SCALAR CURVATURE AND Q-CURVATURE OF RANDOM METRICS.

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ABSTRACT. We define a family of probability measures on the set of Riemannian metrics lying in a fixed conformal class, induced by Gaussian probability measures on the (logarithms of) conformal factors. We control the smoothness of the resulting metric by adjusting the decay rate of the variance of the random Fourier coefficients of the conformal factor. On a compact surface, we evaluate the probability of the set of metrics with non-vanishing Gauss curvature, lying in a fixed conformal class. On higher-dimensional manifolds, we estimate the probability of the set of metrics with non-vanishing scalar curvature (or Q-curvature), lying in a fixed conformal class.

1. INTRODUCTION

The geometry of the space of metrics over a compact Riemannian manifold is naturally related to the behavior of certain functionals of the metric parameter such as volume, scalar curvature and Q-curvature. The aim of this paper is to describe how these functionals behave when the metrics are lying in a fixed conformal class, from a probabilistic point of view.

Our main technical tool is the construction of Gaussian measures on the space of metrics in a given conformal class. Such measures have long been considered in 2-dimensional conformal field theory and quantum gravity ([DS, KPZ, Pol]), random surface models and other fields (see e.g. [Morg]). Let M be a compact Riemannian manifold and let g_0 be a reference Riemanian metric over M. We define Gaussian measures on the conformal class $[g_0]$; this allows us to study the probability of the scalar curvature and Q-curvature (as functionals of the metric parameter $g \in [g_0]$) to change the sign under a (random) conformal perturbation of g_0. Such probabilities are expressed in terms of geometric invariants of (M, g_0). Our techniques are inspired by [AT03, ATT05, AT08, Bl].

Let (M, g) be an n-dimensional compact manifold, with $n \geq 2$. The *Riemann curvature tensor* is defined by $R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, where ∇ denotes the Levi-Civita connection. In local coordinates, $R_{ijkl} := \langle R(\partial_i, \partial_j)\partial_k, \partial_l \rangle$. The *Ricci curvature* of g is given in local coordinates by the formula $R_{jk} = g^{il} R_{ijkl}$.

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The *scalar curvature* of g is then defined by the formula

$$R = g^{ik} R_{ik}.$$

Geometrically, $R(x_0)$ determines the asymptotic deviation of the volume of a (small) radius r ball in M (centered at x_0) from the volume of the Euclidean ball of the same radius:

$$\mathrm{vol}(B_M(x_0, r)) = \mathrm{vol}(B_{\mathbb{R}^n}(r)) \left(1 - \frac{R(x_0) r^2}{6(n+2)} + O(r^4)\right) \quad \text{as } r \to 0.$$

Our paper addresses two main questions:

Question 1.1. *Assuming that the scalar curvature R_0 of the reference metric g_0 doesn't vanish, what is the probability that the scalar curvature of the perturbed metric changes its sign?*

In each conformal class, there exists a *Yamabe metric* with constant scalar curvature $R_0(x) \equiv R_0$ [Yam, Au76, Sch84, Tr], and the sign of R_0 is uniquely determined. Therefore, Question 1.1 can be posed in each conformal class where $R_0 \neq 0$. Question 1.1 is addressed in Section 3 on surfaces (of genus different than one) equipped with metrics that are C^0 with probability one. In Section 4, the probability estimates are significantly improved on the 2-sphere endowed with metrics that are C^2 with probability one. In Section 6, Question 1.1 is addressed for higher dimensional manifolds. In Section 3.1 several comparison theorems are proved for random real-analytic metrics.

It was shown in [CY, DM, N] that in every conformal class satisfying certain *generic* conditions, there exists a metric g_0 with constant Q-curvature. Hence, we may also formulate Question 1.1 for Q-curvature, in each conformal class where $Q_0 \neq 0$. We address this problem in Section 7.6 on compact manifolds of dimension higher than two.

Question 1.2. *What is the probability that the value of the curvature of the perturbed metric differs from the value of the reference one by a constant greater than u (where u is a positive real parameter, subject to some restrictions)?*

We address Question 1.2 both for scalar curvature on surfaces (Section 5), and for Q-curvature on manifolds of dimension higher than two (Section 7.7). In Appendix A we provide a short survey on existence of metrics of positive and negative scalar curvature in conformal classes. Finally, in Appendix B we verify the assumptions needed to apply the results of R. Adler and J. Taylor to answer Question 1.1 on the round metric on the 2-sphere.

1.1. **Motivation: geometric analysis on manifolds of metrics.** The space of all Riemannian metrics on a manifold M can be endowed with a structure of a Riemannian manifold. The differential structure of manifolds of Riemannian metrics, and the corresponding action of the group of diffeomorphisms were considered by D. Ebin in his thesis (cf. [Eb]). The differential geometry of manifolds of metrics was developed in [FrGr, GM]; we refer to [Sm] for a recent summary, and to [Cl] for the discussion of the geometry of their completions.

To further develop geometric analysis on manifolds of metrics, one should define *integration* on those manifolds. One way of doing this is to construct measures on relevant geometric classes of such manifolds, for example on the manifold of all metrics lying in a given conformal class. A reasonable condition to impose on such measures is the invariance under the action of the group of diffeomorphisms. In the present paper we construct a natural class of measures on the manifold of Riemannian metrics lying in a given conformal class, developing the ideas in [Bl]. All the measures considered in this paper are defined using elliptic differential operators, hence diffeomorphism invariance holds automatically.

Our results describe some very natural geometric properties of the spaces of positively or negatively-curved metrics. Specifically, we compute their relative volume in the space of all metrics (with respect to our measures). The measures that we define are localized at the reference metric g_0. Accordingly, we interpret our results as giving us information about the *local* geometry of the manifold of metrics close to g_0.

Studying spaces of positively and negatively-curved metrics in dimension two is related to other interesting questions in geometry and erogdic theory; we address one such question below. It is known that geodesic flows behave very differently on positively and negatively-curved surface. In particular, geodesic flows in non-positive curvature have no conjugate points. We would like to formulate the following conjecture:

Conjecture 1.3. *The probability that the geodesic flow for a random metric g (on a surface Σ_γ of genus $\gamma \geq 2$) has no conjugate points, is strictly* greater *than the probability that g has nonpositive Gauss curvature.*

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2. Random metrics in a conformal class

Consider a conformal class of metrics on a compact Riemannian manifold M of the form

$$g_1 = e^{af} g_0, \tag{1}$$

where g_0 is a reference Riemannian metric on M, a is a constant, and $f \in C^2(M)$. Let $\{\phi_j\}_{j=1}^{\infty}$ denote an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of $-\Delta_0$ (the positive definite Laplace operator associated to g_0) with corresponding eigenvalues λ_j. We set $\lambda_0 = 0$, and without loss of generality assume $\phi_0 = 1$. Define the random conformal multiple f by

$$f(x) = -\sum_{j=1}^{\infty} a_j c_j \phi_j(x), \qquad x \in M, \tag{2}$$

where the c_j's are positive real numbers, and the a_j's are independent, identically distributed, standard Gaussian random variables ($a_j \sim \mathcal{N}(0,1)$).

Assume further that $c_j = F(\lambda_j)$, where $F(t)$ is an eventually monotone decreasing function of t, $F(t) \to 0$ as $t \to \infty$. For example, we may choose $c_j = e^{-\tau\lambda_j}$ or $c_j = \lambda_j^{-s}$ with some τ, $s > 0$. Equivalently we equip the space of functions (distributions) $L^2(M)$ with the probability measure $\nu = \nu_{\{c_k\}_{k=1}^{\infty}}$ generated by the densities on the finite cylinder sets

$$d\nu_{(k_1,k_2,\dots k_l)}(f) = \frac{1}{\prod\limits_{j=1}^{l}(2\pi c_{k_j}^2)^{1/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{l}\frac{f_{k_j}^2}{c_{k_j}^2}\right) df_{k_1}\dots df_{k_l}, \tag{3}$$

where $f_k = \langle f, \phi_k\rangle_{L^2(M)}$ are the Fourier coefficients.

The random field f is a centered Gaussian field with *covariance function*

$$r_f(x,y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2\phi_j(x)\phi_j(y), \tag{4}$$

$x, y \in M$. In particular for every $x \in M$, $f(x)$ is mean zero Gaussian of variance

$$\sigma^2(x) = r_f(x,x) = \sum_{j=1}^{\infty} c_j^2\phi_j(x)^2.$$

For special manifolds such as the 2-dimensional sphere $\mathcal{S}^2 \subseteq \mathbb{R}^3$ it is convenient to parameterize f in a different fashion (see (22)).

The central object of the present study is the scalar curvature resulting from the conformal change of the metric (1). For this purpose it is convenient to introduce the random centered Gaussian field

$$h(x) := \Delta_0 f(x) = \sum_{j=1}^{\infty} a_j c_j \lambda_j \phi_j(x) \tag{5}$$

with covariance function

$$r_h(x,y) = \sum_{j=1}^{\infty} c_j^2\lambda_j^2\phi_j(x)\phi_j(y), \tag{6}$$

$x, y \in M$. In principle, one may derive any property of h in terms of the function r_h and its derivatives by the Kolmogorov theorem (see [CL], Chapter 3.3).

2.1. **Smoothness.** Given a Riemannian compact manifold (M, g_0) and $r \in \mathbb{R}$, the Sovolev Space $H_r(M)$ is the completion of $C^\infty(M)$ relative to the inner product

$$\langle u, v\rangle_r := \langle(-\Delta_{g_0} + I)^r u, v\rangle_{g_0}, \qquad u, v \in C^\infty(M).$$

We refer to [Au98, Ch. 2] for basic facts about Sobolev spaces.

The smoothness of the Gaussian random field (2) is given by the following proposition [Bl, Proposition 1], where, for brevity we use the shortcut *a.s.* (almost surely) for an event that occurs with probability one.

Proposition 2.1. *If $\sum_{j=1}^{\infty}(\lambda_j+1)^r c_j^2 < \infty$, then $f \in H^r(M)$ a.s. Equivalently, the measure ν defined as* (3) *is supported on $H^r(M)$, i.e. $\nu(H^r) = 1$.*

Choosing $c_j = F(\lambda_j) = \lambda_j^{-s}$ translates the hypothesis to $\sum_{j\geq 1} \lambda_j^{r-2s} < \infty$. In dimension n, it follows from Weyl's law ([Ch]) that $\lambda_j \asymp j^{2/n}$ as $j \to \infty$; we find that

$$\textit{If } s > \frac{2r+n}{4}, \quad \textit{then } f \in H^r(M) \textit{ a.s.}$$

By the Sobolev embedding theorem, $H^r \subset C^k$ for $k < r - n/2$. Substituting into the formula above, we find that

$$\textit{If } c_j = O(\lambda_j^{-s}), s > \frac{n+k}{2}, \quad \textit{then } f \in C^k \textit{ a.s.} \tag{7}$$

The cases $k = 0$ and $k = 2$ are the ones of interest for our purposes. Accordingly, we formulate the following corollary.

Corollary 2.2. *If $c_j = O(\lambda_j^{-s}), s > n/2$, then $f \in C^0$ a.s; if $c_j = O(\lambda_j^{-s}), s > n/2 + 1$, then $f \in C^2$ a.s. Similarly, if $c_j = O(\lambda_j^{-s}), s > n/2 + 1$, then $\Delta_0 f \in C^0$ a.s; if $c_j = O(\lambda_j^{-s}), s > n/2 + 2$, then $\Delta_0 f \in C^2$ a.s.*

2.2. **Volume.** Consider the volume of the random metric in (1). The volume element dV_1 corresponding to g_1 is given by

$$dV_1 = e^{naf/2} dV_0, \tag{8}$$

where dV_0 denotes the volume element corresponding to g_0. In the following proposition we prove that a small metric perturbation ($a \to 0$) corresponds to a small perturbation of the manifold volume $V_1 = \mathrm{vol}(M, g_1)$, at least at the expectation level.

Proposition 2.3. *Under the notation as above,*

$$\lim_{a\to 0} \mathbb{E}[V_1(a)] = V_0,$$

where V_0 denotes the volume of (M, g_0).

Proof. Recall that $f(x)$ defined by (2) is a mean zero Gaussian with variance $\sigma(x)^2 = r_f(x,x)$. One may compute explicitly

$$\mathbb{E}[e^{naf(x)/2}] = e^{\frac{1}{8}n^2a^2 r_f(x,x)},$$

so that (8) implies that

$$\mathbb{E}[dV_1(x)] = e^{\frac{1}{8}n^2a^2 r_f(x,x)} dV_0(x).$$

Hence, using Fubini we obtain

$$\mathbb{E}[V_1(a)] = \int_M \mathbb{E}[dV_1] = \int_M e^{\frac{1}{8}n^2a^2 r_f(x,x)} dV_0.$$

Since $r_f(x,x)$ is continuous, as $a \to 0$, the latter converge to V_0 by the dominated convergence theorem. $\square$

Remark 2.4. *The smoothness of the metric $g_1 = e^{af} g_0$ is almost surely determined by the coefficients c_j. The parameter a can be regarded as the radius of a sphere (in an appropriate space of Riemannian metrics on M) centered at g_0. Most of the results in this paper hold in the limit $a \to 0$; thus, we are studying* local *geometry of the space of Riemannian metrics on M.*

2.3. **Scalar curvature in a conformal class.** It is well-known that the scalar curvature R_1 of the metric $g_1 = e^{af} g_0$ is related to the scalar curvature R_0 of the metric g_0 by the following formula ([Au98, §5.2, p. 146])

$$R_1 = e^{-af}\left[R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)4^{-1}|\nabla_0 f|^2\right], \tag{9}$$

where Δ_0 is the (negative definite) Laplacian for g_0, and ∇_0 is the gradient corresponding to g_0. For $n = 2$, the last term in equation (9) vanishes and we get

$$R_1 = e^{-af}[R_0 - a\Delta_0 f]. \tag{10}$$

The smoothness of the scalar curvature for the metric g_1 is then determined by the random field $a(n-1)\Delta_0 f + a^2(n-1)(n-2)4^{-1}|\nabla_0 f|^2$. The following proposition follows easily from (9) and Corollary 2.2.

Proposition 2.5. *If $R_0 \in C^0$ and $c_j = O(\lambda_j^{-s}), s > n/2+1$ then $R_1 \in C^0$ a.s. If $R_0 \in C^2$ and $c_j = O(\lambda_j^{-s})$ with $s > n/2+2$, then $R_1 \in C^2$ a.s.*

Consider the *sign* of the scalar curvature R_1 of the new metric and recall that h is given by (5).

Remark 2.6. *The quantity e^{-af} is always positive, hence the sign of R_1 satisfies*

$$\operatorname{sgn}(R_1) = \operatorname{sgn}[R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)4^{-1}|\nabla_0 f|^2].$$

In particular for $n = 2$, assuming that R_0 has constant sign, we find that

$$\operatorname{sgn}(R_1) = \operatorname{sgn}(R_0 - a\Delta_0 f) = \operatorname{sgn}(R_0 - ah) = \operatorname{sgn}(R_0)\cdot \operatorname{sgn}(1 - ah/R_0).$$

3. Probability that R_1 changes sign on surfaces

In this section, we shall use Borell-TIS inequality given in Theorem 3.1 to estimate the probability that the curvature of a random metric on a compact surface M of genus different than 1 changes sign. We remark that by Gauss-Bonnet Theorem applied to the 2-torus $(\mathbb{T}^2, g)$ we have $\int_M R_g = 0$, so the curvature has to change sign on $\mathbb{T}^2$.

We make the following conventions: given a random field $F : T \to \mathbb{R}$ on a parameter set T we define the random variable

$$\|F\|_T := \sup_{t\in T} F(t).$$

Note that there is no absolute value in the definition of $\|\cdot\|_T$, so that it is by no means a norm; this is in contrast to $\|\cdot\|_\infty$, which denotes the supremum norm. Let Ψ be the error function

$$\Psi(u) = \frac{1}{\sqrt{2\pi}}\int_u^\infty e^{-t^2/2}dt.$$

Denote by $M = M_\gamma$ a compact surface of genus $\gamma \neq 1$. Choose a reference metric g_0 so that R_0 has constant sign (positive if M is the sphere, and negative if M has genus greater than 2). Define the random metric on M_γ by $g_1(a) = e^{af}g_0$ and continue to define f as the gaussian field described in (2). In this section we shall estimate the probability $P_2(a)$ defined by

$$P_2(a) := \operatorname{Prob}\{\exists\, x \in M : \operatorname{sgn} R_1(x) \neq \operatorname{sgn}(R_0)\}, \tag{11}$$

i.e. that the curvature R_1 of the random metric $g_1(a)$ *changes sign* somewhere on M. The probability of the complementary event $P_1(a) = 1 - P_2(a)$ is clearly

$$P_1(a) := \text{Prob}\{\forall x \in M : \text{sgn}(R_1(x)) = \text{sgn}(R_0)\},$$

i.e. the curvature of the random metric $g_1(a)$ *does not* change sign.

Recall that in dimension two $\text{sgn}(R_1) = \text{sgn}(R_0)\,\text{sgn}(1 - ah/R_0)$, where $h = \Delta_0 f$ was defined earlier in (5) (cf. Remark 2.6). Let v denote the random field

$$v(x) = h(x)/R_0(x) \tag{12}$$

with the variance

$$r_v(x,x) := \mathbb{E}[v(x)^2] = r_h(x,x)/[R_0(x)]^2, \tag{13}$$

where $r_h(x,x)$ is as in (6). Let

$$\sigma_v^2 = \sup_{x\in M} r_v(x,x) = \sup_{x\in M} r_h(x,x)/[R_0(x)]^2, \tag{14}$$

and recall that $||v||_M := \sup_{x\in M} v(x)$. It follows from Remark 2.6 that

$$P_2(a) = \text{Prob}\left\{||v||_M > 1/a\right\}. \tag{15}$$

We study $P_2(a)$ in the limit $a \to 0$. Geometrically, that means that $g_1(a) \to g_0$, so $P_2(a)$ should go to zero as $a \to 0$. Below, we shall estimate the *rate*. To do that, we use a strong version of the Borell-TIS inequality ([Bor, TIS]) formulated below. The proof of the following result can be found in [Bor, TIS], or in [AT08, p. 51]

Theorem 3.1 (Borel-TIS). *Let f be a centered Gaussian process, a.s. bounded on M, and $\sigma_M^2 := \sup_{x\in M}\mathbb{E}[f(x)^2]$. Then $\mathbb{E}\{||f||_M\} < \infty$, and there exists a constant α depending only on $\mathbb{E}\{||f||_M\}$ so that for $u > E\{||f||_M\}$ we have*

$$\text{Prob}\{||f||_M > u\} \le e^{\alpha u - u^2/(2\sigma_M^2)}.$$

From now on we assume that $R_0 \in C^0(M)$, and that $c_j = O(\lambda_j^{-s}), s > 2$. For M compact, Proposition 2.5 implies that h and R_1 are a.s. C^0 and hence bounded. Since $h(x) = \sum_{j=1}^{\infty} \lambda_j c_j a_j \phi_j(x)$ for $x \in M$, it follows that the variance of $v = h/R_0$ is equal to

$$r_v(x,x) = \frac{1}{R_0(x)^2}\sum_{j=1}^{\infty} c_j^2 \lambda_j^2 \phi_j(x)^2.$$

Continue to write $\sigma_v^2 = \sup_{x\in M} r_v(x,x)$ and assume that the supremum is attained at $x = x_0$. We shall use (15) to estimate $P_2(a)$ from above and below. To get a lower bound for $\text{Prob}\{\|v\|_M\} > 1/a)$, choose $x = x_0$. Clearly,

$$\text{Prob}\{\|v\|_M\} > 1/a\} \ge \text{Prob}\{v(x_0) > 1/a\}.$$

The random variable $v(x_0)$ is Gaussian with mean 0 and variance σ_v^2. Accordingly,

$$\text{Prob}\{v(x_0) > 1/a\} = \Psi\left(\frac{1}{a\sigma_v}\right). \tag{16}$$

An upper bound is obtained by a straightforward application of Theorem 3.1 on our problem.

Proposition 3.2. *There exist a constant C so that*

$$\text{Prob}\{\|v\|_M) > 1/a\} \le e^{C/a - 1/(2a^2\sigma_v^2)}.$$

Combining Proposition 3.2 with (16) we obtain the following theorem.

Theorem 3.3. *Assume that $R_0 \in C^0(M)$ and that $c_j = O(\lambda_j^{-s}), s > 2$. Then there exist constants $C_1 > 0$ and C_2 such that the probability $P_2(a)$ satisfies*

$$(C_1 a)e^{-1/(2a^2\sigma_v^2)} \le P_2(a) \le e^{C_2/a - 1/(2a^2\sigma_v^2)},$$

as $a \to 0$. In particular

$$\lim_{a\to 0} a^2 \ln P_2(a) = -\frac{1}{2\sigma_v^2}.$$

Remark 3.4. *In Section 4, we shall greatly improve the result of Theorem 3.3 and obtain much more precise estimates of $P_2(a)$ for $M = \mathcal{S}^2$ (see Theorem 4.3 below) using the results of Adler and Taylor described in the next section. To apply Borell-TIS inequality, h is required to be a.s. C^0. To apply the results of Adler-Taylor, h needs to be a.s. C^2. We hope to improve the estimates in Theorem 3.3 in a forthcoming paper.*

3.1. **Random real-analytic metrics and comparison results.** In this section, we let M be a compact orientable surface, of genus different than 1. We shall consider random *real-analytic* conformal deformations; this corresponds to the case when the coefficients c_j in (2) decay *exponentially.* We shall use standard estimates for the heat kernel to estimate the probabilities that appear in the statement of Theorem 3.3.

The kernel of the heat operator $e^{-T\Delta_{g_0}}$ at time $T > 0$ is given by the expression

$$e(x, y, T) = \sum_{j=0}^{\infty} e^{-\lambda_j T}\phi_j(x)\phi_j(y) \qquad x, y \in M,$$

and it is well-known that $e(x, y, T)$ is smooth in $x, y \in M$. Also, $e^*(x, y, T) := e(x, y, T) - 1$ decays exponentially in T, [Ch, §6.4, p. 154], [Gilk].

Fix a real parameter $T > 0$ and choose the coefficients c_j in (2) to be equal to

$$c_j = e^{-\lambda_j T/2}/\lambda_j. \tag{17}$$

Then it follows from (6) that

$$r_h(x, x) = e^*(x, x, T) = \sum_{j:\lambda_j>0} e^{-\lambda_j T}\phi_j(x)^2.$$

3.2. **Comparison Theorem:** $T \to 0^+$**.** The following asymptotic expansion for the heat kernel is standard [Gilk]:

$$e(x, x, T) \sim_{T\to 0^+} \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x)T^{j-n/2};$$

here a_j is the j-th *heat invariant*, and in particular,

$$a_0(x) = 1, \; a_1(x) = R(x)/6.$$

It then follows that,

$$\lim_{T\to 0^+} e(x, x, T)T^{n/2} = \frac{1}{(4\pi)^{n/2}}.$$

Combining with (14), we obtain the following proposition.

Proposition 3.5. *Assume that the coefficients c_j are chosen as in* (17)*. Then as $T \to 0^+$, σ_v^2 is asymptotic to*

$$\frac{1}{(4\pi T)^{n/2} \inf_{x\in M}(R_0(x))^2}.$$

That is, as $T \to 0^+$, the probability $P_2(a)$ is determined by the value of

$$\inf_{x\in M}(R_0(x))^2.$$

Next, we apply Proposition 3.5 to prove a comparison theorem.

Theorem 3.6. *Let g_0 and g_1 be two distinct reference metrics on M, normalized to have equal volume, such that R_0 and R_1 have constant sign, $R_0 \equiv const$ and $R_1 \not\equiv const$. Then there exist $a_0 > 0$ and $T_0 > 0$ (that depend on g_0, g_1) such that for $0 < a < a_0$ and $0 < T < T_0$,*

$$P_2(a, T, g_1) > P_2(a, T, g_0).$$

Proof: It follows from Gauss-Bonnet's theorem that

$$\int_M R_0 dV_0 = \int_M R_1 dV_1.$$

Since $\mathrm{vol}(M, g_0) = \mathrm{vol}(M, g_1)$, and by assumption $R_0 \equiv const$ and $R_1 \not\equiv const$, it follows that

$$b_0 := \min_{x\in M}(R_0(x))^2 = R_0^2 > \inf_{x\in M}(R_1(x))^2 := b_1.$$

Accordingly, as $T \to 0^+$, we have

$$\frac{\sigma_v^2(g_1, T)}{\sigma_v^2(g_0, T)} \asymp \frac{b_0}{b_1} > 1.$$

Theorem 3.6 then follows from Theorem 3.3.

$\square$

It follows that in every conformal class, $P_2(a, T, g_0)$ is minimized in the limit $a \to 0, T \to 0^+$ for the metric g_0 of constant curvature.

3.3. **Comparison Theorem: $T \to \infty$.** Let M be a compact surface, where the scalar curvature R_0 of the reference metric g_0 has constant sign. Let $\lambda_1 = \lambda_1(g_0)$ denote the smallest nonzero eigenvalue of Δ_0. Denote by $m = m(\lambda_1)$ the multiplicity of λ_1, and let

$$F := \sup_{x\in M} \frac{\sum_{j=1}^m \phi_j(x)^2}{R_0(x)^2}. \tag{18}$$

The number F is finite by compactness and the assumption that R_0 has constant sign on M.

Proposition 3.7. *Let the coefficients c_j be as in* (17)*. Denote by $\sigma_v^2(T)$ the corresponding supremum of the variance of v. Then*

$$\lim_{T\to\infty} \frac{\sigma_v^2(T)}{F e^{-\lambda_1 T}} = 1. \tag{19}$$

Proof of Proposition 3.7: Recall that it follows from (6) and (13) that

$$r_v(x,x) = \frac{e^*(x,x,T)}{R_0(x)^2}.$$

Write $e^*(x,x,T) = e_1(x,T) + e_2(x,T)$ for

$$e_1(x,T) = e^{-\lambda_1 T}\sum_{j=1}^{m} \phi_j(x)^2, \qquad \textit{and} \qquad e_2(x,T) = \sum_{j=m+1}^{\infty} e^{-\lambda_j T}\phi_j(x)^2.$$

Clearly, as $T\to\infty$, we have

$$\lim_{T\to\infty} e^{\lambda_1 T}\sup_{x\in M}\frac{e_1(x,T)}{R_0(x)^2} = F,$$

where F was defined in (18). It suffices to show that as $T\to\infty$,

$$\frac{e_2(x,T)}{R_0(x)^2} = o\left(e^{-\lambda_1 T}\right). \tag{20}$$

Note that by compactness, there exists $C_1 > 0$ such that $(1/C_1) \le R_0^2(x) \le C_1$ for all $x \in M$. Accordingly, it suffices to establish (20) for

$$\sup_{x\in M} e_2(x,T).$$

Note that $\lambda_m = \lambda_1$ by the definition of m. Let $c := \lambda_1/\lambda_{m+1} < 1$. We remark that

$$\begin{aligned} e_2(x,T) &= e^{-\lambda_1 T}\sum_{j=m+1}^{\infty} e^{-(\lambda_j-\lambda_1)T}\phi_j(x)^2 \\ &\le\; e^{-\lambda_1 T}\sum_{j=m+1}^{\infty} e^{-\lambda_j T(1-c)}\phi_j(x)^2 \\ &\le e^{-\lambda_1 T}\sup_{x\in M} e^*(x,x,T(1-c)), \end{aligned} \tag{21}$$

and $e^*(x,x,T(1-c)) \to 0$ exponentially fast as $T\to\infty$, and uniformly in x. This establishes (20) for $\sup_{x\in M} e_2(x,T)$ and finishes the proof of Proposition 3.7.

□

Theorem 3.8. *Let g_0 and g_1 be two reference metrics (of equal area) on a compact surface M, such that R_0 and R_1 have constant sign, and such that $\lambda_1(g_0) > \lambda_1(g_1)$. Then there exist $a_0 > 0$ and $0 < T_0 < \infty$ (that depend on g_0, g_1), such that for all $a < a_0$ and $T > T_0$,*

$$P_2(a,T;g_0) < P_2(a,T;g_1).$$

Proof of Theorem 3.8: By Proposition 3.7, we find that for $T > T_1 = T_1(g_0,g_1)$ there exists $C > 0$ such that

$$\frac{1}{C} \le \frac{\sigma_v^2(T,g_1)e^{\lambda_1(g_1)T}}{\sigma_v^2(T,g_2)e^{\lambda_1(g_2)T}} \le C.$$

Accordingly, if we choose T_2 so that $e^{(\lambda_1(g_1)-\lambda_1(g_2))T_2} > C$, and consider $T > \max\{T_1,T_2\}$, we find that Theorem 3.8 follows from the formula above and Theorem 3.3.

□

It was proved by Hersch in [Her] that for the 2-sphere $(\mathcal{S}^2, g_0)$ endowed with the round metric, one has $\lambda_1(g_0) > \lambda_1(g_1)$ for any other metric g_1 on $\mathcal{S}^2$ of equal area. This immediately implies the following corollary.

Corollary 3.9. *Let g_0 be the round metric on $\mathcal{S}^2$, and let g_1 be any other metric of equal area. Then, there exist $a_0 > 0$ and $T_0 > 0$ (depending on g_1) such that for all $a < a_0$ and $T > T_0$ we have $P_2(a, T; g_0) < P_2(a, T; g_1)$.*

It seems interesting to establish comparison results for finite times $0 < T < \infty$. In fact, it was proved in [Mor, Theorem 1] that the heat trace for the round metric on $\mathcal{S}^2$ locally minimizes the heat trace for all metrics on $\mathcal{S}^2$ of the same volume, in an L^∞ neighborhood of the set of conformal factors on $\mathcal{S}^2$; the size of the neighborhood depends on the interval $[a, b] \subset (0, \infty)$, where $T \in [a, b]$. It was also shown in [EI02], that the round metric on $\mathcal{S}^2$ was the unique critical metric on $\mathcal{S}^2$ for the heat trace functional. Accordingly, it seems natural to conjecture that the round metric on $\mathcal{S}^2$ are extremal for $P_2(a, T)$ for all T, in the limit $a \to 0$.

For surfaces of genus $\gamma \geq 2$ the situation is different. Given a metric g on M and $k \in \mathbb{Z}^+$, write $\ell := \dim Ker(-\Delta_g - \lambda_k(g)) - 1$. It is well-known that a metric g_0 that is extremal for $g \mapsto \lambda_k(g)$, among all metrics of the same volume in the same conformal class, admits a *minimal immersion* into the ℓ-sphere, $\mathcal{S}^\ell$, given by eigenfunctions that form an orthonormal basis of the eigenspace $Ker(-\Delta_{g_0} - \lambda_k(g_0))$, [Bryant, EI03, EI08, Nad, Tak].

It was proved in [Bryant, Theorem 2.3] that surfaces of constant negative curvature *cannot* be minimally immersed in $\mathcal{S}^3$; the corresponding result for minimal surfaces in $\mathbb{R}^n$ was established in [Yau74, Thm. 6]. This implies the following result (D.J. first learned about it from S. Wolpert).

Proposition 3.10. *Let g_0 be a hyperbolic metric on a compact orientable surface M of genus $\gamma \geq 2$. Then g_0* does not *maximize λ_1 in its conformal class.*

Strong results about the existence of metrics maximizing λ_1 in a conformal class were established recently in [NS]. The metrics that are extremal for λ_k among *all* metrics of the same volume (and not just in the same conformal class) admit *isometric* minimal immersions into round spheres by the corresponding eigenfunctions, see the above references about the minimal immersions, as well as [Her, LY, YY]. A metric that maximizes λ_1 for surfaces of genus 2 is a branched covering of the round 2-sphere, cf. [JLNNP]. Accordingly, we conclude that on surfaces of genus $\gamma \geq 2$, different metrics maximize $P_2(a, T)$ in the limit $a \to 0, T \to 0$ and in the limit $a \to 0, T \to \infty$, unlike the situation on $\mathcal{S}^2$.

4. Using results of [AT08] on the 2-Sphere

The sphere is a special surface, since the curvature perturbation is *isotropic*, and in particular the variance is constant. In this case a special theorem due to Adler-Taylor gives precise asymptotics for the excursion probability.

4.1. **Random functions on the 2-Sphere, $\mathcal{S}^2$.** For an integer m let $\mathcal{E}_m$ be the space of spherical harmonics of degree m of dimension $N_m = 2m + 1$ associated to the eigenvalue $E_m = m(m + 1)$, and for every m fix an L^2 orthonormal basis $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$ of $\mathcal{E}_m$.

To treat the spectrum degeneracy it is convenient to use a slightly different parametrization of the conformal factor than the usual one (2)

$$f(x) = -\sqrt{|\mathcal{S}^2|} \sum_{m\geq 1,\, k} \frac{\sqrt{c_m}}{E_m\sqrt{N_m}} a_{m,k}\eta_{m,k}(x), \tag{22}$$

where the and c_m's are some (suitably decaying) positive constants, $|\mathcal{S}^2| = 4\pi$ is the surface area of the sphere, and the $a_{m,k}$'s are independent, identically distributed, standard Gaussian random variables ($a_{m,k} \sim \mathcal{N}(0,1)$). For extra convenience we assume in addition that

$$\sum_{m=1}^{\infty} c_m = 1, \tag{23}$$

which has an advantage that the random field h defined below is of unit variance.

Remark 4.1. *For convenience, in the present section, the random fields f and h (see below) are defined differently than in the rest of the paper. The reason for the new definitions is spectral degeneracy on $\mathcal{S}^2$.*

The measure $\nu = \nu_{\{c_m\}_{m=1}^{\infty}}$ corresponding to (3) is generated by the densities on the finite cylinder sets

$$d\nu_{(m_1,k_1),\dots(m_l,k_l)}(f) = \frac{1}{\prod\limits_{i=1}^{l}(2\pi s_{m_i})^{1/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{l}\frac{f^2_{(m_i,k_i)}}{s_{m_i}}\right) df_{(m_1,k_1)}\dots df_{(m_l,k_l)},$$

where $f_{(m,k)} = \langle f, \eta_{m,k}\rangle_{L^2(\mathcal{S}^2)}$ are the Fourier coefficients, and

$$s_m := |\mathcal{S}^2|\frac{c_m}{E_m^2 N_m}.$$

Note that ν is invariant with respect to the choice of the orthonormal basis $\{B_m\}_{m=1}^{\infty}$ of the spaces $\mathcal{E}_m$ of the spherical harmonics, by the invariance of the Gaussian.

Recall that the Sobolev space $H_r(\mathcal{S}^2)$ consists of functions (distributions) $g : \mathcal{S}^2 \to \mathbb{R}$, for which

$$\sum_{m,k}(E_m+1)^r g^2_{m,k} < \infty.$$

In particular, $L^2(\mathcal{S}^2) = H_0(\mathcal{S}^2)$. By proposition 2.1, the measure ν is concentrated on $H_r(\mathcal{S}^2)$, if and only if

$$\sum_{m=1}^{\infty} N_m(E_m+1)^r\frac{c_m}{E_m^2 N_m} < \infty.$$

Since

$$(E_m+1)^r\frac{c_m}{E_m^2} \asymp m^{2r-4}c_m,$$

we have the following lemma.

Lemma 4.2. *Given a sequence c_m satisfying* (23)*, we have $f \in H_r(\mathcal{S}^2)$ ν-a.s. (or equivalently, the measure ν defined above satisfies $\nu(H_r(\mathcal{S}^2)) = 1$) if and only if*

$$\sum_{m=1}^{\infty} m^{2r-4}c_m < \infty.$$

In what follows we always assume that

$$c_m = O\left(\frac{1}{m^s}\right). \tag{24}$$

Thus $f \in H_r(\mathcal{S}^2)$ a.s. precisely for $r < \frac{s}{2} + \frac{3}{2}$. Note that if $c_m = \frac{K}{m^s}$, (23) requires $K = \frac{1}{\zeta(s)}$ where $\zeta(s)$ is the Riemann zeta function.

4.2. **Curvature's probability estimates on the 2-Sphere.**

Theorem 4.3. *Let $s > 7$, and the metric g_1 on $\mathcal{S}^2$ be given by*

$$g_1 = e^{af} g_0$$

where f is given by (22)*. Also, let $c_m \neq 0$ for at least one* odd m*. Then as $a \to 0$, the probability that the curvature is everywhere positive is given by*

$$\begin{aligned}\mathrm{Prob}\{\forall x \in \mathcal{S}^2, R_1(x) > 0\} &= 1 - C_1 \Psi\left(\frac{1}{a}\right) - \frac{C_2}{a}\exp\left(-\frac{1}{2a^2}\right) + o\left(\exp(-\frac{\alpha}{2a^2})\right)\\ &\sim 1 - C_1 \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2a^2}\right) - \frac{C_2}{a}\exp\left(-\frac{1}{2a^2}\right),\end{aligned}$$

where $C_1 = 2$, $C_2 = \frac{1}{\sqrt{2\pi}}\sum_{m\geq 1} c_m E_m$ and $\alpha > 1$.

The curvature corresponding to the random Riemannian metric $g_1 = e^{af}g_0$ is given by

$$R_1 e^{af} = R_0 - a\Delta_{g_0} f = 1 - a\Delta_{g_0} f, \tag{25}$$

where $R_0 \equiv 1$ corresponds to the round metric g_0. To make sense of it we shall assume that $f \in C^2(\mathcal{S}^2)$ a.s., for which we need that $s > 3$ (cf. (24)).

It is then natural to introduce the Gaussian random field (cf. (5))

$$h(x) := \Delta_{g_0} f(x) = \sqrt{|\mathcal{S}^2|}\sum_{m\geq 1,\, k}\frac{\sqrt{c_m}}{\sqrt{N_m}} a_{m,k}\eta_{m,k}(x), \tag{26}$$

so that (25) is

$$R_1 e^{af} = 1 - ah. \tag{27}$$

The random field h is centered unit variance (see (23)) Gaussian isotropic with covariance function $r_h : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathbb{R}$ given explicitly by

$$r_h(x,y) := \mathbb{E}[h(x)h(y)] = \sum_{m=1}^{\infty} c_m P_m(\cos(d(x,y))), \tag{28}$$

where P_m is the Legendre polynomial of degree m and $d(x,y)$ is the (spherical) distance between x and y.

The following lemma follows easily from (26), Proposition 2.1 and Sobolev embedding theorem.

Lemma 4.4. *If $c_m = O(m^{-s}), s > 2k+3$, then h, and hence R_1, are a.s. C^k.*

The condition (24) in particular ensures that the series in (26) is a.s. pointwise convergent; we need stronger conditions to work with *smooth* sample functions. It then follows from (27) that R_1 is everywhere positive if and only if

$$\|h\|_{\mathcal{S}^2} := \sup_{x\in\mathcal{S}^2}\{h(x)\} < \frac{1}{a}.$$

The problem of approximating the excursion probability of

$$E_{h,u} := \left\{ \|h\|_{\mathcal{S}^2} > u := \frac{1}{a} \right\}$$

(i.e., of the complement event) for a given random field h as $u \to \infty$ (i.e. $a \to 0$, small perturbation) is a classical problem in probability. For the constant variance random fields (which follows from the isotropic property of h), there is a special precise result due to Adler-Taylor [AT03]. The latter relates $\mathrm{Prob}\{E_{h,u}\}$ to the expected value of Euler characteristic of the excursion set $h^{-1}([u,\infty])$, giving an explicit expression for the latter, where the answer depends on the Adler-Taylor metric associated to h defined below.

4.3. **The Adler-Taylor metric on $\mathcal{S}^2$.** Let h be an a.s. C^1 random field on $\mathcal{S}^2$. The *Adler-Taylor* Riemannian metric g_h^{AT} on $\mathcal{S}^2$ is defined as follows (cf. [AT08, (12.2.2)]). Let $x \in \mathcal{S}^2$ and $X, Y \in T_x\mathcal{S}^2$; then

$$g_{h;x}^{AT}(X,Y) := \mathbb{E}[Xh \cdot Yh].$$

One may compute g^{AT} in terms of the covariance function as ([AT08], p. 306)

$$g_{h;x}^{AT}(X,Y) = XY r_h(x,y)|_{x=y}.$$

For the covariance as in (28) we obtain an expression for the metric

$$g_{h;x}^{AT}(X,Y) = \sum_{m=1}^{\infty} c_m \left(XY P_m(\cos(d(x,y))) \right)|_{x=y}.$$

The latter metric may be given as a 2×2 matrix if one chooses an orthonormal coordinate frame, i.e. a local choice of an orthonormal basis for $T_x(\mathcal{S}^2)$. It was found [W, W1] that no matter what the coordinate frame is, $g_{h;x}^{AT}(X,Y)$ is the scalar matrix

$$g_{h;x}^{AT} = C I_2, \tag{29}$$

with I_2 the 2×2 identity matrix, and $C = \frac{1}{2}\sum_{m=1}^{\infty} c_m E_m$.

For a general C^2-Riemannian manifold one can define its *Lipschitz-Killing curvatures*, also known as intrinsic volumes. For a general definition see (7.6.2) in [AT08]. On the 2-sphere endowed with the Adler-Taylor metric g_h^{AT}, the Lipschitz-Killing curvatures are the constants (see [AT08], (7.7.9))

$$\mathcal{L}_0(\mathcal{S}^2,h) := 2, \qquad \mathcal{L}_1(\mathcal{S}^2,h) := 0, \qquad \text{and} \qquad \mathcal{L}_2(\mathcal{S}^2,h) := \pi \sum_{m\geq 1} c_m E_m.$$

Theorem 4.5 ([AT08], Theorem 12.4.1)**.** *Let $h : \mathcal{S}^2 \to \mathbb{R}$ be a centered, unit variance Gaussian field on a C^2. Then if h is attainable (see definition B.1 in Appendix B.1),*

$$\mathbb{E}[\chi(u,+\infty)] = \sum_{j=0}^{2} \mathcal{L}_j(\mathcal{S}^2,h)\rho_j(u),$$

where

$$\rho_j(u) = \begin{cases} \Psi(u) & j=0, \\ \frac{1}{(2\pi)^{1/2}} e^{-u^2/2} & j=1, \\ \frac{1}{(2\pi)^{3/2}} u e^{-u^2/2} & j=2. \end{cases}$$

We apply Theorem 4.5 on the random field $h = \Delta_{g_0} f$. Theorem 4.5 allows us to compute the expected Euler characteristic of the excursion set, which is intimately related to the excursion probability. Provided the assumptions on Theorem 14.3.3 in [AT08] hold, according to equation (14.0.2) in [AT08] one has

$$\left|\text{Prob}\{\|h\|_{\mathcal{S}^2} \geq u\} - \mathbb{E}[\chi\left(h^{-1}[u,+\infty)\right)]\right| = O\left(e^{-\alpha u^2/2}\right) \tag{30}$$

for some $\alpha > 1$ (here we used assumption (23); otherwise we need to modify accordingly).

4.4. **Proof of Theorem 4.3.**

Proof of Theorem 4.3. We are interested in the probability that for every $x \in \mathcal{S}^2$

$$h(x) \leq u := \frac{1}{a}$$

for small $a > 0$, or, equivalently, its complement

$$\text{Prob}\{\|h\|_{\mathcal{S}^2} \geq u\}.$$

We employ Theorem 4.5 due to Adler-Taylor to compute the expected value of Euler characteristic of the excursion set explicitly as

$$\mathbb{E}[\chi\left(h^{-1}[u,\infty)\right)] = \sum_{j=0}^{2} \mathcal{L}_j(\mathcal{S}^2, h)\rho_j(u). \tag{31}$$

The statement of Theorem 4.3 then follows from (30), (31), and the values of $\mathcal{L}_j(\mathcal{S}^2, h)$ for $j = 0, 1, 2$. Note that to justify the application of Theorem 4.3 and (30) we have to validate the hypotheses of the corresponding theorems. We do so in Appendix B.

□

5. L^∞ CURVATURE BOUNDS ON SURFACES

5.1. **Definitions and the main result.** In Sections 3 and 4 we studied the probability of the curvature *changing sign* after a small conformal perturbation, on $\mathcal{S}^2$ and on surfaces of genus greater than one. On the 2-torus $\mathbb{T}^2$, however, Gauss-Bonnet theorem implies that the curvature *has* to change sign for every metric, so that question is meaningless.

Accordingly, on $\mathbb{T}^2$ we investigate the probability of another event that is considered very frequently in comparison geometry: the probability that scalar curvature satisfies the L^∞ curvature bounds $||R_1||_\infty < u$, where $u > 0$ is a parameter. Metrics satisfying such bounds for fixed u are called *metrics of bounded geometry.* We then study the probability that $||R_1 - R_0||_\infty < u$ separately for $\mathbb{T}^2$, $\mathcal{S}^2$, and surfaces of genus greater than one. In this section we *do not* assume that $R_0 \equiv$ *const*; nor do we assume that R_0 has constant sign.

Definition 5.1. *We shall consider the following three centered random fields on the surface M:*

i) *The random conformal multiple f given by* (2)*. We denote its covariance function by $r_f(x, y)$, and we define $\sigma_f^2 = \sup_{x \in M} r_f(x, x)$.*
ii) *The random field $h = \Delta_0 f$ defined in* (5)*. We denote its covariance function by $r_h(x, y)$, and we define $\sigma_h^2 = \sup_{x \in M} r_h(x, x)$.*

iii) *The random field* $w = \Delta_0 f + R_0 f = h + R_0 f$. *We denote its covariance function by* $r_w(x,y)$, *and we define* $\sigma_w^2 = \sup_{x\in M} r_w(x,x)$. *Note that when* M *is the flat* $\mathbb{T}^2$, $R_0 \equiv 0$ *and therefore* $h \equiv w$.

The random fields f, h *and* w *have constant variance on round* $\mathcal{S}^2$*; also* f *and* $h = w$ *have constant variance on flat* $\mathbb{T}^2$.

We prove the following theorem:

Theorem 5.2. *Assume that the random metric is chosen so that the random fields* f, h, w *are a.s.* C^0. *Let* $a \to 0$ *and* $u \to 0$ *so that*

$$\frac{u}{a} \to \infty. \tag{32}$$

Then

$$\log \operatorname{Prob}\{\|R_1 - R_0\|_\infty > u\} \sim -\frac{u^2}{2a^2\sigma_w^2}. \tag{33}$$

5.2. **Proof of Theorem 5.2.**

Proof. In the proof, we shall use Theorem 3.1 and this is the reason why the random fields f, h, w are required to be a.s. C^0. Let M denote a compact orientable surface ($\mathcal{S}^2, \mathbb{T}^2$, or of genus $\gamma \geq 2$).

Step 1. Let S be a (large) parameter S that will be chosen later. On $\mathbb{T}^2$, we let B_S denote the "bad" event where f is *large*

$$B_S = \{||f||_\infty > S\}. \tag{34}$$

Applying Theorem 3.1, we find that there exists a constant α_f such that the following estimate holds:

$$\operatorname{Prob}(B_S) = O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right)\right). \tag{35}$$

On $\mathcal{S}^2$ and on surfaces of genus ≥ 2 we modify the definition slightly, and let B_S denote the "bad" event that either f or h is large

$$B_S = \{||f||_\infty > S\} \cup \{||h||_\infty > S\}. \tag{36}$$

By Theorem 3.1 there exist two constants α_f and α_h, such that

$$\operatorname{Prob}(B_S) = O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right) + \exp\left(\alpha_h S - \frac{S^2}{2\sigma_h^2}\right)\right). \tag{37}$$

Denote $A_{u,a}$ the event $\{||R_1 - R_0||_\infty > u\}$; clearly,

$$\operatorname{Prob}(A_{u,a}) = \operatorname{Prob}(A_{u,a} \cap B_S) + \operatorname{Prob}(A_{u,a} \cap B_S^c). \tag{38}$$

We shall choose S later so that

$$\operatorname{Prob}(A_{u,a} \cap B_S) = o\left(\operatorname{Prob}(A_{u,a} \cap B_S^c)\right) \tag{39}$$

(for $a, u \to 0$ with (32) the LHS of (39) is of smaller order compared to the RHS); this is only possible under the assumption (32) of the present theorem. The inequality (39) implies that it is sufficient to evaluate $\operatorname{Prob}(A_{u,a} \cap B_S^c)$.

Consider first the event $A_{u,a} \cap B_S$; we estimate its probability trivially:

$$\operatorname{Prob}(A_{u,a} \cap B_S) \leq \operatorname{Prob}(B_S).$$

Accordingly, it follows from (35) for the torus, and (37) for the sphere or a surface of genus ≥ 2 that in each of the cases

(40)
$$\operatorname{Prob}(A_{u,a} \cap B_S) = \begin{cases} O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right)\right), & M = \mathbb{T}^2; \\ O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right) + \exp\left(\alpha_h S - \frac{S^2}{2\sigma_h^2}\right)\right), & \text{otherwise} \end{cases}.$$

Step 2. We next estimate $\operatorname{Prob}(A_{u,a} \cap B_S^c)$. Recall that in dimension two, it follows from (10) that

$$R_1 - R_0 = R_0(e^{-af} - 1) - ae^{-af}\Delta_0 f = R_0(e^{-af} - 1) - ae^{-af}h. \tag{41}$$

Note that on $\mathbb{T}^2$ we have $R_0 = 0$, and the first term on the right vanishes, hence we get

$$R_1 = -ae^{-af}h$$

in that case.

Step 2a. We start with the case $M = \mathbb{T}^2$. We choose a constant S satisfying

$$aS = o(1). \tag{42}$$

On B_S^c, we have $|f(x)| = O(S)$, hence $e^{-af(x)} = 1 + O(aS)$, so that

$$\begin{aligned} \operatorname{Prob}(A_{u,a} \cap B_S^c) &= \operatorname{Prob}\left(\left\{\|h\|_\infty > \frac{u}{a(1+O(aS))}\right\} \cap B_S^c\right) \\ &= \operatorname{Prob}\left(\left\{\|h\|_\infty > \frac{u}{a(1+O(aS))}\right\}\right) + O\left(\operatorname{Prob}(B_S)\right), \end{aligned} \tag{43}$$

the last summand being already estimated in (40). By (42), we have $\frac{u}{a(1+O(aS))} \sim \frac{u}{a}$. Plugging (40) and (43) into (38) we obtain

$$\operatorname{Prob}(A_{u,a}) = \operatorname{Prob}\left(\left\{\|h\|_\infty > \frac{u}{a(1+O(aS))}\right\}\right) + O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right)\right). \tag{44}$$

It then remains to evaluate

$$\operatorname{Prob}\left(\left\{\|h\|_\infty > \frac{u}{a(1+O(aS))}\right\}\right),$$

and choose S so that the other term is negligible. To this end we note that by symmetry,

$$\begin{aligned} \operatorname{Prob}\left(\left\{\|h\|_{\mathbb{T}^2} > \frac{u}{a(1+O(aS))}\right\}\right) &\leq \operatorname{Prob}\left(\left\{\|h\|_\infty > \frac{u}{a(1+O(aS))}\right\}\right) \\ &\leq 2\operatorname{Prob}\left(\left\{\|h\|_{\mathbb{T}^2} > \frac{u}{a(1+O(aS))}\right\}\right), \end{aligned} \tag{45}$$

and the factor 2 is negligible on the *logarithmic* scale.

To evaluate

$$\operatorname{Prob}\left(\left\{\|h\|_{\mathbb{T}^2} > \frac{u}{a(1+O(aS))}\right\}\right) \tag{46}$$

we note that (32) together with (42) imply that

$$\frac{u}{a(1+O(aS))} \to \infty, \tag{47}$$

so that we may apply Theorem 3.1 to obtain

$$\mathrm{Prob}\left(\left\{\|h\|_{\mathbb{T}^2} > \frac{u}{a(1+O(aS))}\right\}\right) = \\ = O\left(\exp\left[\frac{\alpha_h u}{a(1+O(aS))} - \frac{u^2}{2a^2\sigma_h^2(1+O(aS))^2}\right]\right)$$

To get a lower bound for (46), we proceed as in Section 3 and choose $x_0 \in \mathbb{T}^2$ where $\sigma_h^2 = \sup_{x\in\mathbb{T}^2} r_h(x,x)$ is attained. Clearly, we shall get a lower bound in (46) by evaluating $\mathrm{Prob}\left(\left\{h(x_0) > \frac{u}{a(1+O(aS))}\right\}\right)$, and the latter is equal to $\Psi(u/(a\sigma_h^2(1+O(aS))))$.

Next, we remark that $u/(a(1+O(aS))) \sim u/a$ provided S is chosen so that $aS = o(1)$. Comparing the estimates from above and from below, we find that

$$\log \mathrm{Prob}\left(\left\{\|h\|_{\mathbb{T}^2} > \frac{u}{a(1+O(aS))}\right\}\right) = \frac{-u^2}{2a^2\sigma_h^2}.$$

This concludes the proof of Theorem 5.2 for $M = \mathbb{T}^2$, provided (39) holds (that ensures that the last term gives the dominant contribution to $\mathrm{Prob}(A_{u,a})$). It remains to show that we can choose S satisfying all the constraints we encountered; accordingly, we collect all the inequalities that relate the various parameters in the course of the proof, and make sure that a proper choice for S is possible.

For the applications of Theorem 3.1, we need both $S \to \infty$ and (47); for the latter it is sufficient to require that $aS = o(1)$ or, equivalently, $S = o(1/a)$ (recall that we assume (32)). To make sure that (39) holds, we need $u/a = o(S)$. All in all, we need $u/a = o(S)$ and $S = o(1/a)$ while $S \to \infty$; the assumption $u \to 0$ of the present theorem leaves a handy margin for a possible choice of S, since it implies that u/a is much smaller than $1/a$.

Step 2b. Next consider the case $M = \mathcal{S}^2$ or $M = M_\gamma$ is a surface of genus $\gamma \geq 2$. We want to estimate the probability of the event $\{||R_1 - R_0||_\infty > u\} \cap B_S^c$. Recall from (41) that

$$R_1 - R_0 = R_0(e^{-af} - 1) - ae^{-af}h.$$

By the definition of B_S, on B_S^c, we have for $x \in M$, $|f(x)| = O(S)$ and

$$|h(x)| = |\Delta_0 f(x)| = O(S).$$

Again, we choose S so that $aS = o(1)$, and it follows easily from the Taylor expansion of e^{-af} and the definition of w that

$$R_1 - R_0 = -aw - O(aS)(af + ah) = -aw + O(a^2S^2). \tag{48}$$

On $\mathcal{S}^2$, the isotropic random field w has constant variance σ_w^2 that we shall compute later; on $M_\gamma, \gamma \geq 2$ the variance $r_w(x,x)$ is no longer constant, and we denote by σ_w^2 its supremum $\sup_{x\in M_\gamma} r_w(x,x)$. Therefore (cf. (43)),

$$\begin{aligned} \mathrm{Prob}(A_{u,a} \cap B_S^c) &= \mathrm{Prob}\left(\left\{\|w + O(aS^2)\|_\infty > \frac{u}{a}\right\} \cap B_S^c\right) \\ &= \mathrm{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a} + O(aS^2)\right\}\right) + O(\mathrm{Prob}(B_S)). \end{aligned} \tag{49}$$

Assuming that (39) holds and taking (38) into account, we obtain

$$\begin{aligned}\operatorname{Prob}(A_{u,a}) = \operatorname{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a} + O(aS^2)\right\}\right) \\ + \; O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right) + \exp\left(\alpha_h S - \frac{S^2}{2\sigma_h^2}\right)\right).\end{aligned} \tag{50}$$

We choose S so that $\frac{u}{a} = o(S)$ but $S = o\left(\frac{\sqrt{u}}{a}\right)$, so that this choice is possible since $\sqrt{u}$ is much larger than u, as u is small. We then have

$$aS^2 = o\left(\frac{u}{a}\right),$$

so that

$$\operatorname{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a} + O(aS^2)\right\}\right) = \operatorname{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a}(1 + o(1))\right\}\right).$$

As in Section 3, we shall estimate the quantity $\operatorname{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a}(1 + o(1))\right\}\right)$ from above and below by separate arguments. We let

$$\tau = \tau(u, a, S) := u/a + O(aS^2) = (u/a)(1 + o(1)).$$

By Borel-TIS Theorem 3.1, there exists α_w such that

$$\operatorname{Prob}\left(\{||w||_\infty > \tau\}\right) \le \exp\left(\alpha_w \tau - \frac{\tau^2}{2\sigma_w^2}\right). \tag{51}$$

This concludes the proof of the upper bound in (33) in this case.

To get a lower bound in (33), consider the point $x_0 \in M$ where $r_w(x, x)$ attains its maximum, $r_w(x_0, x_0) = \sigma_w^2$. Consider the event $\{|w(x_0)| > \tau\}$. We find that trivially

$$\operatorname{Prob}\left(\{||w||_\infty > \tau\}\right) \ge \operatorname{Prob}(\{|w(x_0)| > \tau\}) \ge \left(\frac{C_1}{\tau} - \frac{C_2}{\tau^3}\right)\exp\left(-\frac{\tau^2}{2\sigma_w^2}\right). \tag{52}$$

We next pass to the limit $u \to 0, u/a \to \infty$; then $\tau \cdot a/u \to 1$. Taking logarithm in (51) and (52) and comparing the upper and lower bound, we establish (33) for surfaces of genus ≥ 2. This concludes the proof of Theorem 5.2.

□

6. Dimension $n > 2$

Let (M, g_0) be a compact orientable n-dimensional Riemannian manifold, $n > 2$. Let $R_0 \in C^0(M)$ be the scalar curvature of g_0; we assume that R_0 has constant sign. Let $g_1 = e^{af} g_0$ with f as in (2) be a conformal change of metric. The key difference between dimension 2 and dimension $n > 2$ in our calculations is the presence of the (non-Gaussian) gradient term $a^2(n-1)(n-2)|\nabla_0 f|^2/4$ in the equation (9). We shall assume that $c_j = O(\lambda_j^{-s}), s > n/2 + 1$. Then $R_1 \in C^0(M)$ a.s. by Proposition 2.5.

We shall consider the random field $v(x) = (\Delta_0 f)(x)/R_0(x)$. As usual, we let

$$\sigma_v^2 = \sup_{x \in M} r_v(x, x). \tag{53}$$

We let $P_2(a)$ be the probability of the scalar curvature sign change after the conformal metric transformation $g_1 = e^{af} g_0$, as in dimension two.

6.1. **Negative** R_0**.** We first consider the case of $\forall x \in M$, $R_0(x) < 0$.

Proposition 6.1. *Let (M, g_0) be a compact orientable n-dimensional Riemannian manifold, $n > 2$, such that the scalar curvature $R_0 \in C^0(M)$ and for all $x \in M$, $R_0(x) < 0$. Assume that $c_j = O(\lambda_j^{-s}), s > n/2 + 1$, so that $h, R_1 \in C^0(M)$. Then there exists $\alpha > 0$ so that*

$$P_2(a) = O\left(\exp\left(\frac{\alpha}{a(n-1)} - \frac{1}{2a^2(n-1)^2\sigma_v^2}\right)\right).$$

Proof of Proposition 6.1.

Recall that the curvature transformation corresponding to the conformal change $g_1 = e^{af} g_0$ is given by (9), and observe that $R_1 e^{af}$ is not Gaussian because of the presence of the non-Gaussian term $|\nabla_0 f|^2$. Then, in order to prove the assertion of the present proposition, we get rid of the term $|\nabla_0 f|^2$ in (9) by taking advantage of its positivity, so that

$$\{R_0 - a(n-1)\Delta_0 f > 0\} \supseteq \{R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)4^{-1}|\nabla_0 f|^2 > 0\}.$$

Therefore

$$P_2(a) \leq \text{Prob}\{\exists\, x \in M : R_0(x) - a(n-1)(\Delta_0 f)(x) > 0\}.$$

Recall that $h = \Delta_0 f$. We remark that

$$\text{sgn}(R_0(x) - a(n-1)h(x)) = -\,\text{sgn}(1 - (n-1)h(x)/R_0(x)).$$

Accordingly,

$$P_2(a) \leq \text{Prob}\{\exists\, x \in M : 1 - a(n-1)h/R_0 < 0\} = \text{Prob}\{||h/R_0||_M > 1/(a(n-1))\}.$$

It then remains to apply Theorem 3.1 for $u = 1/(a(n-1))$.

□

6.2. **Positive** R_0**.** We next work out the more involved case $R_0 > 0$. The regularity assumptions are the same as in Section 6.1. In this section, we consider the random field $v = h/R_0$ (considered earlier in Section 6.1). We introduce the quantity

$$\sigma_2 = \sup_{x \in M} \frac{\mathbb{E}[|\nabla_0 f(x)|^2]}{R_0(x)}. \tag{54}$$

Proposition 6.2. *Let (M, g_0) be a compact orientable n-dimensional Riemannian manifold, $n > 2$, such that the scalar curvature $R_0 \in C^0(M)$, and for all $x \in M$, $R_0(x) > 0$. Assume that $c_j = O(\lambda_j^{-s})$ with $s > n/2 + 1$. Then, there exists $\beta > 0$ so that*

$$P_2(a) = O\left(\exp\left(\frac{\beta}{a} - \frac{B_1}{a^2}\right)\right),$$

where B_1 is any number less than

$$B = \frac{2 + \omega - \sqrt{\omega^2 + 4\omega}}{\sigma_2 n(n-1)(n-2)}.$$

and

$$\omega = \frac{4\sigma_v^2(n-1)}{\sigma_2 n(n-2)}.$$

Proof of Proposition 6.2.

Note that $\operatorname{sgn} R_1$ is equal to

$$\operatorname{sgn}\left(1-\frac{a(n-1)h}{R_0}-\frac{a^2(n-1)(n-2)|\nabla_0 f|^2}{4R_0}\right);$$

here we have used the assumption that for all $x\in M$, $R_0(x)>0$. Recall that $P_2(a)$ denotes the probability that $\exists x\in M$ with $R_1(x)<0$. We define a random field u to be

$$u:=\frac{(n-1)h}{R_0}+\frac{a(n-1)(n-2)|\nabla_0 f|^2}{4R_0}.$$

Then $R_1<0$ is equivalent to

$$\|u\|_M>\frac{1}{a}.$$

For every $0\le\delta\le 1$, we have the event $\left\{\|u\|_M\ge\frac{1}{a}\right\}$ is contained in

$$\left\{\left\|\frac{(n-1)h}{R_0}\right\|_M\ge\frac{\delta}{a}\right\}\bigcup\left\{\left\|\frac{a(n-1)(n-2)|\nabla_0 f|^2}{4R_0}\right\|_M\ge\frac{1-\delta}{a}\right\},$$

and therefore

$$\begin{aligned}\operatorname{Prob}\left\{\|u\|_M\ge\frac{1}{a}\right\}\le\operatorname{Prob}&\left\{\left\|\frac{(n-1)h}{R_0}\right\|_M\ge\frac{\delta}{a}\right\}\\&+\operatorname{Prob}\left\{\left\|\frac{a(n-1)(n-2)|\nabla_0 f|^2}{4R_0}\right\|_M\ge\frac{1-\delta}{a}\right\}.\end{aligned}\tag{55}$$

The probability $\operatorname{Prob}\left\{\left\|\frac{(n-1)h}{R_0}\right\|_M\ge\frac{\delta}{a}\right\}$ can be estimated in a straightforward way using Theorem 3.1. Indeed, define the random field $v:=h/R_0$ (as in Section 6.1); as before, let σ_v^2 be defined by (53). Then $\left\{\left\|\frac{(n-1)h}{R_0}\right\|_M\ge\frac{\delta}{a}\right\}$ is equivalent to

$$\|v\|_M>\frac{\delta}{a(n-1)},$$

and the latter can be bounded by Theorem 3.1 (letting $u=\delta/(a(n-1))$) as

$$\operatorname{Prob}\left\{\left\|\frac{(n-1)h}{R_0}\right\|_M\ge\frac{\delta}{a}\right\}\le\exp\left(\frac{\beta_1\delta}{a}-\frac{\delta^2}{2(a(n-1))^2\sigma_v^2}\right),\tag{56}$$

for some constant $\beta_1>0$.

Set

$$\kappa:=\frac{4(1-\delta)}{a^2(n-1)(n-2)}.\tag{57}$$

To bound

$$\operatorname{Prob}\left\|\frac{a(n-1)(n-2)|\nabla_0 f|^2}{4R_0}\right\|_M\ge\frac{1-\delta}{a}=\operatorname{Prob}\left\|\frac{|\nabla_0 f|^2}{R_0}\right\|_M\ge\kappa$$

we need to work harder, as the random field $|\nabla_0 f(x)|^2/R_0(x)$ is not Gaussian. The key observation is that we may represent this field as *locally* Gaussian subordinated. Namely, let

$$\{U_i:\ i=1,\dots,m\}$$

be a finite covering of M by small open sets. We chose a geodesic frame $\{E_1^i,\dots,E_n^i\}$ defined on U_i, say geodesic normal frame at $p_i\in U_i$. At $x=p_i$, we have

$|\nabla_0 f(x)|^2 = \sum_{i=1}^n (E_k^i f(x))^2$. Given $\epsilon > 0$, we can choose U_i small enough so that on U_i we shall have

$$1-\epsilon \leq \frac{|\nabla_0 f(x)|^2}{\sum_{i=1}^n (E_k^i f(x))^2} \leq 1+\epsilon. \tag{58}$$

Accordingly,

$$\frac{|\nabla_0 f(x)|^2}{R_0(x)} \leq (1+\epsilon) \sum_{k=1}^n \frac{(E_k^i f(x))^2}{R_0(x)},$$

Observe that $G_{i,k}(x) := (E_k^i f(x))/\sqrt{R_0(x)}$ are *centered Gaussian* random fields defined on U_i. For each i, k and $x \in M$ we have by (58) and (54)

$$\mathbb{E}\{G_{i,k}(x)^2\} \leq \mathbb{E}\left\{\frac{|\nabla f(x)|^2}{R_0(x)}\right\} \leq \frac{\sigma_2}{1-\epsilon}. \tag{59}$$

We have

$$\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_M \leq (1+\epsilon) \max_i \left\|\sum_{k=1}^n G_{i,k}(x)^2\right\|_{U_i}.$$

Clearly,

$$\begin{aligned}\operatorname{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_M \geq \kappa\right\} &\leq \sum_{i=1}^m \operatorname{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_{U_i} \geq \kappa\right\} \\ &\leq m \operatorname{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_{U_{i_0}} \geq \kappa\right\}\end{aligned} \tag{60}$$

where $i_0 = i_0(a)$ maximizes the probability

$$\operatorname{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_{U_i} \geq \kappa\right\}$$

for $1 \leq i \leq m$.

Therefore we need to bound

$$\begin{aligned}\operatorname{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_{U_{i_0}} \geq \kappa\right\} &= \operatorname{Prob}\left\{\left\|\sum_{k=1}^n \frac{(E_k^{i_0} f(x))^2}{R_0(x)}\right\|_{U_{i_0}} \geq \frac{\kappa}{1+\epsilon}\right\} \\ &< \sum_{k=1}^n \operatorname{Prob}\left\{\left\|\frac{|E_k^{i_0} f(x)|}{\sqrt{R_0(x)}}\right\|_{U_{i_0}} \geq \sqrt{\frac{\kappa}{n(1+\epsilon)}}\right\}.\end{aligned} \tag{61}$$

We may bound each of the summands using the Borel-TIS inequality as

$$\operatorname{Prob}\left\{\left\|\frac{|E_k^{i_0} f(x)|}{\sqrt{R_0(x)}}\right\|_{U_{i_0}} \geq \sqrt{\frac{\kappa}{n(1+\epsilon)}}\right\} \leq \exp\left(\frac{\beta_2}{a} - \frac{\kappa(1-\epsilon)}{2\sigma_2 n(1+\epsilon)n}\right),$$

where we exploited (59); the constant β_2 absorbs the 2 factor coming from the possibility that we might have either a positive or negative sign. Plugging the last estimate into (61) and the resulting bound into (60), we finally obtain, possibly choosing a larger constant β_2 to absorb the constants in front of the exponent,

$$\operatorname{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_M \geq \kappa\right\} \leq \exp\left(\frac{\beta_2}{a} - \frac{\kappa(1-\epsilon)}{2\sigma_2 n(1+\epsilon)}\right). \tag{62}$$

We next choose δ in an optimal way, so that the negative exponents in (62) and (56) match. To get the best possible estimate, we should let $\epsilon \to 0$ in (58). This allows us to get the "limiting" value B in Proposition 6.2. Since actually $\epsilon > 0$, the actual estimate involves an arbitrary $B_1 < B$. The matching condition (pretending $\epsilon = 0$) reads

$$\frac{\delta^2}{2(n-1)^2\sigma_v^2} = \frac{2(1-\delta)}{\sigma_2 n(n-1)(n-2)}. \tag{63}$$

or, letting $\omega = \frac{4\sigma_v^2(n-1)}{\sigma_2 n(n-2)}$,

$$\delta^2 + \omega\delta - \omega = 0.$$

It is easy to check that the root $\delta_0 = (\sqrt{\omega^2 + 4\omega} - \omega)/2$ satisfies the required inequality $0 < \delta < 1$ and thus gives an admissible solution to (63). Substituting δ_0, we find that the exponents in (63) are both equal to

$$B = \frac{2 + \omega - \sqrt{\omega^2 + 4\omega}}{\sigma_2 n(n-1)(n-2)}.$$

Substituting into (62) and (56) finishes the proof of Proposition 6.2.

□

7. Q-curvature on Manifolds of even dimension

The Q-curvature was first studied by Branson and later by Gover, Orsted, Fefferman, Graham, Zworski, Chang, Yang, Djadli, Malchiodi and others. We refer to [BG] for a detailed survey.

7.1. **Conformally covariant operators.** Here we summarize some useful results in [BG]. Let (M, g) be a Riemannian manifold of *even* dimension $n \geq 3$. There is a generalization of the Yamabe and Paneitz operators constructed by Graham, Jenne, Mason and Sparling, cf. [GJMS]. For $m = 1, \ldots, \frac{n}{2}$, they introduced an operator of order m

$$P_{g,m} : C^\infty(M) \to C^\infty(M).$$

We shall restrict to the critical case $m = \frac{n}{2}$ and, to shorten notation, we denote the corresponding operator $P_{g,\frac{n}{2}}$ by P_g. The operator P_g is formally self-adjoint. The leading order term of P_g is $\Delta_g^{n/2}$, and therefore P_g is strongly elliptic. Under a conformal change of metric $\tilde{g} = e^{2\omega}g$, the operator P_g changes as $P_{\tilde{g}} = e^{-n\omega}P_g$. The operator P_g has a polynomial expression in the Levi-Civita connection and the scalar curvature.

7.2. **Q-curvature and its key properties.** The Q-curvature in dimension 4 was defined by Paneitz as follows:

$$Q_g = -\frac{1}{12}\left(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2\right). \tag{64}$$

In higher dimensions, Q-curvature is a local scalar invariant associated to the operator P_g. It was introduced by T. Branson in [Bran] and alternative constructions were provided in [FG, FH] using the *ambient metric* construction. Under a conformal change of the metric $\tilde{g} = e^{2\omega}g$, the Q-curvature transforms following [BG, (4)]

$$P_g\omega + Q_g = Q_{\tilde{g}}e^{n\omega}. \tag{65}$$

A natural problem is the existence of metrics with constant Q-curvature in a given conformal class. In the following proposition, we summarize results due to Chang and Yang, and Djadli and Malchiodi in dimension 4, and to Ndiaye in arbitrary even dimension $n > 4$ [CY, DM, N].

Proposition 7.1. *Let (M, g) be a compact Riemannian manifold of even dimension $n \geq 4$, and assume that M satisfies the following "generic" assumptions:*

i) *In dimension $n = 4$, the assumptions are ([DM]): $\ker P_{g,2} = \{const\}$, and $\int_M Q_g dV_g \neq 8\pi^2 k, k = 1, 2, \ldots$*
ii) *In even dimension $n > 4$, the assumptions are ([N]): $\ker P_{g,\frac{n}{2}} = \{const\}$, and $\int_M Q_g dV_g \neq (n-1)!\omega_n k, k = 1, 2, \ldots,$ where $(n-1)!\omega_n = \int_{\mathcal{S}^n} Q_g dV_g$, the integral of Q-curvature for the round $\mathcal{S}^n$.*

Then there exists a metric g_Q on M in the conformal class of g with constant Q-curvature. If $n = 4, \int_M Q_g dV_g < 8\pi^2$, $P_{g,2} \geq 0$ and $\ker P_{g,2} = \{const\}$, then g_Q is unique, [CY, Thm 2.2].

If g has positive scalar curvature and $M \neq \mathcal{S}^4$, then the assumption $\int_M Q_g dV_g < 8\pi^2$ is satisfied; if in addition $\int_M Q_g dV_g \geq 0$, then the assumptions $P_{g,2} \geq 0$ and $\ker P_{g,2} = \{const\}$ are also satisfied.

7.3. **Generalizing the results for scalar curvature.** Consider a manifold M with a "reference" metric g_0 such that Q-curvature has constant sign and a conformal perturbation $g_1 = e^{2af} g_0$ where a is a positive number; we expand f in a series of eigenfunctions of P. Next, we use formula (65) to study the induced curvature Q_1. Finally, we use methods of Adler-Taylor to prove sharper estimates for the probability for manifolds with constant Q-curvature. We remark that in every conformal class where the generic conditions of [DM, N] hold, there exist metrics with Q-curvature of constant sign.

7.4. **Q-curvature in a conformal class.** Let M be a manifold of even dimension n, and let g_0 be a metric with Q-curvature Q_0. In the Fourier expansions considered below, we shall restrict our summation to *nonzero* eigenvalues of $P_{g_0} = P_{g_0,\frac{n}{2}}$. Under the assumptions of Proposition 7.1, $\ker P_{g_0} = \{const\}$.

Let P_{g_0} have k negative eigenvalues (counted with multiplicity)

$$-\mu_k \leq -\mu_2 \leq \ldots \leq -\mu_1 < 0,$$

and denote the corresponding eigenfunctions by ψ_j for $1 \leq j \leq k$ (that is $P_{g_0}\psi_j = -\mu_j \psi_j$). The other nonzero eigenvalues are positive and are denoted by

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

with respective eigenfunctions ϕ_j, satisfying $P_{g_0}\phi_j = \lambda_j \phi_j$, for $j \geq 1$.

Consider the change of metric $g_1 = e^{2af} g_0$, where we let

$$f = \sum_{i=1}^{k} b_i \psi_i + \sum_{j=1}^{\infty} a_j \phi_j, \tag{66}$$

and where $b_i \sim \mathcal{N}(0, t_i^2)$ and the $a_j \sim \mathcal{N}(0, c_j^2)$ are independent. We define $h := -P_{g_0} f$, and substituting into (65), we find that

$$Q_1 e^{naf} = Q_0 - ah = Q_0 + a\left(\sum_{j=1}^{\infty} \tilde{a}_j \phi_j - \sum_{i=1}^{k} \tilde{b}_i \psi_i\right), \tag{67}$$

where $\tilde{a}_j \sim \mathcal{N}(0, \lambda_j^2 c_j^2)$ and $\tilde{b}_i \sim \mathcal{N}(0, t_i^2 \mu_i^2)$.

Remark 7.2. *For $x \in M$, it follows that $Q_1 e^{naf}(x)$ is a Gaussian random variable with expectation $Q_0(x)$ and covariance function*

$$a^2 \cdot r_h(x,y) = a^2 \left(\sum_{i=1}^{k} t_i^2 \mu_i^2 \psi_i(x)\psi_i(y) + \sum_{j=1}^{\infty} \lambda_j^2 c_j^2 \phi_j(x)\phi_j(y) \right). \tag{68}$$

7.5. **Regularity.** It is easy to see that the regularity of the random field in (66) is determined by the principal symbol of the GJMS operator $P_{g_0} = P_{g_0, \frac{n}{2}}$ which agrees with that of $\Delta_{g_0}^{n/2}$. The following Proposition is then a straightforward extension of Proposition 2.1:

Proposition 7.3. *Let f be defined as in* (66)*. If $c_j = O(\lambda_j^{-t})$ and $t > 1 + \frac{k}{n}$, then $f \in C^k$. Similarly, if $c_j = O(\lambda_j^{-t})$ and $t > 2 + \frac{k}{n}$, then $P_{g_0} f \in C^k$.*

7.6. **Estimating the probability that the Q-curvature changes sign.** Consider a metric g_0 where Q_0 has constant sign. We remark that such metric always exists in the conformal class of g_0 if Proposition 7.1 holds.

Let f be as in equation (66) and such that $P_{g_0} f$ is a.s. C^0. We remark that it follows from Proposition 7.3 that this happens if $c_j = O(\lambda_j^{-t})$ with $t > 2$.

Let $g_1 = e^{2af} g_0$. Denote the Q-curvature of g_1 by Q_1; then it follows from (65) that

$$\operatorname{sgn}(Q_1) = \operatorname{sgn}(Q_0) \operatorname{sgn}(1 - ah/Q_0) \tag{69}$$

It follows that Q_1 changes sign iff $\sup_{x \in M} h(x)/Q_0(x) > 1/a$.

Denote by $v(x)$ the random field $h(x)/Q_0$. It follows from (68) that the covariance function of $v(x)$ is equal to

$$r_v(x,y) = \frac{1}{Q_0(x)Q_0(y)} \left(\sum_{i=1}^{k} t_i^2 \mu_i^2 \psi_i(x)\psi_i(y) + \sum_{j=1}^{\infty} \lambda_j^2 c_j^2 \phi_j(x)\phi_j(y) \right). \tag{70}$$

We let

$$\sigma_v^2 := \sup_{x \in M} r_v(x,x). \tag{71}$$

The following definition is the Q-curvature analogue for the sign change probability in the scalar curvature case.

Definition 7.4. *Denote by $P_2(a)$ the probability that the Q-curvature Q_1 of the metric $g_1 = g_1(a)$ changes sign.*

Theorem 7.5. *Assume that $Q_0 \in C^0(M)$ and that $c_j = O(\lambda_j^{-t}), t > 2$. Then there exist constants $C_1 > 0$ and C_2 such that the probability $P_2(a)$ satisfies*

$$(C_1 a) e^{-1/(2a^2\sigma_v^2)} \le P_2(a) \le e^{C_2/a - 1/(2a^2\sigma_v^2)},$$

as $a \to 0$. In particular,

$$\lim_{a \to 0} a^2 \ln P_2(a) = -\frac{1}{2\sigma_v^2}.$$

Proof of Theorem 7.5. It follows from the assumptions of the theorem and from Proposition 7.3 that $v \in C^0(M)$ a.s., and hence the Borell-TIS theorem applies. The rest of the proof follows the proof of Theorem 3.3.

∎

7.7. L^∞ **bounds for the Q-curvature.** Here we extend the results in Section 5 to Q-curvature. We have not pursued similar questions for the scalar curvature in dimension $n \geq 3$ due to the presence of the gradient term in the transformation formula (9). For the Q-curvature, there is no gradient term in the corresponding transformation formula (65), which allows us to establish the following theorem.

Theorem 7.6. *Let (M, g_0) be an n-dimensional compact orientable Riemannian manifold, with n even. Assume that $Q_0 \in C^0(M)$, and that $c_j = O(\lambda_j^{-t})$, $t > 2$. Let $w := h - nQ_0 f$, denote by $r_w(x,y)$ its covariance function and set*

$$\sigma_w^2 := \sup_{x \in M} r_w(x,x).$$

Let $a \to 0$ and $u \to 0$ so that

$$\frac{u}{a} \to \infty.$$

Then

$$\log \mathrm{Prob}(\|Q_1 - Q_0\|_\infty > u) \sim -\frac{u^2}{2a^2\sigma_w^2}.$$

Proof. The proof of this theorem is very similar to the one presented in step (2c) of Theorem 5.2. We have to deal with the fact that neither f, h or w have constant variance. We start by defining the "bad" event B_S, for $S > 0$,

$$B_S = \{||f||_\infty > S\} \cup \{||h||_\infty > S\}.$$

By Theorem 3.1, there exist two constants α_f and α_h such that

$$\mathrm{Prob}(B_S) = O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right) + \exp\left(\alpha_h S - \frac{S^2}{2\sigma_h^2}\right)\right).$$

for $\sigma_f^2 := \sup_{x\in M} r_f(x,x)$ and $\sigma_h^2 := \sup_{x\in M} r_h(x,x)$, where r_f and r_h are the covariance functions of f and h respectively.
As before, we denote $A_{u,a}$ the event $\{||Q_1 - Q_0||_\infty > u\}$ and observe that $\mathrm{Prob}(A_{u,a}) = \mathrm{Prob}(A_{u,a} \cap B_S) + \mathrm{Prob}(A_{u,a} \cap B_S^c)$. We estimate $\mathrm{Prob}(A_{u,a} \cap B_S)$ trivially: $\mathrm{Prob}(A_{u,a} \cap B_S) \leq \mathrm{Prob}(B_S)$. This implies that

$$\mathrm{Prob}(A_{u,a} \cap B_S) = O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right) + \exp\left(\alpha_h S - \frac{S^2}{2\sigma_h^2}\right)\right).$$

In order to estimate $\mathrm{Prob}(A_{u,a} \cap B_S^c)$, note that it follows from (65) that

$$Q_1 - Q_0 = Q_0(e^{-naf} - 1) + ahe^{-naf}.$$

By the definition of B_S, on B_S^c, we have for $x \in M$ that

$$|f(x)| = O(S) \quad \text{and} \quad |h(x)| = O(S).$$

Choose S so that $aS = o(1)$. It follows easily from the Taylor expansion of e^{-af} and the definition of w that

$$Q_1 - Q_0 = ah - anQ_0 f + O(a^2S^2) = aw + O(a^2S^2).$$

Therefore,

$$\begin{aligned}\mathrm{Prob}(A_{u,a} \cap B_S^c) &= \mathrm{Prob}\left(\left\{\|w + O(aS^2)\|_\infty > \frac{u}{a}\right\} \cap B_S^c\right)\\ &= \mathrm{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a} + O(aS^2)\right\}\right) + O(\mathrm{Prob}(B_S)).\end{aligned}$$

The notation was conveniently chosen so that the rest of the proof be identical to the argument that follows equation (49) in Step 2b of Theorem 5.2.

□

Appendix A. Metrics with positive and negative scalar curvature

In this section we review some results about the spaces of metrics of positive and negative scalar curvature. In dimension two, $\mathcal{S}^2$ admits the metric of positive curvature, and surfaces of genus ≥ 2 admit metrics of negative curvature. For connected manifolds M of dimension $n \geq 3$, Kazdan and Warner [KW] proved the following "trichotomy" theorem:

i) If M admits a metric of nonnegative and not identically 0 scalar curvature, then any $f \in C^\infty(M)$ can be realized as a scalar curvature of some Riemannian metric.
ii) If M is not in (i) and admits a metric of vanishing scalar curvature, then $f \in C^\infty(M)$ can be realized as a scalar curvature provided $f(x) < 0$ for some $x \in M$, or else $f \equiv 0$.
iii) If M is not in (i) or (ii), then $f \in C^\infty(M)$ can be realized as a scalar curvature provided $f(x) < 0$ for some $x \in M$.

A.1. **Negative scalar curvature.** Denote by $S^-(M)$ the space of metrics of negative scalar curvature on a manifold M of dimension $n \geq 3$; it follows from results of Aubin and Kazdan-Warner that $S^-(M)$ is always nonempty. A fundamental theorem about the structure of $S^-(M)$ was proved by J. Lohkamp [Lo], who showed that $S^-(M)$ is connected and aspherical (and hence is contractible). He also showed that the space $S_{-1}(M)$ of metrics of constant curvature -1 is contractible. It is shown in [Kat] that on a Haken manifold, the moduli space $S_{-1}(M)/Diff_0(M)$ (where $Diff_0(M)$ denotes the the group of diffeomorphisms isotopic to the identity) is also contractible, similarly to Teichmuller spaces for surfaces of genus ≥ 2 in dimension 2. In a different paper, Lockhamp showed that $S^-(M)$ and $S_{-1}(M)$ are never convex.

A.2. **Positive scalar curvature.** Questions about existence and spaces of metrics of positive scalar curvature are more complicated than similar questions for negative scalar curvature. Here we recall some of the less technical results in recent Rosenberg's survey [Ros06]. We make no attempt to give a complete survey, we just want to list some examples of manifolds where the results of our paper hold.

There are several techniques for proving results about non-existence of metrics of positive scalar curvature on a given manifold. We assume that M is compact, closed, oriented manifold.

i) For spin manifolds with positive scalar curvature, it follows from the work of Lichnerowicz that all harmonic spinors (lying in the kernel of the Dirac operator) have to vanish; Therefore, it follows from the work of Lichnerowicz and Hitchin that any manifold with nonvanishing Hirzebruch genus $\hat{A}(M)$ has no metrics of positive scalar curvature. We refer to [Ros86] and [Ros06] for further non-existence results that use index theory of Dirac operator, and for relations to Novikov conjectures.
ii) It follows from the work of Schoen and Yau on minimal surfaces [SY79-1, SY79-2, SY82] that if N is a stable $(n-1)$-dimensional submanifold of an n-dimensional manifold M with positive scalar curvature, and if N is

dual to a nonzero element in $H^1(M,\mathbb{Z})$, then N also admits a metric of positive scalar curvature. It was shown in [SY79-2] that if on a 3-manifold $\pi_1(M)$ contains a product of two cyclic groups, or a subgroup isomorphic to the fundamental group of a compact Riemann surface of genus > 1, then M cannot have a metric of positive scalar curvature. Moreover, it was shown in [SY87] that a closed aspherical 4-manifold cannot admit a metric of positive scalar curvature.

iii) Furhter negative results for 4-manifolds can be obtained using Seiberg-Witten theory. It was shown by Witten and Morgan that on a 4-manifold with $b_2^+(M) > 1$, if the Seiberg-Witten invariant $SW(\xi) \neq 0$ for some *spin*^{c} structure ξ, then M does not admit a metric of positive scalar curvature. Taubes showed that existence of a symplectic structure on a 4-manifold with $b_2^+(M) > 1$ implies the previous condition. We refer to [Ros06] for a summary of results in case $b_2^+(M) = 1$.

In the positive direction, it was shown by Gromov-Lawson and Schoen-Yau [GL, SY79-1] that if M_0 is a manifold (not necessarily connected) of positive scalar curvature, then any manifold M_1 obtained from M_0 by a surgery in codimension ≥ 3 also admits a metric of positive scalar curvature. In dimension $n \geq 5$, the condition $w_2(M) \neq 0$ (where $w_2(M)$ is the second Stiefel-Whitney class of M) implies the existence of metrics with positive scalar curvature.

A.3. **Moduli spaces of metrics of positive scalar curvature.** Denote by $S^+(M)$ the space of metrics of negative scalar curvature on a manifold M of dimension $n \geq 3$ (the space $S^+(\mathcal{S}^2)$ is contractible). In general, $S^+(M)$ is not connected. For example, Hitchin [Hit] showed that on a n-dimensional spin manifold M admitting a metric of positive scalar curvature, $\pi_0(S^+(M)) \neq 0$ if $n \equiv 0$ or $1 \mod 8$, and $\pi_1(S^+(M)) \neq 0$ if $n \equiv 0$ or $-1 \mod 8$. For more general results, we refer to the results of Stolz [Ros06, Thm. 2.3]. Gromov and Lawson proved that $S^+(\mathcal{S}^7)$ has infinitely many components. The same result holds for $M = \mathcal{S}^{4k-1}, k > 2$, cf. [Ros06]. we refer to [Ros06, Thm. 2.7, 2.8] for further results. In dimension 4, Ruberman showed that there exists a simply-connected M^4 with infinitely many metrics of positive scalar curvature that are *concordant* (i.e. restrictions to $s = 0$ and $s = 1$ of a metric of positive scalar curvature on $M \times [0,1]$), but not isotopic.

Appendix B. Validity of applying Adler-Taylor for $h = \Delta_0 f$ on $\mathcal{S}^2$

In this appendix we justify the application of two results. In section B.1 we prove that h satisfies the conditions of Theorem 4.5 due to Adler-Taylor, namely that h is attainable. In section B.2 we prove that the sufficient conditions for (30) hold (i.e. the hypotheses of [AT08], Theorem 14.3.3).

B.1. **Attainability of $h = \Delta_0 f$ on $\mathcal{S}^2$.** We first introduce the definition of attainability for an arbitrary Gaussian random field on the sphere.

Definition B.1. *Let $f : \mathcal{S}^2 \to \mathbb{R}$ a smooth centered Gaussian random field with covariance function $r_f(x,y)$. We say that f is* attainable*, if there exists a countable atlas $\mathcal{A} = (U_\alpha, \psi_\alpha)_{\alpha\in I}$ on $\mathcal{S}^2$, such that for every $\alpha \in I$, $f^\alpha := f \circ \psi_\alpha^{-1}$ defined on $\psi(U_\alpha) \subseteq \mathbb{R}^2$, satisfies:*

(1) *For each $t \in \psi(U_\alpha)$, the joint distributions of*

$$(f_i^\alpha(t), f_{ij}^\alpha(t))_{i<j} \in \mathbb{R}^5$$

are nondegenerate[1], *where* f^α_i *and* f^α_{ij} *are the corresponding partial derivatives of* f^α *of first and second order respectively.*

(2) *We have (cf.* [AT08], *(11.3.1))*

$$\max_{i,j}\left|r_{h_{ij}}(t,t)+r_{h_{ij}}(s,s)-2r_{h_{ij}}(s,t)\right|\le K_\alpha[\ln|t-s|]^{-(1+\beta)}$$

for some $\beta>0$.

The goal of the present section is to prove that the random field $h=\Delta f$ on the 2-dimensional sphere, given by (26), is attainable. As in Theorem 4.3, we assume that the coefficients c_m decay as

$$c_m=O(m^{-s})\text{ for }s>7. \tag{72}$$

First, Lemma 4.4 and the assumptions on the decay of c_m imply that h is C^2 a.s. In fact, from the $C^{k,\beta}$ version of the Sobolev embedding theorem (cf. [Au98, Thm. 2.10, 2nd part]), it follows from the strict inequality $s>7$ that there exists $\beta>0$ such that

$$h\in C^{2,\beta}(\mathcal{S}^2)\text{ a.s.} \tag{73}$$

Next we check conditions (1) and (2) of Definition B.1 of attainability.

For each $y\in\mathcal{S}^2$ let $\delta:T\to\mathcal{S}^2$ be the spherical coordinates with pole at y, where $T=[0,\pi]\times[0,2\pi]$. Namely, we let (θ_y,ϕ_y) be the standard spherical coordinates of y and define

$$\delta_y(\theta,\phi)=(\sin(\theta-\theta_y)\cos(\phi-\phi_y),\sin(\theta-\theta_y)\sin(\phi-\phi_y),\cos(\theta-\theta_y)),$$

and $\psi_y:=\delta_y^{-1}$. Let $x\in\mathcal{S}^2$ be a point and (U_x,ψ_y) be any small chart with $x\in U_x$, $\psi_y(U_x)\subseteq\mathbb{R}^2$ for some $y\in\mathcal{S}^2$. We claim that choosing y appropriately, a sufficiently small chart U satisfies condition (1) of attainability. This is, of course, sufficient to form a finite atlas, by the compactness of the sphere.

First, at any point $t\in U_x$, the random vector

$$H(t)=(h_i(t),h_{ij}(t))_{i<j}\in\mathbb{R}^5$$

is mean zero Gaussian (here the derivatives are w.r.t. the cartesian coordinates in $\mathbb{R}^2$). Therefore we have to check that its covariance matrix $C_{H(t)}\in M_5(\mathbb{R})$ is non-degenerate; by the locality it is sufficient to check that $C_{H(x)}$ is nonsingular. The matrix $C_{H(x)}$ depends, in general, on the choice of y; we are free to choose y as we wish.

It turns out that for y for which $\phi=\frac{\pi}{2}$, $C_{H(x)}$ is of a particularly simple form. For this choice of y, we compute $C_{H(x)}=\sum\limits_{m\ge1}c_mC_m$ (with finite entries), where the single eigenspace covariance matrices C_m are given explicitly by

$$C_m=\begin{pmatrix}\frac{E_m}{2}I_2 & 0_{2\times3}\\ 0_{3\times2} & \Omega^m_{3\times3}\end{pmatrix},$$

with

$$\Omega^m_{3\times3}=\frac{E_m}{8}\begin{pmatrix}3E_m-2 & E_m+2 & \\ E_m+2 & 3E_m-2 & \\ & & E_m-2\end{pmatrix},$$

[1]Here the nondegeneracy means that the Gaussian measure on $\mathbb{R}^5$ is not supported on a proper linear subspace of $\mathbb{R}^5$; equivalently, the covariance matrix is nonsingular.

and it is then easy to use (72) in order to check that the entries of $C_{H(x)}$ are finite and the matrix is nonsingular.

Remark B.2. *A priori, it seems that non-degeneracy of H in one point is sufficient, thanks to the isotropic property of h. However, one should bear in mind that introducing a chart breaks the symmetry, so that the second derivatives are no longer isotropic, being dependent on the local properties of the corresponding frame. This is unlike the first (directional) derivatives, which depend only on the direction of the frame at the given point.*

As for condition (2) of Definition B.1, it follows easily from (73), the latter implying

$$r_{h_{ij}}(\cdot,t), r_{h_{ij}}(t,\cdot) \in C^{0,\beta}(\mathcal{S}^2)$$

for every $t \in \mathcal{S}^2$.

B.2. **Relation of the expected Euler characteristic of the excursion set and the excursion probability.** The goal of the present section is to justify the application of [AT08], Theorem 14.3.3 on $h = \Delta f$ given by (26). Recall that the covariance function of h is r_h, given by (28).

In addition to the assumptions already validated in the previous section we are required to show that

$$r_h(x,y) = 1 \quad \Leftrightarrow \quad x = y \tag{74}$$

(recall that for every $x \in \mathcal{S}^2$ we have $r(x,x) = 1$ by the assumption (23)). This condition rules out degeneracies such as periodic processes.

We claim that (74) holds if and only if there exists an *odd* m_0 so that

$$c_{m_0} > 0. \tag{75}$$

That is guaranteed by one of the assumptions in Theorem 4.3.

To see that we note ([Sz, (7.21.1)]) that for every $m \geq 1$, $|P_m(t)| \leq 1$ for $t \in [-1,1]$, $P_m(1) = 1$;

$$|P_m(t)| = 1 \Leftrightarrow t = \pm 1,$$

and P_m is even or odd, for m even or odd respectively. Thus we have by (28)

$$|r_h(x,y)| \leq \sum_{m=1}^{\infty} c_m = 1$$

by (23), and the equality may hold only if $\cos(d(x,y)) = \pm 1$, i.e. $x = \pm y$. In case $x = -y$ this may not hold by (75).

References

[AT03] R. Adler and J. Taylor. *Euler Characteristics for Gaussian fields on manifolds.* The Annals of Probability 31 (2003), No. 2, 533–563.

[ATT05] R. Adler, A. Takemura and J. Taylor. *Validity of the expected Euler characteristic heuristic.* The Annals of Probability 33 (2005), No. 4, 1362–1396.

[AT08] R. Adler and J. Taylor. *Random fields and geometry.* Springer, 2008.

[And05] M. Anderson. *On uniqueness and differentiability in the space of Yamabe metrics.* Commun. Contemp. Math. 7 (2005), no. 3, 299–310.

[Au76] T. Aubin. *The scalar curvature.* In *Differential geometry and relativity*, Mathematical Phys. and Appl. Math., Vol. 3, Reidel, Dordrecht, 1976, 5–18.

[Au98] T. Aubin. *Some nonlinear problems in Riemannian geometry.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[Bes] A. Besse. *Einstein Manifolds*, Springer Verlag, Berlin, 1987.

[Bl] D. Bleecker. *Non-perturbative conformal field theory.* Class. Quantum Grav. 4 (1987), 827–849.

[Bor] C. Borell. *The Brunn-Minkowski inequality in Gauss space.* Invent. Math. 30 (1975), 205–216.

[Bran] T. Branson. *Differential operators canonically associated to a conformal structure.* Math. Scand. 57 (1985), 293–345.

[BG] T. Branson and A. Rod Gover. *Origins, applications and generalisations of the Q-curvature.* Acta Appl. Math. 102 (2008), no. 2-3, 131–146.

[Bryant] R. Bryant. *Minimal Surfaces of Constant Curvature in S^n.* Transactions of the AMS, 290, No. 1 (1985), 259–271.

[CL] H. Cramer and M. R. Leadbetter. *Stationary and related stochastic processes.* John Wiley & Sons, 1967, New York.

[CY] S.Y.A. Chang and P. Yang. *Extremal metrics of zeta function determinants on 4-manifolds.* Ann. of Math. 142 (1995), 171–212.

[Ch] I. Chavel. *Eigenvalues in Riemannian geometry.* Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984.

[Cl] B. Clarke. *The metric geometry of the manifold of Riemannian metrics over a closed manifold.* Calc. Var. Partial Differential Equations 39 (2010), no. 3-4, 533–545.

[DM] Z. Djadli and A. Malchiodi. *Existence of conformal metrics with constant Q-curvature.* Ann. of Math. (2) 168 (2008), no. 3, 813–858.

[DS] B. Duplantier and S. Sheffield. *Liouville Quantum Gravity and KPZ.* Invent. Math. 185 (2011), no. 2, 333–393.

[Eb] D. Ebin. *The manifold of Riemannian metrics.* 1970 Global Analysis, Proc. Symp. Pure Math., Vol. XV (1968), 11–40, AMS.

[EI02] A. El Soufi and S. Ilias. *Critical metrics of the trace of the heat kernel on a compact manifold.* J. Math. Pures Appl. 81 (2002), 1053-1070.

[EI03] A. El Soufi and S. Ilias. *Extremal metrics for the first eigenvalue of the Laplacian in a conformal class.* Proc. Amer. Math. Soc. 131 (2003), no. 5, 1611–1618.

[EI08] A. El Soufi and S. Ilias. *Laplacian eigenvalue functionals and metric deformations on compact manifolds.* J. Geom. Phys. 58 (2008), no. 1, 89–104.

[FG] C. Fefferman and R. Graham. *Q-curvature and Poincaré metrics.* Math. Res. Lett. 9 (2002), 139–151.

[FH] C. Fefferman and K. Hirachi. *Ambient metric construction of Q-curvature in conformal and CR geometries.* Math. Res. Lett. 10 (2003), 819–832.

[FrGr] D. Freed and D. Groisser. *The basic geometry of the manifold of Riemannian metrics and of its quotient by the diffeomorphism group.* Michigan Math. J. 36 (1989), no. 3, 323–344.

[Gilk] P. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* Mathematics Lecture Series, 11. Publish or Perish, Inc., Wilmington, DE, 1984.

[GM] O. Gil-Medrano and P. Michor. *The Riemannian manifold of all Riemannian metrics.* Quart. J. Math. Oxford Ser. (2) 42 (1991), no. 166, 183–202.

[Gov] A. R. Gover. *Laplacian operators and Q-curvature on conformally Einstein manifolds.* Math. Ann. 336 (2006), no. 2, 311–334.

[GJMS] C.R. Graham, R. Jenne, L.J. Mason and G.A. Sparling. *Conformally invariant powers of the Laplacian, I: Existence.* J. Lond. Math. Soc. 46 (1992), 557–565.

[GZ] C. Graham and M. Zworski. *Scattering matrix in conformal geometry.* Invent. Math. 152 (2003), 89–118.

[GL] M. Gromov and H. Blaine Lawson. *The classification of simply connected manifolds of positive scalar curvature.* Ann. of Math. (2) 111 (1980), no. 3, 423–434.

[Her] J. Hersch. *Quatre propriétés isopérimétriques de membranes sphériques homogènes.* C. R. Acad. Sci. Paris Sér. A-B 270 (1970), A1645–A1648.

[Hit] N. Hitchin. *Harmonic spinors.* Advances in Math. 14 (1974), 1–55.

[JLNNP] D. Jakobson, M. Levitin, N. Nadirashvili, N. Nigam and I. Polterovich. *How large can the first eigenvalue be on a surface of genus two?* Int. Math. Res. Not. 2005, no. 63, 3967–3985.

[Kah] J.-P. Kahane. *Random Series of Functions,* 2nd ed. Cambridge Studies in Advanced Mathematics 5, Cambridge Univ. Press, 1985.

[Kat] M. Katagiri. *On the topology of the moduli space of negative constant scalar curvature metrics on a Haken manifold.* Proc. Japan Acad. 75 (A), 126–128.

[KW] J. Kazdan and F. Warner. *Scalar curvature and conformal deformation of Riemannian structure.* J. Differential Geometry 10 (1975), 113–134.

[KPZ] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov. *Fractal structure of 2D-quantum gravity.* Modern Phys. Lett. A, 3, no. 8 (1988), 819–826.

[LY] P. Li and S. T. Yau. *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces.* Invent. Math. 69 (1982), 269–291.

[Li] A. Lichnerowicz. *Spineurs harmoniques.* C. R. Acad. Sci. Paris 257 (1963), 7–9.

[Lo] J. Lohkamp. *The space of negative scalar curvature metrics.* Invent. Math. 110 (1992), 403–407.

[Morg] F. Morgan. *Measures on Spaces of Surfaces.* Archive for Rational Mechanics and Analysis 78, no. 4 (1982), 335–359.

[Mor] C. Morpurgo. *Local Extrema of Traces of Heat Kernels on S^2.* Jour. Func. Analysis 141 (1996), 335–364.

[Nad] N. Nadirashvili. *Berger's isoperimetric problem and minimal immersions of surfaces.* Geom. Funct. Anal. 6 (1996), no. 5, 877–897.

[NS] N. Nadirashvili and Y. Sire. *Conformal Spectrum and Harmonic maps.* arXiv:1007.3104.

[N] C. B. Ndiaye. *Constant Q-curvature metrics in arbitrary dimension.* J. Funct. Anal. 251 (2007), no. 1, 1–58.

[Pol] A. M. Polyakov. *Quantum geometry of bosonic strings.* Phys. Lett. B, 103, no. 3 (1981), 207-210.

[Ros86] J. Rosenberg. *C^*-algebras, positive scalar curvature, and the Novikov conjecture. III* Topology 25 (1986), no. 3, 319–336.

[Ros06] J. Rosenberg. *Manifolds of positive scalar curvature: a progress report.* Surveys in differential geometry. Vol. XI, 259–294, Surv. Differ. Geom., 11, Int. Press, Somerville, MA, 2007.

[Sch84] R. Schoen. *Conformal deformation of a Riemannian metric to constant scalar curvature.* J. Differential Geom. 20 (1984), no. 2, 479–495.

[Sch87] R. Schoen. *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, In Lecture Notes in Mathematics, Vol. 1365, Springer Verlag, 1987, 120–154.

[SY79-1] R. Schoen and S. T. Yau. *On the structure of manifolds with positive scalar curvature.* Manuscripta Math. 28 (1979), no. 1-3, 159–183.

[SY79-2] R. Schoen and S. T. Yau. *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature.* Ann. of Math. (2) 110 (1979), no. 1, 127–142.

[SY82] R. Schoen and S. T. Yau. *Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature, Seminar on Differential Geometry.* Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 209–228.

[SY87] R. Schoen and S. T. Yau. *The structure of manifolds with positive scalar curvature.* Directions in partial differential equations (Madison, WI, 1985), Publ. Math. Res. Center Univ. Wisconsin, vol. 54, Academic Press, Boston, MA, 1987, 235–242.

[Sm] N. Smolentsev. Spaces of Riemannian metrics. Russian: Sovrem. Mat. Prilozh. No. 31, Geometriya (2005), 69–147; translation in J. Math. Sci. (N. Y.) 142 (2007), no. 5, 2436–2519.

[Sz] G. Szegö. *Orthogonal Polynomials.* AMS Colloquium Publications, Vol. 23, 2003.

[Tak] , T. Takahashi. *Minimal immersions of Riemannian manifolds.* J. Math. Soc. Japan 18 (1966), 380–385.

[Tr] N. Trudinger. *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds.* Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 265–274.

[TIS] B. Tsirelson, I. Ibragimov and V. Sudakov. *Norms of Gaussian sample functions.* Proceedings of the Third Japan-USSR Symposium on Probability Theory (Tashkent, 1975), pp. 20–41. Lecture Notes in Math., Vol. 550, Springer, Berlin, 1976.

[W] I. Wigman. *On the distribution of the nodal sets of random spherical harmonics.* arXiv:0805.2768

[W1] I. Wigman. Fluctuations of the nodal length of random spherical harmonics. arXiv:0907.1648

[Yam] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. 12 (1960), 21–37.

[YY] P. Yang and S. T. Yau. *Eigenvalues of the laplacian on compact Riemann surfaces and minimal submanifolds.* Ann. Sc. Norm. Sup. de Pisa 7 (1980), 55–63.

[Yau74] S. T. Yau. *Submanifolds with Constant Mean Curvature.* American Jour. of Math. 96, No. 2 (1974), pp. 346–366.

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