SCALAR CURVATURE AND Q-CURVATURE OF RANDOM METRICS.

YAIZA CANZANI, DMITRY JAKOBSON, AND IGOR WIGMAN

Abstract. We study Gauss curvature for random Riemannian metrics on a compact surface, lying in a fixed conformal class; our questions are motivated by comparison geometry. We next consider analogous questions for the scalar curvature in dimension $n > 2$, and for the Q-curvature of random Riemannian metrics.

1. Introduction

The goal of the authors in this paper is to initiate the study of standard questions in *comparison geometry* for random Riemannian metrics lying in the same conformal class.

Since the 19th century, many results have been established comparing geometric and topological properties of manifolds where the (sectional or Ricci) curvature is bounded from above or from below, with similar properties of manifolds of constant curvature.

When studying such questions for random Riemannian metrics, the first natural question is to estimate the *probability* of the metric satisfying certain curvature bounds, in a suitable regime. The present paper addresses such questions for *scalar curvature*, and also for Branson's *Q-curvature*. We study the behavior of scalar curvature for random Riemannian metrics in a fixed conformal class, where the conformal factor is a random function possessing certain smoothness; our random metrics are close to a "reference" metric that we denote g_0.

The paper addresses two main questions (our manifold M is always assumed to be compact and orientable):

Question 1.1. *Assuming that the scalar curvature R_0 of the reference metric g_0 doesn't vanish, what is the probability that the scalar curvature of the perturbed metric changes sign?*

We remark that in each conformal class, there exists a *Yamabe metric* with constant scalar curvature $R_0(x) \equiv R_0$ [Yam, Au76, Sch84, Tr]; the sign of R_0 is uniquely determined. Question 1.1 can be posed in each conformal class where $R_0 \neq 0$ (e.g. in dimension two, for $M \ncong \mathbf{T}^2$). Also, it was shown in [CY, DM, N] that in every conformal class satisfying certain *generic* conditions, there exists a

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metric g_0 with constant Q-curvature, $Q_0(x) \equiv Q_0$. Question 1.1 can be posed in each conformal class where $Q_0 \neq 0$.

Question 1.2. *What is the probability that the curvature of the perturbed metric changes by more than u (where u is a positive real parameter, subject to some restrictions)?*

We study Question 1.2 for Gauss curvature (equal to 1/2 the scalar curvature) in dimension 2, and for Q-curvature in higher dimensions. Our techniques are inspired by [AT08, Bl].

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2. Random metrics in a conformal class

We consider a conformal class of metrics on a Riemannian manifold M of the form

$$g_1 = e^{af} g_0, \tag{1}$$

where g_0 is a "reference" Riemannian metric on M, a is a constant, and $f = f(x)$ is a C^2 function on M.

Given a metric g_0 on M and the corresponding Laplacian Δ_0, let $\{\lambda_j, \phi_j\}$ denote an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of $-\Delta_0$; we let $\lambda_0 = 0, \phi_0 = 1$. We define a random conformal multiple $f(x)$ by

$$f(x) = -\sum_{j=1}^{\infty} a_j c_j \phi_j(x), \tag{2}$$

where $a_j \sim \mathcal{N}(0, 1)$ are i.i.d standard Gaussians, and c_j are positive real numbers, and we use the minus sign for convenience purposes only. We assume that $c_j = F(\lambda_j)$, where $F(t)$ is an eventually monotone decreasing function of t, $F(t) \to 0$ as $t \to \infty$.

The *random field* $f(x)$ is a centered Gaussian field with covariance function

$$r_f(x, y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)\phi_j(y),$$

$x, y \in M$. In particular for every $x \in M$, $f(x)$ is mean zero Gaussian of variance

$$\sigma^2(x) = r_f(x, x) = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)^2.$$

In the study of scalar curvature, it is convenient to work with the random centered Gaussian field

$$h(x) := \Delta_0 f(x) = \sum_{j=1}^{\infty} a_j c_j \lambda_j \phi_j(x) \tag{3}$$

having the covariance function

$$r_h(x,y) = \sum_{j=1}^{\infty} c_j^2 \lambda_j^2 \phi_j(x)\phi_j(y) \tag{4}$$

$x, y \in M$.

The smoothness of the Gaussian random field (2) is given by the following proposition, [Bl, Proposition 1]:

Proposition 2.1. *If $\sum_{j=1}^{\infty}(\lambda_j+1)^r c_j^2 < \infty$, then $f(x) \in H^r(M)$ a.s.*

Choosing $c_j = F(\lambda_j) = \lambda_j^{-s}$ translates to $\sum_{j\geq 1} \lambda_j^{r-2s} < \infty$. In dimension n, it follows from Weyl's law that $\lambda_j \asymp j^{2/n}$ as $j \to \infty$; we find that

$$If\ s > \frac{2r+n}{4}, \quad then\ f(x) \in H^r(M)\ a.s.$$

By the Sobolev embedding theorem, $H^r \subset C^k$ for $k < r - n/2$. Substituting into the formula above, we find that

$$If\ c_j = O(\lambda_j^{-s}), s > \frac{n+k}{2}, \quad then\ f(x) \in C^k\ a.s. \tag{5}$$

We will be mainly interested in $k = 0$ and $k = 2$. Accordingly, we formulate the following

Corollary 2.2. *If $c_j = O(\lambda_j^{-s}), s > n/2$, then $f \in C^0$ a.s; if $c_j = O(\lambda_j^{-s}), s > n/2+1$, then $f \in C^2$ a.s. Similarly, if $c_j = O(\lambda_j^{-s}), s > n/2+1$, then $\Delta_0 f \in C^0$ a.s; if $c_j = O(\lambda_j^{-s}), s > n/2+2$, then $\Delta_0 f \in C^2$ a.s.*

We next consider the volume of the random metric in (1). The volume element dV_1 corresponding to g_1 is given by

$$dV_1 = e^{naf/2} dV_0, \tag{6}$$

where dV_0 denotes the volume element corresponding to g_0.

Consider the random variable $V_1 = \mathrm{vol}(M, g_1)$.

Proposition 2.3. *Notation as above,*

$$\lim_{a\to 0} \mathbb{E}[V_1(a)] = V_0,$$

where V_0 denotes the volume of (M, g_0).

Remark 2.4. *The constant a can be regarded as the radius of a sphere (in a space $\mathcal{M}$ of Riemannian metrics on M), centered at g_0. The regime $a \to 0$ can thus be considered as studying* local *geometry of $\mathcal{M}$.*

It is well-known that the scalar curvature R_1 of the metric g_1 in (1) is related to the scalar curvature R_0 of the metric g_0 by the following formula ([Au98, §5.2, p. 146])

$$R_1 = e^{-af}[R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)|\nabla_0 f|^2/4], \tag{7}$$

where Δ_0 is the (negative definite) Laplacian for g_0, and ∇_0 is the gradient corresponding to g_0. We observe that the last term vanishes when $n = 2$:

$$R_1 = e^{-af}[R_0 - a\Delta_0 f]. \tag{8}$$

Substituting (2), we find that

$$R_1(x)e^{af(x)} = R_0(x) - a\sum_{j=1}^{\infty}\lambda_j a_j c_j \phi_j(x). \tag{9}$$

Proposition 2.5. *If $R_0 \in C^0$ and $c_j = O(\lambda_j^{-s}), s > n/2 + 1$ then $R_1 \in C^0$ a.s. If $R_0 \in C^2$ and $c_j = O(\lambda_j^{-s}), s > n/2 + 2$ then $R_1 \in C^2$ a.s.*

Consider the *sign* of the scalar curvature R_1 of the new metric. We make a remark that will be important later:

Remark 2.6. *Note that the quantity e^{-af} is positive so that the sign of R_1 satisfies*

$$\operatorname{sgn}(R_1) = \operatorname{sgn}[R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)|\nabla_0 f|^2/4],$$

in particular for $n = 2$, assuming that R_0 has constant sign, we find that

$$\operatorname{sgn}(R_1) = \operatorname{sgn}(R_0 - a\Delta_0 f) = \operatorname{sgn}(R_0 - ah) = \operatorname{sgn}(R_0)\cdot\operatorname{sgn}(1 - ah/R_0).$$

3. Studying Question 1.1

We denote by $M = M_\gamma$ a compact surface of genus $\gamma \neq 1$. We choose a reference metric g_0 so that R_0 has constant sign (positive if $M = S^2$, and negative if M has genus ≥ 2).

Define the random metric on M_γ by $g_1 = e^{af}g_0$, (as in (1)) and f is given by (2). Let $P_2(a)$ the probability

$$P_2(a) := \operatorname{Prob}\{\exists x \in M : \operatorname{sgn} R_1(g_1(a), x) \neq \operatorname{sgn}(R_0)\}, \tag{10}$$

i.e. the probability that the curvature R_1 of the random metric $g_1(a)$ *changes sign* somewhere on M. The probability of the complementary event $P_1(a) = 1 - P_2(a)$ is clearly $P_1(a) := \operatorname{Prob}\{\forall x \in M : \operatorname{sgn}(R_1(g_1(a), x)) = \operatorname{sgn}(R_0), \}$ i.e. the probability that the curvature of the random metric $g_1(a)$ *does not* change sign.

By Remark 2.6, in dimension two $\operatorname{sgn}(R_1) = \operatorname{sgn}(R_0)\operatorname{sgn}(1 - ah/R_0)$, where $h = \Delta_0 f$ was defined earlier in (3). We let v denote the random field

$$v(x) = h(x)/R_0(x) \tag{11}$$

Note that

$$r_v(x, x) = r_h(x, x)/[R_0(x)]^2, \tag{12}$$

and let

$$\sigma_v^2 = \sup_{x\in M} r_v(x, x) = \sup_{x\in M} r_h(x, x)/[R_0(x)]^2. \tag{13}$$

For a random field $F : M \to \mathbb{R}$ that is a.s. bounded we introduce the random variable $\|F\|_M := \sup_{x\in M} F(x)$. It follows from Remark 2.6 that

$$P_2(a) = \operatorname{Prob}\{||v||_M > 1/a\}. \tag{14}$$

We shall estimate $P_2(a)$ in the limit $a \to 0$. Geometrically, that means that $g_1(a) \to g_0$, so $P_2(a)$ should go to zero as $a \to 0$; below, we estimate the *rate.*

To do that, we use a strong version of the Borell-TIS inequality, cf. [Bor, TIS] or [AT08, p. 51].

From now on we shall assume that $R_0 \in C^0(M)$, and that $c_j = O(\lambda_j^{-s}), s > 2$. Then Proposition 2.5 implies that h and R_1 are a.s. C^0 and hence bounded, since M is compact.

Theorem 3.1. *Assume that $R_0 \in C^0(M)$ and that $c_j = O(\lambda_j^{-s}), s > 2$. Then there exist constants $C_1 > 0$ and C_2 such that the probability $P_2(a)$ satisfies*

$$(C_1 a) e^{-1/(2a^2\sigma_v^2)} \leq P_2(a) \leq e^{C_2/a - 1/(2a^2\sigma_v^2)},$$

as $a \to 0$. In particular

$$\lim_{a\to 0} a^2 \ln P_2(a) = \frac{-1}{2\sigma_v^2}.$$

Let M be a compact orientable surface, $M \ncong \mathbf{T}^2$. Consider random *real-analytic* conformal deformations; this corresponds to the case when the coefficients c_j in (2) decay *exponentially.*

We fix a real parameter $T > 0$ and choose the coefficients c_j in (2) to be equal to

$$c_j = e^{-\lambda_j T/2}/\lambda_j. \tag{15}$$

Here $T/2$ can be regarded as the radius of analyticity of g_1.

Then it follows from (4) that

$$r_h(x, x) = e^*(x, x, T) = \sum_{j:\lambda_j>0} e^{-\lambda_j T}\phi_j(x)^2,$$

where $e^*(x, x, T)$ denotes the *heat kernel* on M *without the constant term*, evaluated at x at time T.

Proposition 3.2. *Assume that the coefficients c_j are chosen as in* (15)*. Then as $T \to 0^+$, σ_v^2 is asymptotic to*

$$\frac{1}{(4\pi T)^{n/2} \inf_{x\in M}(R_0(x))^2}.$$

Theorem 3.3. *Let g_0 and g_1 be two distinct reference metrics on M, normalized to have equal volume, such that R_0 and R_1 have constant sign, $R_0 \equiv const$ and $R_1 \not\equiv const$. Then there exists $a_0, T_0 > 0$ (that depend on g_0, g_1) such that for any $0 < a < a_0$ and for any $0 < t < T_0$, we have $P_2(a, T, g_1) > P_2(a, T, g_0)$.*

It follows that in every conformal class, $P_2(a, T, g_0)$ is minimized in the limit $a \to 0, T \to 0^+$ for the metric g_0 of constant curvature.

Let $\lambda_1 = \lambda_1(g_0)$ denote the smallest nonzero eigenvalue of Δ_0. Denote by $m = m(\lambda_1)$ the multiplicity of λ_1, and let

$$F := \sup_{x\in M} \frac{\sum_{j=1}^m \phi_j(x)^2}{R_0(x)^2}. \tag{16}$$

Proposition 3.4. *Let the coefficients c_j be as in* (15)*. Denote by $\sigma_v^2(T)$ the corresponding supremum of the variance of v. Then*

$$\lim_{T\to\infty} \frac{\sigma_v^2(T)}{F e^{-\lambda_1 T}} = 1. \tag{17}$$

Theorem 3.5. *Let g_0 and g_1 be two reference metrics (of equal area) on a compact surface M, such that R_0 and R_1 have constant sign, and such that $\lambda_1(g_0) > \lambda_1(g_1)$. Then there exist $a_0 > 0$ and $0 < T_0 < \infty$ (that depend on g_0, g_1), such that for all $a < a_0$ and $T > T_0$ we have $P_2(a, T; g_0) < P_2(a, T; g_1)$.*

It was proved by Hersch in [Her] that for $M = S^2$, if we denote by g_0 the round metric on S^2, then $\lambda_1(g_0) > \lambda_1(g_1)$ for any other metric g_1 on S^2 of equal area. This immediately implies the following

Corollary 3.6. *Let g_0 be the round metric on S^2, and let g_1 be any other metric of equal area. Then, there exist $a_0 > 0$ and $T_0 > 0$ (depending on g_1) such that for all $a < a_0$ and $T > T_0$ we have $P_2(a, T; g_0) < P_2(a, T; g_1)$.*

It seems natural to conjecture ([Mor, EI02]) that the round metric on S^2 will be extremal for $P_2(a, T)$ for all T, in the limit $a \to 0$.

For surfaces of genus $\gamma \geq 2$ the situation is different. It follows from [Yau74] and standard results about extremal metrics for λ_1 that

Proposition 3.7. *Let g_0 be a hyperbolic metric on a compact orientable surface M of genus $\gamma \geq 2$. Then g_0* does not *maximize λ_1 in its conformal class.*

A metric that maximizes λ_1 for surfaces of genus 2 is a branched covering of the round 2-sphere, cf. [JLNNP]. Accordingly, we conclude that on surfaces of genus $\gamma \geq 2$, different metrics maximize $P_2(a, T)$ in the limit $a \to 0, T \to 0$ and in the limit $a \to 0, T \to \infty$, unlike the situation on S^2.

The 2-sphere is special in that the curvature perturbation is *isotropic*, so that in particular the variance is constant. In this case a special theorem due to Adler-Taylor gives a precise asymptotics for the excursion probability.

For an integer m let $\mathcal{E}_m$ be the space of spherical harmonics of degree m of dimension $N_m = 2m + 1$ associated to the eigenvalue $E_m = m(m + 1)$, and for every m fix an L^2 orthonormal basis $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$ of $\mathcal{E}_m$.

To treat the spectrum degeneracy it will be convenient to use a slightly different parametrization of the conformal factor than the usual one (2)

$$f(x) = -\sqrt{|\mathcal{S}^2|} \sum_{m \geq 1,\, k} \frac{\sqrt{c_m}}{E_m \sqrt{N_m}} a_{m,k} \eta_{m,k}(x), \tag{18}$$

where $a_{m,k}$ are standard Gaussian i.i.d. and $c_m > 0$ are some (suitably decaying) constants. For extra convenience we will assume in addition that

$$\sum_{m=1}^{\infty} c_m = 1. \tag{19}$$

Lemma 3.8. *Given a sequence c_m satisfying* (19)*, we have $f(x) \in H_r(\mathcal{S}^2)$ a.s. if and only if*

$$\sum_{m=1}^{\infty} m^{2r-4} c_m < \infty.$$

In what follows we will always assume that

$$c_m = O\left(\frac{1}{m^s}\right) \tag{20}$$

for some $s > 0$.

Using the two-dimensional case of [AT08, Thm. 12.4.1], we show that

Theorem 3.9. *Let $s > 7$, and the metric g_1 on $\mathcal{S}^2$ be given by*

$$g_1 = e^{af} g_0$$

where f is given by (18)*. Also, let $c_m \neq 0$ for at least one* odd m*. Then as $a \to 0$, the probability that the curvature is negative somewhere is given by*

$$\begin{aligned} P_2(a) &= \frac{C_2}{a} \exp\left(-\frac{1}{2a^2}\right) + C_1 \Psi\left(\frac{1}{a}\right) + o\left(\exp(-\frac{\alpha}{2a^2})\right) \\ &\sim \frac{C_2}{a} \exp\left(-\frac{1}{2a^2}\right) + \frac{C_1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2a^2}\right), \end{aligned}$$

where $C_1 = 2$, $C_2 = \frac{1}{\sqrt{2\pi}} \sum_{m \geq 1} c_m E_m$ and $\alpha > 1$.

Here $\Psi(u)$ denotes the error function $\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-t^2/2} dt$.

4. L^∞ curvature bounds

On the torus $\mathbf{T}^2$, Gauss-Bonnet theorem implies that the curvature has to change sign for every metric, so that question about the probability of the curvature changing sign is meaningless. We study another question instead, which can be formulated for arbitrary reference metric g_0 on any compact orientable surface: estimate the probability that $||R_1 - R_0||_\infty > u$, where R_1 denotes the curvature of the metric $g_1 = e^{af} g_0$, and $u > 0$ is a real parameter.

Recall that the random field f was defined by (2) (with $\sigma_f^2 = \sup_{x \in M} r_f(x, x)$); $h = \Delta_0 f$ was defined in (3) (with $\sigma_h^2 = \sup_{x \in M} r_h(x, x)$).

Definition 4.1. *Let $w = \Delta_0 f + R_0 f = h + R_0 f$. We denote its covariance function by $r_w(x, y)$, and we define $\sigma_w^2 = \sup_{x \in M} r_w(x, x)$.*

Note that on flat $\mathbf{T}^2$, $h \equiv w$ and $\sigma_h = \sigma_w$. Note also that the random fields f, h and w have constant variance on round S^2; also f and $h = w$ have constant variance on flat $\mathbf{T}^2$.

Theorem 4.2. *Assume that the random metric is chosen so that the random fields f, h, w are a.s. C^0. Let $a \to 0$ and $u \to 0$ so that*

$$\frac{u}{a} \to \infty. \tag{21}$$

Then

$$\log \operatorname{Prob}\{\|R_1 - R_0\|_\infty > u\} \sim -\frac{u^2}{2a^2 \sigma_w^2}. \tag{22}$$

5. Dimension $n > 2$

Let (M, g_0) be a compact orientable n-dimensional Riemannian manifold, $n > 2$. Let $R_0 \in C^0(M)$ be the scalar curvature of g_0; we assume that R_0 has constant sign. Let $g_1 = e^{af} g_0$ with f as in (2) be a conformal change of metric. The key difference between dimension 2 and dimension $n > 2$ in our calculations is the presence of the gradient term. We shall assume that $c_j = O(\lambda_j^{-s}), s > n/2 + 1$. Then $R_1 \in C^0(M)$ a.s. by Proposition 2.5. We let $P_2(a)$ denote the probability that the scalar curvature R_1 of g_1 will change sign.

Below, we shall consider a random field $v = (\Delta_0 f)/R_0 = h/R_0$. As usual, we let $\sigma_v^2 = \sup_{x\in M} r_v(x,x)$. We shall also consider the quantity

$$\sigma_2 = \sup_{x\in M} \frac{\mathbb{E}[|\nabla_0 f(x)|^2]}{R_0(x)}.$$

Proposition 5.1. *Let (M, g_0) be a compact orientable n-dimensional Riemannian manifold, $n > 2$, such that the scalar curvature $R_0 \in C^0(M)$ and $R_0(x) \neq 0, \forall x \in M$. Assume that $c_j = O(\lambda_j^{-s}), s > n/2+1$, so that $h, R_1 \in C^0(M)$.*

1) *Assume that $\forall x \in M. R_0(x) < 0$. Then there exists $\alpha > 0$ so that*

$$P_2(a) = O\left(\exp\left(\frac{\alpha}{a} - \frac{1}{2a^2(n-1)^2\sigma_v^2}\right)\right).$$

2) *Assume that $\forall x \in M. R_0(x) > 0$. Then there exists $\beta > 0$ so that*

$$P_2(a) = O\left(\exp\left(\frac{\beta}{a} - \frac{B}{a^2}\right)\right),$$

where

$$B = \frac{2+\kappa-\sqrt{\kappa^2+4\kappa}}{\sigma_2 n(n-1)(n-2)}.$$

and

$$\kappa = \frac{4\sigma_v^2(n-1)}{\sigma_2 n(n-2)}.$$

6. Q-curvature

The Q-curvature was first studied by Branson and later by Gover, Orsted, Fefferman, Graham, Zworski, Chang, Yang, Djadli, Malchiodi and others. We refer to [BG] for a detailed survey.

We start by discussing conformally covariant operators, first considered by Paneitz in dimension 4: the *Paneitz operator* is $P_4 = \Delta_g^2 + \delta[(2/3)R_g g - 2\mathrm{Ric}_g]d$. Below we summarize the relevant results from [BG]. Let M be a compact orientable manifold of dimension $n \geq 3$. Let m be even, and $m \notin \{n+2, n+4, \ldots\} \Leftrightarrow m - n \notin 2\mathbf{Z}^+$. Then there exists on M a conformally covariant elliptic operator P_m of oreder $2m$ (GJMS operators of Graham-Jenne-Mason-Sparling, cf. [GJMS]). We shall restrict ourselves to even n, and to $m = n$. We shall denote the corresponding operator P_n simply by P.

It satisfies the following properties: $P = \Delta^{n/2} +$*lower order terms*. P is formally self-adjoint (Graham-Zworski [GZ], Fefferman-Graham [FG]). Under a conformal transformation of metric $\tilde{g} = e^{2\omega}g$, the operator P changes as follows: $\widetilde{P} = e^{-n\omega}P$.

We next discuss Q-curvature and its key properties. In dimension 4, it was defined by Paneitz as follows:

$$Q_g = -\frac{1}{12}\left(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2\right). \tag{23}$$

In higher dimensions, Q-curvature is a local scalar invariant associated to the operator P. It was introduced by T. Branson in [Br]; alternative constructions were provided in [FG, FH] using the *ambient metric* construction.

Q-curvature is equal to $1/(2(n-1))\Delta^{n/2}R$ modulo nonlinear terms in curvature. Under a conformal transformation of variables $\tilde{g} = e^{2\omega}g$ on M^n, the Q-curvature transforms as follows [BG, (4)]:

$$P\omega + Q = \tilde{Q}e^{n\omega}. \tag{24}$$

Integral of the Q-curvature is conformally invariant.

A natural problem is the existence of metrics with constant Q-curvature in a given conformal class. It was established by Chang and Yang, and Djadli and Malchiodi in dimension 4, and by Ndiaye in arbitrary even dimension $n > 4$ [CY, DM, N], assuming that certain "generic" assumptions are satisfied.

To generalize our results for scalar curvature to Q-curvature, consider a manifold M with a "reference" metric g_0 such that Q-curvature $Q_0(x)$ has constant sign, and a conformal perturbation $g_1 = e^{2af}g_0$; expand f in a series of eigenfunctions of P, and use formula (24) to study the induced curvature Q_1.

In the Fourier expansions considered below, we shall restrict our summation to *nonzero* eigenvalues of P. Let P have k negative eigenvalues (counted with multiplicity); denote the corresponding spectrum by $P\psi_j = -\mu_j\psi_j$, for $1 \le j \le k$, where $0 > -\mu_1 \ge -\mu_2 \ge \ldots \ge -\mu_k$. The other nonzero eigenvalues are positive, and the corresponding spectrum is denoted by $P\phi_j = \lambda_j\phi_j$, for $j \ge 1$, where $0 < \lambda_1 \le \lambda_2 \le \ldots$.

Consider the transformation of metric $g_1 = e^{2af}g_0$, where we let

$$f = \sum_{i=1}^{k} b_i\psi_i + \sum_{j=1}^{\infty} a_j\phi_j, \tag{25}$$

and where $b_i \sim \mathcal{N}(0, t_i^2)$ and $a_j \sim \mathcal{N}(0, c_j^2)$.

We define $h := -Pf$, and substituting into (24), we find that

$$Q_1 e^{naf} = Q_0 - ah = Q_0 + a\left(\sum_{j=1}^{\infty} \tilde{a}_j\phi_j - \sum_{i=1}^{k} \tilde{b}_i\psi_i\right), \tag{26}$$

where $\tilde{a}_j \sim \mathcal{N}(0, \lambda_j^2 c_j^2)$ and $\tilde{b}_i \sim \mathcal{N}(0, t_i^2\mu_i^2)$.

It is easy to see that the regularity of the random field in (25) is determined by the principal symbol $\Delta^{n/2}$ of the GJMS operator P. The following Proposition is then a straightforward extension of Proposition 2.1:

Proposition 6.1. *Let f be defined as in* (25)*. If $c_j = O(\lambda_j^{-t})$ and $t > 1 + \frac{k}{n}$, then $f \in C^k$. Similarly, if $c_j = O(\lambda_j^{-t})$ and $t > 2 + \frac{k}{n}$ then $Pf \in C^k$.*

Let f be as in equation (25) and such that Pf is a.s. C^0. We remark that it follows from Proposition 6.1 that this happens if $c_j = O(\lambda_j^{-t})$ where $t > 2$.

Let $g_1 = e^{2af}g_0$. Denote the Q-curvature of g_1 by Q_1; then it follows from (24) that

$$\operatorname{sgn}(Q_1) = \operatorname{sgn}(Q_0)\operatorname{sgn}(1 - ah/Q_0) \tag{27}$$

It follows that Q_1 changes sign iff $\sup_{x\in M} h(x)/Q_0(x) > 1/a$.

We denote by $v(x)$ the random field $h(x)/Q_0(x)$. The covariance function of $v(x)$ is equal to

$$r_v(x,y) = \frac{1}{Q_0(x)Q_0(y)}\left(\sum_{i=1}^{k} t_i^2\mu_i^2\psi_i(x)\psi_i(y) + \sum_{j=1}^{\infty}\lambda_j^2 c_j^2\phi_j(x)\phi_j(y)\right). \tag{28}$$

We let

$$\sigma_v^2 := \sup_{x\in M} r_v(x,x). \tag{29}$$

As for the scalar curvature, we make the following

Definition 6.2. *Denote by $P_2(a)$ the probability that the Q-curvature Q_1 of the metric $g_1 = g_1(a)$ changes sign.*

Theorem 6.3. *Assume that $Q_0 \in C^0(M)$ and that $c_j = O(\lambda_j^{-t}), t > 2$. Then there exist constants $C_1 > 0$ and C_2 such that the probability $P_2(a)$ satisfies*

$$(C_1 a)e^{-1/(2a^2\sigma_v^2)} \le P_2(a) \le e^{C_2/a - 1/(2a^2\sigma_v^2)},$$

as $a \to 0$. In particular

$$\lim_{a\to 0} a^2 \ln P_2(a) = \frac{-1}{2\sigma_v^2}.$$

Next, we extend the results in section 4 to Q-curvature.

Theorem 6.4. *Let (M, g_0) be an n-dimensional compact orientable Riemannian manifold, with n even. Assume that $Q_0 \in C^0(M)$, and that $c_j = O(\lambda_j^{-t})$, $t > 2$, so that by Proposition 6.1 the random fields f and h are a.s. C^2. Let $w := h - nQ_0 f$, denote by $r_w(x,y)$ its covariance function and set*

$$\sigma_w^2 := \sup_{x\in M} r_w(x,x).$$

Let $a \to 0$ and $u \to 0$ so that

$$\frac{u}{a} \to \infty.$$

Then

$$\log \mathrm{Prob}\{\|Q_1 - Q_0\|_\infty > u\} \sim -\frac{u^2}{2a^2\sigma_w^2}.$$

7. Conclusion

There are numerous questions that were not addressed in the present paper. We concentrated on the study of *local* geometry of spaces of positively- or negatively-curved metrics (see Remark 2.4), but it seems extremely interesting to study *global* geometry of these spaces, [GL, Kat, Lo, Ros06, Sch87, SY79-1, SY82, SY87].

Another interesting question that seems tractable concerns the study of the *nodal set* of R_1 i.e. its zero set. That set, like the sign of R_1, only depends on the quantity $R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)|\nabla_0 f|^2/4$ (or $R_0 - a\Delta_0 f$ in dimension two). It also seems interesting to study other characteristics of the curvature (whether it changes sign or not), such as its L^p norms, the structure of its nodal domains (if it changes sign), and of its sub- and super-level sets.

Also, it seems quite interesting to study related questions for Ricci and sectional curvatures in dimension $n \ge 3$. Another important question concerns an appropriate definition of measures on the space of Riemannian metrics not restricted to a single conformal class. A very important question concerns the study of metrics of lower regularity than in the present paper, appearing e.g. in 2-dimensional Liouville quantum gravity, cf. [DS].

In addition, it seems very interesting to study various questions about random metrics that are influenced by curvature, such as various geometric invariants (girth, diameter, isoperimetric constants etc); spectral invariants (small eigenvalues of Δ, determinants of Laplacians, estimates for the heat kernel, statistical properties of eigenvalues and of the spectral function, etc); as well as various questions related to the geodesic flow or the frame flow on M, such as existence of conjugate points, ergodicity, Lyapunov exponents and entropy, etc.

We plan to address these and other questions in subsequent papers.

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Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Str. West, Montréal QC H3A 2K6, Canada.

E-mail address: canzani@math.mcgill.ca

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Str. West, Montréal QC H3A 2K6, Canada.

E-mail address: jakobson@math.mcgill.ca

Centre de recherches mathématiques (CRM), Université de Montréal C.P. 6128, succ. centre-ville Montréal, Québec H3C 3J7, Canada
currently at
Institutionen för Matematik, Kungliga Tekniska högskolan (KTH), Lindstedtsvägen 25, 10044 Stockholm, Sweden

E-mail address: wigman@kth.se