ k+1\right)$ and ${}_3F_2(\frac{1}{2}+iu,\frac{1}{2}-iu,\frac{1}{2}+k+it;1+2it,k+1)$ uniformly in t. Since $|(1/2+it)_n/(1+2it)_n| < 1$ for every n, we can conclude (as in section 2) that

$${}_3F_2(\frac{1}{2}+iu,\frac{1}{2}-iu,\frac{1}{2}+it;1+2it,k+1;1) \ < \ {}_2F_1(\frac{1}{2}+iu,\frac{1}{2}-iu;k+1;1) \quad ,$$

where ${}_2F_1$ is the usual Gauss hypergeometric function F. By a well-known property of F ([G-R] 9.122.1, p. 1042)

$${}_2F_1(\frac{1}{2}+iu,\frac{1}{2}-iu;k+1;1) \ = \ \frac{\Gamma(k+1)\Gamma(k)}{\Gamma(k+1/2-iu)\Gamma(k+1/2+iu)} \quad ,$$

and the last expression by Stirling's formula is $\ll_k e^{\pi u}u^{-2k}$.

Estimating ${}_3F_2(\frac{1}{2}+iu,\frac{1}{2}-iu,\frac{1}{2}+k+it;1+2it,k+1)$ requires a more careful analysis of individual terms and the rates of convergence of hypergeometric series. One can estimate it crudely as follows:

Claim 3.4.

$${}_3F_2(\frac{1}{2}+iu,\frac{1}{2}-iu,\frac{1}{2}+k+it;1+2it,k+1) \ \ll_k \ e^{\pi u}u^{2k+\frac{3}{2}} \ ,$$

The proof is given in the Appendix.[6]

[6] Note that the Claim allows to take the limit as $t \to \infty$ in the series (2.8) defining the above hypergeometric function term by term (as in §2) yielding (for u fixed) the expression ${}_3F_2(\frac{1}{2}+iu,\frac{1}{2}-iu;k+1;\frac{1}{2})$ as the limit. The same answer, of course, is obtained when we take the limit as $t \to \infty$ of the function ${}_3F_2(\frac{1}{2}+iu,\frac{1}{2}-iu,\frac{1}{2}+it;1+2it;k+1)$.

Now one can complete the proof of lemma 3.2 by considering the above integral applying the above estimates, inequalities (2.4) Stirling's formula and Weyl's bound for ζ-function on a critical line:

$$\zeta(1/2+iv) \ll_\varepsilon |v|^{\frac{1}{6}+\varepsilon}$$

To complete the proof of Proposition 3.1, it remains to consider the residues at $s = 1 \pm 2it$ of (3.4) together with the "top" and the "bottom" integrals coming from shifting the line of integration in (3.4). The residues at $1 \pm 2it$ can be shown to decay arbitrarily fast since $\psi(y) \in C_0^\infty((0,\infty))$. The "top" and the "bottom" integrals can be estimated similarly to the integral above (the calculations are analogous to those in the Appendix) and found to decay as $t \to \infty$. This completes the proof of proposition (3.1).

4. Proof of Theorem 1. In this section we formulate and prove Proposition 4.1 (the analogue of W. Luo and P. Sarnak's result (0.3)), which implies Theorem 1 by an approximation argument. To prove it, we need the estimates obtained in sections 2 and 3.

Proposition 4.1. *Let f be a continuous function with compact support in $\Gamma\backslash G$. Then*

$$\frac{1}{\ln t}(f, d\mu_t^F) \sim \frac{48}{\pi}\int_{\Gamma\backslash G} f\, d\omega ,$$

where $d\omega$ is a Liouville measure on $\Gamma\backslash G$ (cf. (0.2)).

Before proving the Proposition, we state what we mean by "asymptotic equivalence" of $d\mu_t$ and $d\mu_t^F$ (cf. Introduction): it is proved in [Z2], Proposition 3.8 that for $g \in C_0^\infty(\Gamma\backslash G)$ (a smooth function with compact support) and for every $\varepsilon > 0$,

$$(g, d\mu_t - d\mu_t^F) \ll_\varepsilon \frac{\ln t}{t^{1-\varepsilon}} \tag{4.1}$$

This implies that one can substitute $d\mu_t^F$ for $d\mu_t$ in Propositions 2.1, 2.2 and 3.1 and they will remain true. Also, results in [L-S] and formula (4.1) prove the Proposition for f in the space of incomplete Eisenstein series of weight zero (and hence for arbitrary weight). We now turn to the proof.

Proof of Proposition 4.1. One can show using Sobolev inequalities for L^∞ and L^2 norms (cf. [Z3], §6) that for any function $f \in C_0^\infty(\Gamma\backslash G)$ there exist functions g_1 (a finite linear combination of incomplete Eisenstein series of different weghts), and g_2 (a finite sum of cusp forms and holomorphic cusp forms of different weights) such that $||f - g_1 - g_2||_\infty < \varepsilon$ (compare to [L-S] and [D-R-S], Lemma 2.4). Then, as in [L-S], one can show that there exists a function $h \in C^\infty((0,\infty))$ which is positive, rapidly decreasing and such that if we let (in the coordinates (0.1))

$$H(z,\theta) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} h(y(\gamma z)) \tag{4.2}$$

then $H(z,\theta) > |(f - g_1 - g_2)(z,\theta)|$ for every (z,θ) and $\int_{\Gamma\backslash G} H\, d\omega < 5\varepsilon$. By previous remarks then,

$$\limsup_{t\to\infty} \frac{1}{\ln t}\left(h, d\mu_t^F\right) \le 5\varepsilon ,$$

and the proposition follows.

Appendix. Proof of Claim 3.4. We want to estimate the expression ${}_3F_2(\frac{1}{2}+iu, \frac{1}{2}-iu, \frac{1}{2}+k+it; 1+2it, k+1; 1)$ uniformly in t. We shall first estimate this expression when u is fixed, and then turn to arbitrary u.

By definition (2.8) of generalized hypergeometric series the above expression is equal to

$$1+\sum_{n=1}^{\infty}\frac{(\frac{1}{2}+iu)_n(\frac{1}{2}-iu)_n}{(k+1)_n\, n!}\frac{(\frac{1}{2}+k+it)_n}{(1+2it)_n} \tag{A.1}$$

If we "forget" about the t-dependent terms in (A.1) we get the expansion (2.8) of the usual Gauss hypergeometric function ${}_2F_1(\frac{1}{2}+iu, \frac{1}{2}-iu; k+1)$, and the n-th term in this series can be shown to satisfy (for arbitrary $0<\alpha<1$)

$$\frac{(\frac{1}{2}+iu)_n(\frac{1}{2}-iu)_n}{(k+1)_n\, n!} \ll_{u,k} n^{-1-k\alpha} \tag{A.2}$$

using the ratio test. The constant α will be chosen later.

Unfortunately, the t-dependent term $(\frac{1}{2}+k+it)_n/(1+2it)_n$ is no longer bounded (which makes estimating (A.1) more difficult than estimating ${}_3F_2(\frac{1}{2}+iu, \frac{1}{2}-iu, \frac{1}{2}+it; 1+2it, k+1)$). However, we can still get the bound which will depend on n. Namely, using "arithmetic mean $>$ geometric mean" inequality, one can show that for $n\geq 2t/\sqrt{3}$

$$\frac{(\frac{1}{2}+k+it)_n}{(1+2it)_n} \leq \left[1+(k-\frac{1}{2})\frac{\ln n}{n}\right]^n , \tag{A.3}$$

and for $n\leq 2t/\sqrt{3}$ one can show that $(\frac{1}{2}+k+it)_n/(1+2it)_n<1$ provided $t>T(k)$, where $T(k)$ is a constant depending only on k. We do not claim that the above estimates are the best possible, but they suffice to prove the uniform bound for (A.1).

Namely, by logarithmic test the sum of positive terms $\sum_{n=1}^{\infty}a_n$ converges provided
$\frac{\ln 1/a_n}{\ln n}>\beta>1,\ n\geq N_0$ for some β. By (A.2) and (A.3), we know that

$${}_3F_2\left(\begin{matrix}\frac{1}{2}+iu, \frac{1}{2}-iu, \frac{1}{2}+k+it\\ 1+2it, k+1\end{matrix}\right) \ll_{k,u} \sum_{n=1}^{\infty} n^{-1-k\alpha}\left[1+(k-\frac{1}{2})\frac{\ln n}{n}\right]^n , \tag{A.4}$$

and the series on the right converges by the logarithmic test if we chose $1-\frac{1}{2k}<\alpha<1$. This completes the computations for fixed u.

We now turn to obtaining a bound in u. We want to make the inequality in (A.2) explicit. We fix some $1-\frac{1}{2k}<\alpha_0<1$ (say, $\alpha_0=1-\frac{1}{4k}$) and consider the ratio of the n-th and $(n-1)$-st terms on the right and left hand sides of (A.2). One can check that for $n\geq(2u)^2+(2k+1)^2$ the following holds:

$$\left(1-\frac{1}{n}\right)^{1+k\alpha_0} \geq 1-\frac{1+k\alpha_0}{n} \geq \frac{(n-\frac{1}{2}+iu)(n-\frac{1}{2}-iu)}{n(n+k)}$$

which means that starting from $n = (2u)^2 + (2k+1)^2$ the right-hand side of (A.2) decreases slower than the left-hand side. To simplify calculations we assume without loss of generality that $u > 2k+1$, and so the above equality holds for $n > 5u^2$.

It now remains to estimate the terms in (A.1) for $n < 5u^2$. As before, we proceed separately for t-dependent and t-independent factors. Using (A.3) and the estimate $[1 + (k - \frac{1}{2})\frac{\ln n}{n}]^n < C_1(k)n^k$, we find that for $n < 5u^2$, t-dependent factors in (A.1) are all less than $C_1(k)(5u^2)^k = C_2(k)u^{2k}$.

We estimate the sum of t-independent factors in (A.1) up to $n = [5u^2]$ and the $[5u^2]$-th term itself trivially by the expression ${}_2F_1(\frac{1}{2} + iu, \frac{1}{2} - iu; k+1) < C_3(k)e^{\pi u}u^{-2k}$ (cf. end of section 3). Finally, using the above estimates we majorize (A.1) by

$$C_2(k)C_3(k)e^{\pi u}\left(1 + (5u^2)^{k+\frac{3}{4}} \sum_{n=[5u^2]}^{\infty} n^{-1-k\alpha_0}\left[1 + (k - \frac{1}{2})\frac{\ln n}{n}\right]^n\right) < C_4(k)e^{\pi u}u^{2k+\frac{3}{2}} \ ,$$

which finishes the proof of Claim 3.4.

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Department of Mathematics, Princeton University, Princeton, NJ 08544