EQUIDISTRIBUTION OF CUSP FORMS ON $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathrm{PSL}_2(\mathbf{R})$

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Abstract. We prove a microlocal version of the equidistribution theorem for Wigner distributions associated to cusp forms on $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathrm{PSL}_2(\mathbf{R})$. This generalizes a recent result of W. Luo and P. Sarnak who prove equidistribution on $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$.

Résumé. Nous donnons la preuve d'une version microlocale d'un resultat de W. Luo et P. Sarnak concernant la répartition asymptotique des fonctions de Wigner associées aux formes paraboliques sur $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathrm{PSL}_2(\mathbf{R})$.

1. Introduction

Let X be a compact Riemannian manifold with ergodic geodesic flow, Δ the corresponding Laplace-Beltrami operator, φ_j an orthonormal basis of eigenfunction with eigenvalues $0=\lambda_0<\lambda_1\le\lambda_2\le\dots$. Let $d\mu_j$ denote the probability measures $|\varphi_j|^2$ dv where dv is the volume form on X. Shnirelman, Zelditch and Colin de Verdiere ([Sn74], [Sn93], [Ze87], [CdV]) have proved that there exists a subsequence λ_{j_k} of λ_j of the full density (i.e. $\#\{k : j_k\le n\}\sim n$ as $n\to\infty$) such that for every $f\in C^\infty(X)$, $\int f\ d\mu_{j_k}\to\int f\ \mathrm{dv}$. Zelditch in [Ze92] extended that result to non-compact $X=\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$. He also proved that for every $f\in C_0^\infty(X)$ such that $\int_X f\ \mathrm{dv}=0$

$$\sum_{\lambda_j\le\lambda}|\int f\ d\mu_j|^2\ \ll_f\ \lambda/\log\lambda \tag{1}$$

(here φ_j in the definition of $d\mu_j$ form a basis of the space of cusp forms – L^2-eigenfunctions of the hyperbolic Laplacian Δ on X).

In [LS] Luo and Sarnak proved (assuming that φ_j-s are Hecke eigenforms) the following

THEOREM 1. *Let $f\in C_0^\infty(X)$ and let $\int_X f\ \mathrm{dv}=0$. Then*

$$\sum_{\lambda_j\le\lambda}|\int f\ d\mu_j|^2\ \ll_{\varepsilon,f}\ \lambda^{1/2+\varepsilon} \tag{2}$$

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for any $\varepsilon > 0$.

This improves (1) (proved for more general surfaces). Luo and Sarnak also remark that the exponent 1/2 in (2) is the best possible.

Theorem 1 is proved without considering $S^*X = \mathrm{PSL}_2(\mathbf{Z})\backslash\mathrm{PSL}_2(\mathbf{R}) = \Gamma\backslash G$ (unlike (1) which is a corollary of a statement about S^*X) so it is natural to generalize the theorem to a statement about $\Gamma\backslash G$, which we do in this paper.

To formulate an analogue of Theorem 1 one has to "lift" the measures $d\mu_j$ on X to measures $\mathrm{d}\omega_j$ on S^*X. This is done as follows: given a pseudo-differential operator A of order zero with symbol σ_A, one defines the *Wigner distribution* $\mathrm{d}\omega_j$ by

$$\langle A\varphi_j, \varphi_j\rangle \;=\; \int_{S^*X} \sigma_A \;\mathrm{d}\omega_j \tag{3}$$

In the usual coordinates $\{(x, y, \theta) : y > 0, 0 \le \theta < 2\pi\}$ of the Iwasawa decomposition[1] on $\mathrm{SL}_2(\mathbf{R})$ $\mathrm{d}\omega_j$ is given (cf. [Ze91]) by

$$d\omega_j \;=\; \varphi_j(z)\bar{u}_j(z,\theta)\,\mathrm{d}\omega, \qquad u_j \;\sim\; \sum_{k\in\mathbf{Z}} \varphi_{j,k}(z)e^{2ik\theta} \tag{4}$$

where the Liouville measure $\mathrm{d}\omega$ on $\Gamma\backslash G$ is given by $(\mathrm{dxdy}\,\mathrm{d}\theta)/(2\pi y^2)$, where $\varphi_{j,k}(z)$ are "shifted" Maass cusp forms of weight $2k$. $\varphi_{j,k}e^{-2ik\theta}$ is an eigenfunctions of Casimir operator with the same eigenvalue $\lambda_j = 1/4 + r_j^2$ for every k. The Fourier expansion of φ_j and $\varphi_{j,k}$-s was computed in [Ja94].

The Fourier expansion of φ_j in (x, y, θ) coordinates is given by

$$\varphi_j(z) \;=\; \sum_{n\neq 0} \frac{c_j(|n|)}{\sqrt{|n|}} W_{0,ir_j}(4\pi|n|y)\mathbf{e}(nx), \tag{5}$$

where $\lambda_j = 1/4 + r_j^2$, W_{0,ir_j} is a Whittaker function, $\mathbf{e}(nx)$ denotes $e^{2\pi i nx}$ and $c_j(n) = c_j(1)\lambda_j(n)$ where $\lambda_j(n)$-s are Hecke eigenvalues of φ_j.

The Fourier expansion of $\varphi_{j,k}$ for $k > 0$ (weight $2k$) is given by

$$\begin{aligned}\varphi_{j,k}(z) \;=\; & \frac{(-1)^k\Gamma(1/2 + ir_j)}{\Gamma(\frac{1}{2} + k + ir_j)} \sum_{n>0} \frac{c_j(|n|)}{\sqrt{|n|}} W_{k,ir_j}(4\pi|n|y)\mathbf{e}(nx) + \\ & \frac{(-1)^k\Gamma(1/2 + ir_j)}{\Gamma(\frac{1}{2} - k + ir_j)} \sum_{n<0} \frac{c_j(|n|)}{\sqrt{|n|}} W_{-k,ir_j}(4\pi|n|y)\mathbf{e}(nx)\end{aligned} \tag{6}$$

Here W_{k,ir_j} is a Whittaker function and $c_j(n)$-s are as before.

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[1]We use the same notation as in [Ja94], so that $x + iy = z \in \mathbf{H}$, the hyperbolic metric is $|\,\mathrm{dz}\,|/y$, $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ and $\mathrm{dv} = \mathrm{dxdy}\,/y^2$.

The Fourier expansion of $\varphi_{j,-k}, k>0$ (weight $-2k$) is given by

$$\varphi_{j,-k}(z) \;=\; \frac{(-1)^k\Gamma(1/2+ir_j)}{\Gamma(\frac{1}{2}+k+ir_j)}\sum_{n<0}\frac{c_j(|n|)}{\sqrt{|n|}}\,W_{k,ir_j}(4\pi|n|y)\,\mathbf{e}(nx)\;+ \tag{7}$$
$$\frac{(-1)^k\Gamma(1/2+ir_j)}{\Gamma(\frac{1}{2}-k+ir_j)}\sum_{n>0}\frac{c_j(|n|)}{\sqrt{|n|}}\,W_{-k,ir_j}(4\pi|n|y)\,\mathbf{e}(nx)$$

The analogue of Theorem 1 can now be stated as follows:

THEOREM 2. *Let f be a finite sum of functions of even weight and of compact support on $\Gamma\backslash G$, and let $\int_{\Gamma\backslash G} f\ \mathrm{d}\omega = 0$. Then for every $\varepsilon>0$*

$$\sum_{\lambda_j\le\lambda}\Big|\int f\ \mathrm{d}\omega_j\Big|^2 \;\ll_{\varepsilon,f}\; \lambda^{1/2+\varepsilon} \tag{8}$$

Wigner distribution $\mathrm{d}\omega_j$ is not a positive distribution. In some problems it is necessary to consider its positive counterpart which we shall call $d\nu_j$ and which is defined by

$$\langle\sigma, d\nu_j\rangle \;=\; \langle\sigma^F, \mathrm{d}\omega_j\rangle \tag{9}$$

where σ^F is a Friedrichs symmetrization of σ (cf. [Ze87], [Ze91]). One can show ([Ze91, Prop. 3.8]) that $\sigma-\sigma^F$ is a symbol of order $-1+\varepsilon$ for any $\varepsilon>0$, so

$$|\langle f, d\nu_j\rangle - \langle f, \mathrm{d}\omega_j\rangle| \;\ll_\varepsilon\; |\lambda_j|^{-1/2+\varepsilon}$$

This implies that the estimate (8) also holds with $\mathrm{d}\omega_j$-s replaced by $d\nu_j$-s.

Theorem 2 is proved by an approximation argument. It is necessary to study the asymptotics of $\langle f, \mathrm{d}\omega_j\rangle$ with f being an incomplete Eisenstein series or Poincare series on $\Gamma\backslash G$. Studying the asymptotics can be reduced to estimating exponential sums by using Petersson-Kuznetsov trace formula (cf. [LS]). Also, the bounds of Iwaniec and Hoffstein-Lockhart for Fourier coefficients of cusp forms are used. The calculations for incomplete Eisenstein series are similar to those in [Ja94], though the exposition is simplified following [Ja95]. In the estimates in Propositions 1 and 2 we are not able to make the constants grow polynomially with $|k|$ (the constants grow exponentially in our calculations). This prevents us from proving (8) for arbitrary function of compact support on $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathrm{PSL}_2(\mathbf{R})$.

We end this section by recalling the Fourier expansions of Eisenstein series of various weights computed in [Ja94]. The Fourier expansion of $E(z,1/2+it)$ (Eisenstein series of weight zero) is given by

$$E(z,1/2+it) \;=\; y^{1/2+it}+\phi(1/2+it)y^{1/2-it}+ \tag{10}$$

$$\frac{1}{2\xi(1+2it)}\sum_{n\neq 0}|n|^{-\frac{1}{2}+it}\sigma_{-2it}(|n|)W_{0,it}(4\pi|n|y)\mathbf{e}(nx)$$

where $\phi(s) = \xi(2s-1)/\xi(2s)$, $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, and where $\sigma_\nu(n) = \sum_{d|n} d^\nu$.

The Eisenstein series (of weight $2k$, $k>0$) $E_{-2k}(z, 1/2+it)$ is given by

$$E_{-2k}(z,1/2+it) = \frac{(-1)^k\Gamma^2(\frac{1}{2}+it)}{\Gamma(\frac{1}{2}-k+it)\Gamma(\frac{1}{2}+k+it)}\phi(1/2+it)y^{1/2-it}+ \tag{11}$$

$$y^{1/2+it} + \frac{(-1)^k\Gamma(1/2+it)}{2\Gamma(\frac{1}{2}+k+it)\xi(1+2it)}\sum_{n>0}\frac{|n|^{it}\sigma_{-2it}(|n|)}{\sqrt{|n|}}W_{k,it}(4\pi|n|y)\mathbf{e}(nx)+$$
$$\frac{(-1)^k\Gamma(1/2+it)}{2\Gamma(\frac{1}{2}-k+it)\xi(1+2it)}\sum_{n<0}\frac{|n|^{it}\sigma_{-2it}(|n|)}{\sqrt{|n|}}W_{-k,it}(4\pi|n|y)\mathbf{e}(nx)$$

An analogous calculation yields the expansion of $E_{2k}(z,1/2+it)$ (of weight $-2k$, $k>0$):

$$E_{2k}(z,1/2+it) = \frac{(-1)^k\Gamma^2(\frac{1}{2}+it)}{\Gamma(\frac{1}{2}-k+it)\Gamma(\frac{1}{2}+k+it)}\phi(1/2+it)y^{1/2-it}+ \tag{12}$$

$$y^{1/2+it} + \frac{(-1)^k\Gamma(1/2+it)}{2\Gamma(\frac{1}{2}+k+it)\xi(1+2it)}\sum_{n<0}\frac{|n|^{it}\sigma_{-2it}(|n|)}{\sqrt{|n|}}W_{k,it}(4\pi|n|y)\mathbf{e}(nx)+$$
$$\frac{(-1)^k\Gamma(1/2+it)}{2\Gamma(\frac{1}{2}-k+it)\xi(1+2it)}\sum_{n>0}\frac{|n|^{it}\sigma_{-2it}(|n|)}{\sqrt{|n|}}W_{-k,it}(4\pi|n|y)\mathbf{e}(nx)$$

2. Incomplete Eisenstein series

In this section we shall estimate the expression $\langle F_h, \mathrm{d}\omega_j\rangle$, where $F_h(z,\theta)$ is the incomplete Eisenstein series.

Given a function $h \in C^\infty(y_0,\infty)$ let

$$H(s) = \int_0^\infty h(y)y^{-s}\frac{dy}{y}$$

be its Mellin transform.

The incomplete Eisenstein series (of weight $-2k$) corresponding to h is defined by

$$F_h(z,\theta) = \frac{e^{2ik\theta}}{2\pi i}\int_{\Re(s)=s_1>1} H(s)E_{2k}(z,s)ds \tag{13}$$

We let

$$|h^{(i)}(y)| \;\leq\; \frac{C_{i,j}}{y^j}, \quad i,j \geq 0 \tag{14}$$

The estimate (2) has been proved for the incomplete Eisenstein series of weight zero by Luo and Sarnak in [LS]. Here we shall prove

PROPOSITION 1. *Let F_h be an incomplete Eisenstein series of weight $2k \neq 0$. Then*

$$\sum_{r_j \leq R} |\langle F_h, \mathrm{d}\omega_j \rangle|^2 \;\ll_{k,\varepsilon}\; C_{2,2} C_{7,7}\, R^{1+\varepsilon}$$

for any $\varepsilon > 0$ and $R \geq 1$.

Proof of Proposition 1. We shall prove the Proposition for functions of weight $-2k$, $k > 0$. The proof for functions of weight $2k$ is analogous.

Using the formulas (3) and (13), we can "unfold" the expression $\langle F_h, \mathrm{d}\omega_j \rangle$ and get

$$\langle F_h, \mathrm{d}\omega_j \rangle \;=\; \frac{1}{2\pi i} \int_{\Re(s)=s_1} H(s)\, ds \int_{\Gamma\backslash \mathbf{H}} E_{2k}(z,s) \varphi_j(z) \varphi_{j,k}(z)\, \frac{\mathrm{dxdy}}{y^2}$$

Using the Fourier expansions (12), (5) and (6), we can change variables, integrate out x and get

$$\frac{1}{2\pi i} \int_{\Re(s)=s_1} H(s)\, ds \frac{(-1)^k \Gamma(1/2 + i r_j)}{(4\pi)^{s-1}} \left(\sum_{n=1}^{\infty} \frac{c_j(n)^2}{n^s} \right) I_1^k(s, r_j) \tag{15}$$

where $c_j(n)$-s are Hecke eigenvalues and $I_1^k(s,t)$ is defined by

$$\int_0^\infty W_{0,-it}(u)\, u^{s-2} \left[\frac{W_{k,it}(u)}{\Gamma(1/2+k+it)} + \frac{W_{-k,it}(u)}{\Gamma(1/2-k+it)} \right]\, du \tag{16}$$

This integral was evaluated in [Ja94], but here we will evaluate it differently,[2] simplifying the calculations later.

We shall consider a more general integral

$$\int_0^\infty y^v \left[\frac{W_{k,i\alpha}(y)}{\Gamma(1/2+k+i\alpha)} + \frac{W_{-k,i\alpha}(y)}{\Gamma(1/2-k+i\alpha)} \right] W_{0,i\beta}(y)\, dy \tag{17}$$

Let

$$F(k,\alpha,y) \;=\; \left[\frac{W_{k,i\alpha}(y)}{\Gamma(1/2+k+i\alpha)} + \frac{W_{-k,i\alpha}(y)}{\Gamma(1/2-k+i\alpha)} \right] \tag{18}$$

[2] As in [Ja95]

be the expression from (17). To evaluate the integral (16) we shall write the expression (18) as a combination of K-Bessel functions and the formula [GR, p. 693].

Using the formula [MOS, p. 431]

$$\int_{-\infty}^{\infty}(a-ix)^{-\mu}(b+ix)^{-\nu}e^{ixy}dx = \frac{2\pi y^{\frac{\nu+\mu-1}{2}}e^{\frac{y(b-a)}{2}}}{\Gamma(\nu)(a+b)^{\frac{\nu+\mu}{2}}}W_{\frac{\nu-\mu}{2},\frac{1-\nu-\mu}{2}}(ay+by) \tag{19}$$

we can get the following integral representation for the Whittaker function:

$$\frac{W_{k,i\alpha}(y)}{\Gamma(1/2+k+i\alpha)} = \frac{y^{-i\alpha}}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{4}+x^2\right)^{-1/2-i\alpha}\left(\frac{1/2-ix}{1/2+ix}\right)^{k} e^{ixy}\,dx \tag{20}$$

for every $k \in \mathbf{Z}$ (here we used the formula $W_{k,i\alpha}(y) = W_{k,-i\alpha}(y)$).

With a different choice of parameters in (19), we can write

$$\frac{W_{0,-i\alpha-l}(y)}{\Gamma(1/2+l+i\alpha)} = \frac{y^{-i\alpha-l}}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{4}+x^2\right)^{-1/2-i\alpha-l} e^{ixy}\,dx \tag{21}$$

We introduce the following notation:

$$u = 1/4+x^2, \quad e^{i\theta} = \left(\frac{1/2+ix}{1/2-ix}\right), \tag{22}$$

where

$$\theta = \pm\arccos\left(\frac{1/2-u}{u}\right), \quad \Re\left[\left(\frac{1/2+ix}{1/2-ix}\right)^{k}\right] = \cos(k\theta)$$

It follows that

$$\cos(k\theta) = T_k\left(-1+\frac{1}{2u}\right) \tag{23}$$

where T_k is a Chebyshev polynomial of the first kind.

Using the well-known property of Chebyshev polynomials

$$T_n(-x) = (-1)^n\, T_n(x)$$

and the formula [MOS, p. 257]

$$T_n(x) = F\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right)$$

(where F is the Gauss hypergeometric function), we can rewrite (23) as

$$T_k\left(-1+\frac{1}{2u}\right) = (-1)^k\, T_k\left(1-\frac{1}{2u}\right) = (-1)^k\, F\left(-k, k; \frac{1}{2}; \frac{1}{4u}\right) \tag{24}$$

Accordingly, using the formulas (20), (21) and (24), we can write

$$F(k,\alpha,y) \;=\; 2(-1)^k \sum_{l=0}^{k} \frac{(-k)_l\,(k)_l\,y^l}{(1/2)_l\,4^l\,l!}\,\frac{W_{0,-i\alpha-l}(y)}{\Gamma(1/2+l+i\alpha)} \tag{25}$$

where $(x)_l$ is defined by

$$(x)_l \;=\; x(x+1)...(x+l-1); \quad (x)_0 \;=\; 1 \tag{26}$$

Finally, using the formulas (25), $W_{0,\mu}(y)=\sqrt{(y/\pi)}K_\mu(y/2)$ and [GR, p. 693], we can evaluate the integral (17). It is equal to

$$\frac{4^{v+1}(-1)^k}{\pi}\sum_{l=0}^{k}\frac{(-k)_l\,(k)_l}{(\frac{1}{2})_l\,l!\,\Gamma(\frac{1}{2}+l+i\alpha)\Gamma(v+2+l)}\Gamma\left(\frac{v+2l+2+i\alpha+i\beta}{2}\right) \tag{27}$$

$$\Gamma\left(\frac{v+2l+2+i\alpha-i\beta}{2}\right)\Gamma\left(\frac{v+2-i\alpha+i\beta}{2}\right)\Gamma\left(\frac{v+2-i\alpha-i\beta}{2}\right)$$

Remark: The integral (17) can be evaluated as above if the second indices of Whittaker functions in (17) are not purely imaginary. We have not derived the general versions of the formulas (25) and (27) since we are not using the general versions here.

Substituting the parameters from $I_1^k(s,r_j)$ into (27), we find that the integral (16) is equal to

$$\frac{4^{s-1}(-1)^k\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}-ir_j)}{\pi}\sum_{l=0}^{k}\frac{(-k)_l\,(k)_l\,\Gamma(\frac{s}{2}+l)\Gamma(\frac{s}{2}+l+ir_j)}{(\frac{1}{2})_l\,\Gamma(s+l)\Gamma(\frac{1}{2}+l+ir_j)\,l!} \tag{28}$$

It is convenient for later calculations to rewrite the above formula as

$$\frac{4^{s-1}(-1)^k\Gamma^2(\frac{s}{2})\Gamma(\frac{s}{2}-ir_j)\Gamma(\frac{s}{2}+ir_j)}{\pi\Gamma(s)\Gamma(\frac{1}{2}+ir_j)}\,{}_4F_3\left(\begin{array}{c}-k,k,\frac{s}{2},\frac{s}{2}+ir_j\\ s,\frac{1}{2},\frac{1}{2}+ir_j\end{array}\right) \tag{29}$$

where ${}_4F_3$ is a (terminating, Saalschützian) generalized hypergeometric series of the unit argument defined by

$${}_4F_3\left(\begin{array}{c}x,y,z,-k\\ u,v,w\end{array}\right) \;=\; {}_4F_3\left(\begin{array}{c}x,y,z,-k\\ u,v,w\end{array};\,1\right) \;=\; \sum_{l=0}^{k-1}\frac{(-k)_l\,(x)_l\,(y)_l\,(z)_l}{(u)_l\,(v)_l\,(w)_l\,l!}$$

and where $(x)_l$ is defined by (26).

Applying the transformation formula for terminating Saalschützian hypergeometric functions twice [Sl, 4.3.5.1] and making the obvious simplifications, we can write

$${}_4F_3\left(\begin{array}{c}x,y,z,-k\\ u,v,w\end{array}\right) \;=$$

$$\frac{(v-z)_k\,(w-z)_k}{(v)_k\,(w)_k}\,{}_4F_3\left(\begin{array}{c} u-x, u-y, z, -k \\ u, 1-v+z-k, 1-w+z-k \end{array}\right) =$$
$$\frac{(u+v-z-x)_k\,(u+w-z-x)_k}{(v)_k\,(w)_k}\,{}_4F_3\left(\begin{array}{c} u-x, y, u-z, -k \\ u, u+v-x-z, u+w-x-z \end{array}\right)$$

Substituting the function ${}_4F_3$ from (29) into the above formula, we get a new expression for this function:

$$\frac{(1-s)_k\,(\frac{1}{2}-it)_k}{(s)_k\,(\frac{1}{2}+it)_k}\,{}_4F_3\left(\begin{array}{c} \frac{1-s}{2}, \frac{1-s}{2}-ir_j, k, -k \\ 1-s, \frac{1}{2}, \frac{1}{2}-ir_j \end{array}\right) \tag{30}$$

Note that ${}_4F_3$ in (30) is defined when $s=1$, since the factors divisible by $1-s$ in the expression

$$\frac{(1/2-s/2)_l}{(1-s)_l}$$

(coming from the "top" and the "bottom" arguments of ${}_4F_3$) cancel each other. This proves that

$$I_1^k(1, r_j) \;=\; 0 \tag{31}$$

(since $(1-s)_k = 0$ when $s=1$, hence in front of ${}_4F_3$ in (30) is equal to zero, while ${}_4F_3$ itself is well-defined at $s=1$).[3]

Denote by

$$L(\varphi_j \otimes \varphi_j, s)$$

the infinite sum in (15).

We shift the contour of integration from $\Re(s) = s_1 > 1$ to $\Re(s) = 1/2$. The function $L(\varphi_j \otimes \varphi_j, s)$ has a simple pole at $s=1$ (cf. [LS]), while the expression $I_1^k(s, r_j)$ has a simple zero there (cf. (31)). Accordingly, we don't pass through any poles when we shift the integration contour.

Using the Cauchy's inequality and formula (28), we can estimate $|\langle F_h, \mathrm{d}\omega_j\rangle|^2$ by a constant multiple of

$$\int_{(1/2)} |H(s)|\,|ds|\;(k+1)|\Gamma(\frac{1}{2}+ir_j)|^2 \left(\sum_{l=0}^{k} \frac{(-k)_l\,(k)_l}{(\frac{1}{2})_l\, l!}\right. \tag{32}$$
$$\left.\int_{(1/2)} |H(s)|\,|L(\varphi_j\otimes\varphi_j, s)|^2 \frac{|\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}-ir_j)\Gamma(\frac{s}{2}+l)\Gamma(\frac{s}{2}+l+ir_j)|^2}{|\Gamma(s+l)\Gamma(\frac{1}{2}+l+ir_j)|^2}\,|ds|\right)$$

It is easy to show using the definition of the Mellin transform that

$$|H(s)| \;\ll\; \frac{C_{l,l}}{s(s-1)\ldots(s-l+1)} \tag{33}$$

[3]Formula (31) could be proven easier, but the expression (30) makes it easier to estimate $\lim_{s\to 1} I_k^1(s,t)/(s-1)$ and can be used ([Ja95]) to simplify some of the calculations in [Ja94].

Also, it has been shown in [LS] that

$$\sum_{r_j \le R} \frac{1}{\cosh(2\pi r_j)} \, |L(\varphi_j \otimes \varphi_j, 1/2 + iu)|^2 \;\ll\; R^{2+\varepsilon} \, |u|^{11/2+\varepsilon} \tag{34}$$

for every $\varepsilon > 0$.

To prove Proposition 1, it suffices to sum the expression (32) for $R < r_j \le 2R$ (also denoted by $r_j \sim R$). Using formula (33) we can etsimate

$$\int_{(1/2)} |H(s)| \, |ds| \;\ll\; C_{2,2}$$

We need to estimate the second integral in (32). We estimate the expressions

$$\sum_{r_j \sim R} \int_{\Im(s) \le R/10} \tag{35}$$

and

$$\sum_{r_j \sim R} \int_{\Im(s) \ge R/10} \tag{36}$$

separately.

Using (34), (33) and Stirling's formula, we can bound both the sum in (35) and the sum in (36) by $C_{7,7} R^{1+\varepsilon}$. This implies that for any $\varepsilon > 0$,

$$\sum_{r_j \sim R} |\langle F_h, \mathrm{d}\omega_j \rangle|^2 \;\ll_{k,\varepsilon}\; C_{2,2} C_{7,7} \, R^{1+\varepsilon}$$

finishing the proof of Proposition 1. □

3. Incomplete Poincare series

In this section we estimate the expressions $\langle P_{h,m}, \mathrm{d}\omega_j \rangle$ (where $P_{h,m}(z,\theta)$ is the incomplete Poincare series defined below) and finish the proof of Theorem 2.

Let $h \in C^\infty(y_0, \infty)$ be as in the definition of the incomplete Eisenstein series. For $m \in \mathbf{Z}$, the incomplete Poincare series $P_{h,m,2k} = P_{m,2k}$ of weight $-2k$ are defined (in the usual coordinates (x, y, θ)) by

$$P_{m,2k}(z,\theta) \;=\; e^{2ik\theta} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(y(\gamma z)) \, (\epsilon_\gamma(z))^{2k} \, \mathbf{e}(m x(\gamma z)) \tag{37}$$

where $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$,

$$\Gamma_\infty \;=\; \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : \; m \in \mathbf{Z} \right\}$$

and $\epsilon_\gamma(z) = (cz+d)/|cz+d|$ for

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$$

For $m=0$, $P_{m,2k}$ becomes the incomplete Eisenstein series of the same weight.

Theorem 2 has been proved for the incomplete Poincare series of weight zero by Luo and Sarnak in [LS]. We shall prove

PROPOSITION 2. *Let $P_{m,2k}$ be an incomplete Poincare series of weight $-2k \neq 0$. Then*

$$\sum_{r_j \le R} |\langle P_{m,2k}, \mathrm{d}\omega_j\rangle|^2 \ \ll_{k,\varepsilon}\ m^2\left(C_{0,0}^2 + C_{2,2}^2\right)\ R^{1+\varepsilon}$$

for any $\varepsilon > 0$, $m \neq 0$ and $R \geq 1$.

Proof. We shall prove the Proposition for $k>0, m>0$ only; the proofs in other cases are analogous. The proof will be by induction on k.

After "unfolding" $\langle P_{m,2k}, \mathrm{d}\omega_j\rangle$ we get the expression

$$\int_{\Gamma_\infty\backslash\mathbf{H}} h(y)\varphi_{j,k}(z)\varphi_j(z)\mathbf{e}(mx)\frac{\mathrm{dxdy}}{y^2}$$

If we use the formlas (5) and (6) and integrate out x the last expression becomes

$$\frac{(-1)^k\Gamma(\frac12+ir_j)}{\Gamma(\frac12+k+ir_j)}\sum_{n>0}\frac{c_j(n)c_j(n+m)}{\sqrt{n(n+m)}}\int_0^\infty W_{k,ir_j}(4\pi ny)W_{0,ir_j}(4\pi(n+m)y)\frac{dy}{y^2}\ +$$

$$\frac{(-1)^k\Gamma(\frac12+ir_j)}{\Gamma(\frac12-k+ir_j)}\sum_{\substack{n<0\\ n\neq -m}}\frac{c_j(n)c_j(n+m)}{\sqrt{|n(n+m)|}}\int_0^\infty W_{-k,ir_j}(4\pi|n|y)W_{0,ir_j}(4\pi|n+m|y)\frac{dy}{y^2}$$

We shall only consider even cusp forms (i.e. $c_j(-n)=c_j(n)$); the calculations for odd cusp forms are similar. After using the formula $c_j(n) = c_j(1)\lambda_j(n)$ and the property

$$\lambda_j(p)\lambda_j(q)\ =\ \sum_{d|(p,q)}\lambda_j(pq/d^2)$$

and after making a change of variables in the last integral we get

$$4\pi(-1)^k\Gamma(1/2+ir_j)c_j(1)\sum_{d|m}\sum_{l\neq 0,-m/d}\frac{c_j(l^2+lm/d)}{\sqrt{|1+m/(ld)|}} \tag{38}$$

$$\frac{1}{\Gamma(1/2+k\cdot\mathrm{sgn}(l)+ir_j)}\int_0^\infty W_{k\cdot\mathrm{sgn}(l),ir_j}(y)W_{0,ir_j}\left(y\left|1+\frac{m}{ld}\right|\right)h\left(\frac{y}{4\pi|ld|}\right)\ \frac{dy}{y^2}$$

Our proof of Proposition 2 for $k \neq 0$ will follow the proof of the same result for $k = 0$ in [LS] making the modifications needed for $k \neq 0$.

As in [LS] we remark that by dyadic partition it suffices to estimate (38) for $r_j \sim R$ and $l \sim L, L > 0$. Also, it is easy to show that it suffices to consider $dL \leq AR$ (for a sufficiently large A) and $m \ll R$. One can easily show that the square of the integral in (38) is

$$\ll_{k,\varepsilon} C_{0,0}^2 \int_0^\infty |W_{0,ir_j}(y)|^2 y^\varepsilon \frac{dy}{y^2} \int_0^\infty |W_{k\cdot\mathrm{sgn}(l),ir_j}(y)|^2 y^\varepsilon \frac{dy}{y^2} \tag{39}$$

where $\varepsilon > 0$ is small. We shall estimate (39) for $l > 0$ only; the estimates for $l < 0$ are similar.

Define

$$I_k(r,s) = \int_0^\infty |W_{k,ir}(y)|^2 y^s \frac{dy}{y^2}$$

Then

$$I_0(r,s) = \frac{\Gamma^2(s/2)}{8\pi^2\Gamma(s)}\Gamma(s/2+ir)\Gamma(s/2-ir)$$

Also, using [Ze91, 3.15.3] (which relates I_k and I_{k-1}) we find that for $k > 0$

$$I_k(r,\varepsilon) \ll_{k,\varepsilon} r^2 I_{k-1}(r,\varepsilon) \tag{40}$$

Now, using (39), (40) (with $r = r_j$), Stirling's formula, [Kuz, 2.29] and (38) we can show that if we consider only the terms with $l \sim L$ in (38) then

$$\left(\sum_{r_j\sim R} |\langle P_{m,2k}, \mathrm{d}\omega_j\rangle|^2\right)_{l\sim L} \ll_{k,\varepsilon} C_{0,0}^2 L^2 R^\varepsilon$$

for any $\varepsilon > 0$. Hence (as in [LS]) we can assume that $L \gg R^{1/2}$ and $m \ll L^{2/3}$.

Expressing $h(y)$ in terms of its Mellin transform $H(s)$, we can rewrite the integral in (38) as

$$\frac{1}{2\pi i}\int_{\Re(s)>1} \frac{H(s)}{|4\pi l d|^s}\, ds \int_0^\infty y^{s-1} W_{k,ir_j}(y) W_{0,ir_j}\left(y\left|1+\frac{m}{ld}\right|\right)\,\frac{dy}{y}$$

which in turn is equal to

$$\frac{1}{2\pi i}\left(\frac{|1+\frac{m}{ld}|}{2\pi}\right)^{\frac{1}{2}} \int_{\Re(s)>1} \frac{H(s)}{|2\pi l d|^s}\, ds \int_0^\infty y^{s-\frac{3}{2}} W_{k,ir_j}(2y) K_{ir_j}\left(y\left|1+\frac{m}{ld}\right|\right) dy \tag{41}$$

Denote the second integral in (41) by $A_k(s)$ (where we have suppressed the dependence on r_j, m, l, d). When $k = 0$, the integral becomes

$$\sqrt{\frac{2}{\pi}} \int_0^\infty y^{s-1} K_{ir_j}(y) K_{ir_j}\left(y\left|1+\frac{m}{ld}\right|\right)\, dy$$

which was evaluated in [LS]. The integral in the formula above is equal to

$$2^{s-3}\Gamma\left(\frac{s+2ir_j}{2}\right)\Gamma\left(\frac{s-2ir_j}{2}\right)(1+m/dl)^{ir_j} \tag{42}$$

$$\int_0^1 \tau^{s/2-1}(1-\tau)^{s/2-1}\left(1+\frac{2\tau m}{dl}+\tau\left(\frac{m}{dl}\right)^2\right)^{-s/2-ir_j} d\tau$$

If we evaluate $A_1(s)$, we can evaluate $A_k(s)$ for every k using (42) and the following well-known relation between Whittaker functions

$$W_{k+1,it}(2y) \;=\; (-2k+2y)W_{k,it}(2y) - (t^2 + (k-1/2)^2)W_{k-1,it}(2y)$$

resutling in

$$A_{k+1}(s) \;=\; -2kA_k(s) + 2A_k(s+1) - [(k-1/2)^2 + r_j^2]A_{k-1}(s) \tag{43}$$

Using the standard properties of K-Bessel and Whittaker functions ([GR, 8.486.13] and [GR, 9.234.3]) we see that

$$W_{1,ir_j}(2y) \;=\; \sqrt{\frac{2}{\pi}}\left[y^{3/2}K_{ir_j}(y) - y^{1/2}(1/2+ir_j)K_{ir_j}(y) + y^{3/2}K_{ir_j+1}(y)\right]$$

Substituting into the expression for $A_1(s)$ we get

$$A_1(s) = A_0(s+1) - (1/2+ir_j)A_0(s) + \sqrt{2/\pi}B(s) \tag{44}$$

where

$$B(s) \;=\; \int_0^\infty y^s K_{ir_j+1}(y) K_{ir_j}\left(y\left|1+\frac{m}{ld}\right|\right)\, dy \tag{45}$$

Using the formulas [GR, 6.576.4] and the integral representation [GR, 9.111] of the hypergeometric function, we may write $B(s)$ as

$$B(s) \;=\; 2^{s-2}\left(1+\frac{m}{dl}\right)^{ir_j}\Gamma(s/2-ir_j)\Gamma(s/2+1+ir_j) \tag{46}$$

$$\int_0^1 \tau^{s/2-1}(1-\tau)^{s/2}\left(1+\frac{2\tau m}{dl}+\tau\left(\frac{m}{dl}\right)^2\right)^{-(s/2+ir_j+1)} d\tau$$

We now estimate (38) for $k \geq 1$. For $k = 1$ we estimate three terms in (44) separately; for $k \geq 2$ we do the same for (43). We shall also multiply the expressions in (38) and (44) by the ratio of Γ-factors coming from (38). We start with the second term in (44); it contributes

$$\frac{\Gamma(1/2+ir_j)(1/2+ir_j)A_0(s)}{\Gamma(3/2+ir_j)} \;=\; A_0(s)$$

so the estimates can proceed exactly as for $k=0$.

Similarly, the contribution of the third term in (38) is

$$\frac{\Gamma(1/2+ir_j)A_{k-1}(s)}{\Gamma(k-1/2+ir_j)} \cdot \frac{\Gamma(k-1/2+ir_j[(k-1/2)^2+r_j^2])}{\Gamma(k+3/2+ir_j)} \tag{47}$$

The first factor in (47) was estimated by the induction assumption for $k-1$; since the absolute value of the second factor is always less then 1, the same estimate works for (47).

We now see that by induction and the previous remarks one only needs to estimate the terms $A_0(s+k)$ and $B(s+k)$ for $k \geq 1$ to prove Proposition 2.

We proceed to estimate the contribution (in (38)) of the terms of the type $A_0(s+k)$. Taking into account the ratio of Γ-factors in (38), we see that the estimates need to be made for $A_0(s+k)/(1/2+ir_j)_k$. The estimates proceed similarly to those in [LS], namely, by shifting the line of integration in (41) to $\Re(s)=\varepsilon$ where $0<\varepsilon\ll 1$ and using Cauchy's inequality. To that end, we evaluate the integral

$$\int_0^1 [\tau(1-\tau)]^{(s+k)/2-1}\, d\tau \;=\; \frac{\Gamma^2((s+k)/2)}{\Gamma(s+k)} \tag{48}$$

using the formula [GR, 3.191.3].

One can now show that it suffices to prove the following analogue of the estimate (37) of [LS] (we tried to make our notation similar to that of (37) in [LS]):

$$\sum_{r_j\sim R}\left|\sum_{l\sim L} a_l v_j(l^2+lm/d) f(l,r_j)\right|^2 \;\ll\; mR^{2+\varepsilon}L \tag{49}$$

where $v_j(n)$ is given by

$$v_j(n)\cosh \pi r_j/2 \;=\; c_j(n),$$

$f(l,r_j)$ is defined by

$$f(m,d,l,\tau,r_j) \;=\; \left(\frac{1+m/(dl)}{1+2\tau m/(dl)+\tau[m/(dl)]^2}\right)^{ir_j}$$

and finally $a_l = a_l(m, d, \tau, s)$ is given by

$$a_l \;=\; l^{-s}\left[1 + \frac{2\tau m}{dl} + \tau\left(\frac{m}{dl}\right)^2\right]^{-(s+k)/2}$$

If (49) is proved, one makes the estimates (needed in the proof of Proposition 2) for $A_0(s+k)$ by using (48), Stirling formula, (33) and by integrating separately for $\Im(s) \le R/10$ and $\Im(s) \ge R/10$ (as Luo and Sarnak do in [LS]).[4] The proof of (49) is a straightforward modification of the proof of (37) in [LS] (cf. pp. 225-227).

It remains to estimate the terms $B(s+k)$ appearing in (38) for $k \ge 1$. Taking Γ-factors from (38), we see that the estimates need to be made for $B(s+k)/(1/2+ir_j)_{k+1}$. The estimates proceed as those for $A_0(s+k)$, by shifting the line of integration in (41) to $\Re(s) = \varepsilon$ where $0 < \varepsilon \ll 1$ and using Cauchy's inequality. Evaluating the τ-integral gives

$$\int_0^1 \tau^{(s+k)/2-1}(1-\tau)^{(s+k)/2-1}\,d\tau \;=\; \frac{\Gamma((s+k)/2)\Gamma((s+k)/2+1)}{\Gamma(s+k+1)} \tag{50}$$

From the formula (46) that it now suffices to prove (49); the only changing definition is that of a_l:

$$a_l \;=\; l^{-s}\left[1 + \frac{2\tau m}{dl} + \tau\left(\frac{m}{dl}\right)^2\right]^{-(s+k)/2-1}$$

The proof (49) for $B(s+k)$ is an appropriate modification of the proof of its counterpart for $A_0(s+k)$. The remaining estimates are made using (50), Stirling formula, (33) and by integrating separately for $\Im(s) \le R/10$ and $\Im(s) \ge R/10$. This finishes the proof of Proposition 2. $\square$

We can now use the results of Propositions 1 and 2 to prove Theorem 2. It suffices to prove the theorem for functions of a fixed even nonzero weight (the result for weight zero were proved in [LS] and the general result follows easily). The proof proceeds as in [LS, §4]. Namely, we first decompose $X = \mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H} = \cup_j C_j$ into neighborhoods of the cusp, elliptic points and the neighborhoods not containing elliptic fixed points. We choose a partition of unity subordinate to the decomposition.

Consider a smooth function F on X of weight $-2k \ne 0$; without loss of generality we may assume that F is supported in a single neighborhood C. Denote by $\tilde{C}$ the lift of C into the fundamental domain in $\mathbf{H}$ and by f the Γ_∞-periodic function of weight $-2k$ coinciding with F in $\tilde{C}$. Let $W = 1$ if C doesn't have elliptic vertices, and let it be the order

[4]The constant $C_{2,2}^2$ in the statement of Proposition 2 comes from that integration.

of the stabilizer otherwise. Then (compare [LS, (38)])

$$F(z) \;=\; \frac{1}{W}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma z)(\epsilon_\gamma(z))^{2k} \tag{51}$$

where $\epsilon_\gamma(z)$ was defined in (37).

Expanding f into the Fourier series

$$f(z) \;=\; \sum_{m=-\infty}^{\infty} h_m(y)\mathbf{e}(mx)$$

we see from (51) that F can be expressed,

$$F(z) \;=\; \frac{1}{W}\left(F_{h_0}(z) + \sum_{m\neq 0} P_{h_m}(z)\right), \tag{52}$$

as a sum of the incomplete Eisenstein series $F_0 = F_{h_0}$ and the incomplete Poincare series $P_m = P_{h_m}$ of weight $-2k$. Also, the proofs of the estimates for the decay (in m) of h_m-s and their derivatives proceed exactly as in [LS, §4] via integration by parts.

Now, (52) implies

$$W\langle F, \mathrm{d}\omega_j\rangle \;=\; \langle F_0, \mathrm{d}\omega_j\rangle + \sum_{m\neq 0}\langle P_m, \mathrm{d}\omega_j\rangle$$

As in [LS] we use Cauchy's inequality with weight $\alpha_m > 0$, $\alpha_0 = 1$ to write

$$|\langle F, \mathrm{d}\omega_j\rangle|^2 \;\ll\; \left(\sum_m \frac{1}{\alpha_m}\right)\left(|\langle F_0, \mathrm{d}\omega_j\rangle|^2 + \sum_{m\neq 0}\alpha_m|\langle P_m, \mathrm{d}\omega_j\rangle|^2\right) \tag{53}$$

Theorem 2 is proved (as Theorem 1 in [LS]) by summing (53) over $r_j \leq R$, putting $\alpha_m = (|m|+1)^{3/2}$ and using the estimates of Propositions 1 and 2. □

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