ZERO AND NEGATIVE EIGENVALUES OF THE CONFORMAL LAPLACIAN

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This paper is dedicated to the memory of Yuri Safarov

Abstract. We show that zero is not an eigenvalue of the conformal Laplacian for generic Riemannian metrics. We also discuss non-compactness for sequences of metrics with growing number of negative eigenvalues of the conformal Laplacian.

1. Introduction

In [CGJP14, CGJP13], the authors studied spectra and eigenfunctions of conformally covariant operators on compact manifolds of dimension $n \geq 3$. They showed, in particular, that the number of negative eigenvalues of conformal Laplacian is unbounded on any such manifold. One of the questions left open in those papers was whether for a generic Riemannian metric g on a compact n-dimensional manifold M, 0 is an eigenvalue of the conformal Laplacian $Y_g := -\Delta_g + c_n R_g$. The operator Y_g is also called the Yamabe operator. Here $-\Delta_g$ is the nonnegative-definite Laplacian for the metric g, $c_n := (n-2)/(4(n-1))$, and R_g denotes the scalar curvature of g. In this paper we address that question.

Our first main result is

Theorem 1.1. *For generic smooth metrics g on M, zero is not an eigenvalue of Y_g.*

It follows from a transformation formula for Y_g that if 0 is an eigenvalue of Y_{g_0}, then it is also an eigenvalue of Y_{g_1} for all metrics g_1 in the conformal class $[g_0]$. Accordingly, one needs to change conformal class to find metrics for which 0 is *not* an eigenvalue of Y_g.

Also, 0 is an eigenvalue of Y_g for conformally *scalar flat* metrics g, i.e. those metrics lying in a conformal class $[g_0]$ of a scalar-flat metric g_0, such that $R_{g_0} \equiv 0$. The corresponding eigenfunction u is given by $u(g_0) \equiv 1$, and by a suitable power of the conformal factor one obtains the eigenfunction for $g \in [g_0]$.

It is also clear that 0 is not an eigenvalue of Y_g for metrics g with *positive* scalar curvature R_g, and hence in the corresponding conformal classes. However, it is known that some manifolds do not admit metrics with positive scalar curvature, [KW75]. Accordingly, in the current paper we restrict ourselves to metrics lying in conformal classes of metrics with *negative* scalar curvature.

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Remark 1.2. We remark that our proof of Theorem 1.1 works *verbatim* to show an analogous statement for operators $P_{g,c} = \Delta_g + cR_g$, where $c \in \mathbf{R}$ is any constant satisfying $c \neq 0, c \neq 1/2$.

The crucial result is Proposition 4.1, and the only part of the proof where the numerical value of c becomes important is the argument after the equation (3), where it is necessary that $2c - 1 \neq 0$, hence $c \neq 1/2$. Also, we assume that the corresponding eigenfunctions are orthogonal to constants, hence we require that $c \neq 0$. Note that, $P_{g,c}$ is only conformally covariant for $c = c_n$, so for other values of c an argument using conformal perturbations (as in [BW80]) should also work. The case $c = 0$ corresponds to the very-well studied of eigenvalues of the Laplacian Δ_g.

Perturbation theory of conformally covariant operators was previously considered in [Can14, Pon]; see also [Mai97, Dah03, Dah08] for the corresponding results for the Dirac operator. Applications to eigenvalue multiplicity of nonzero eigenvalues were discussed in [Can14]; arbitrary eigenvalues were considered in [Pon], but the question of whether $\ker Y_g$ is generically empty was not settled. It seems interesting to understand whether 0 is generically a *simple* eigenvalue of Y_g, among those metrics for which it *is* an eigenvalue of Y_g.

Another question considered in this paper concerns the study of sequence of metrics g_k such that the number of negative eigenvalues of Y_{g_k} increases. Recall that it was shown in [CGJP14] that the results in [Loh96] imply that the number of negative eigenvalues of Y_g can become arbitrarily large for metrics g on any compact manifold of dimension ≥ 3; thus, it seems natural to ask what is the geometric significance of the increasing number of negative eigenvalues of Y_g.

In Section 5, we show that a sequence of metrics g_k, such that the number of negative eigenvalues of Y_{g_k} increases, cannot satisfy two natural "pre-compactness" conditions (see Proposition 5.3).

2. The space of conformal structures

Let M be a compact orientable manifold of dimension $n \geq 3$; we denote by $\mathcal{M}$ the space of all Riemannian metrics on M. For simplicity, we only consider C^∞ metrics on M, although the regularity can be lowered significantly.

Definition 2.1. Given $k \geq 1$, we denote by $\mathcal{M}_{0,k}$ the set of all metrics g on M s.t. the multiplicity of 0 as an eigenvalue of Y_g is at least k.

As we remarked in section 1, if $g_0 \in \mathcal{M}_{0,k}$, then so is every metric g in the conformal class $[g_0]$; also that condition is invariant under composition with diffeomorphisms of M. Consider the action on $\mathcal{M}$ of the group $\mathcal{P}$ of (pointwise) conformal transformations (multiplication by positive functions), as well as by the group $\mathcal{D}$ of diffeomorphisms; we shall denote by $\mathcal{D}_0$ the subgroup of $\mathcal{D}$ of diffeomorphisms isotopic to identity. It seems natural to consider the *Teichmuller space of conformal structures*

$$\mathcal{T}(M) = \frac{\mathcal{M}/\mathcal{P}}{\mathcal{D}_0},$$

or the *Riemannian moduli space of conformal structures*

$$\mathcal{R}(M) = \frac{\mathcal{M}/\mathcal{P}}{\mathcal{D}},$$

in the terminology of Fischer and Monkrief, [FM96, FM97].[1]

Definition 2.2. We denote by $\mathcal{T}_{0,k}(M)$ the Teichmuller space of conformal structures corresponding to metrics $g_0 \in \mathcal{M}_{0,k}$, i.e. the projection of $\mathcal{M}_{0,k}$ into $\mathcal{T}(M)$.

The meaning of Theorem 1.1 is the following, and we prove this in Section 4

Theorem 2.3. *The complement $\mathcal{T}_{0,1}^c$ of the set $\mathcal{T}_{0,1}(M)$ in $\mathcal{T}(M)$ is open and dense in $\mathcal{T}(M)$.*

3. Curves of metrics

Let g_0 be a metric on M such that 0 is an eigenvalue of Y_{g_0} with multiplicity m, so, $g_0 \in \mathcal{M}_{0,m}$ (recall the definition 2.1). We note that it was shown in [BD03, Lemma 3.4] that the eigenvalues of Y_g depend continuously on g in the C^1-topology (see also [KS60]). Thus, $\mathcal{M}_{0,m_2}$ is a closed submanifold of $\mathcal{M}_{0,m_1}$ for $0 \leq m_1 < m_2$, in the C^k topology for any $k \geq 1$. We would like to compute the tangent space to $\mathcal{M}_{0,m}$, at g_0, for $m \geq 1$.

Denote by E_0 the zero eigenspace of Y_{g_0}; it has dimension m. Let Π_0 denote the orthogonal projection into E_0 with respect to $L^2(M, dV_{g_0})$. Consider a curve g_t of metrics on M passing through g_0 at $t = 0$; denote the t-derivative by $\cdot$

Let $\dot{g}(0) = h$, i.e. $g(t) = g_0 + th + o(t)$. We denote by $Q_{g_0,h}$ the operator

$$Q_{g_0,h} := \Pi_0 \dot{Y_g} = \Pi_0(c_n \dot{R} - \dot{\Delta}) : E_0 \to E_0, \tag{1}$$

Sometimes when the dependence on the metric g_0 is clear, we shall omit the subscript g_0 and simply write Q_h.

We have the following:

Proposition 3.1. *The tangent space to $\mathcal{M}_{0,k}$ at g_0 consists of all the tensors*

$$\mathcal{H}_{0,k} := \{h : \ 0 \textit{ is an eigenvalue of } Q_{g_0,h} \textit{ of multiplicity } \geq k\}. \tag{2}$$

Proof of Proposition 3.1. We refer to [Rel69, page 74] and the discussion in [Can14, §4,5] for basic results about the perturbation theory of conformally covariant operators; see also [DWW05], where some important formulas that we use in our argument were derived.

It follows from general theory that for a real–analytic family of self-adjoint operators $\{Y_{g(t)}\}$, eigenvalue and eigenfunction branches can be chosen to depend analytically on the perturbation parameter t, for t small enough, see for example [Ber73, Lemma 3.15]. Moreover, there is a positive constant ϵ such that, for t small enough, the number of eigenvalues of $Y_{g(t)}$ in the interval $(-\epsilon, \epsilon)$ is equal to m, where $m \geq k$ is the multiplicity of 0 for Y_{g_0}; and the eigenvalue derivatives are equal to the eigenvalues of $Q_{g_0,h}$. In particular, $Q_{g_0,h} \equiv 0$ for any real–analytic perturbation of g_0 when $m = k$. The result is now immediate from the definition of $\mathcal{M}_{0,k}$ and $\mathcal{H}_{0,k}$.

□

It is well-known (see e.g. [BE69, FM77]) that the tangent space to $\mathcal{T}(M)$ at g_0 may be identified with the space of all *transverse traceless* symmetric tensors h

[1]If M is an orientable two-dimensional manifold, then $\mathcal{T}(M)$ (resp. $\mathcal{R}(M)$) are the usual Teichmuller (resp. moduli) spaces. In [FM97], the space $\mathcal{T}(M)$ for Haken 3-manifolds M of degree 0 is proposed as a configuration space for a Hamiltonian reduction of Einstein's vacuum field equations.

satisfying $\mathrm{tr}_{g_0} h = 0, \delta h = 0$. Clearly, the tangent space to $\mathcal{T}_{0,k}(M)$ at g_0 consists of all transverse traceless tensors lying in $\mathcal{H}_{0,k}$.

4. Nullspace of Y_g

In this section we prove Theorem 2.3. We keep the notation from section 3. For convenience we shall assume that g_0 is a *Yamabe* metric, i.e. that $R_{g_0} \equiv -1$.

Theorem 2.3 will follow from the following important result:

Proposition 4.1. *There exists a symmetric tensor h such that $Q_{g_0,h} \not\equiv 0$.*

We postpone the proof that Proposition 4.1 implies Theorem 2.3 until later, and first prove the Proposition.

Proof of Proposition 4.1.

Let g_t be a curve of metrics real-analytic in t, and $\psi \not\equiv 0$ be an element of E_0 that belongs to the Rellich basis of g_t, i.e. ψ is an eigenvector of the operator Q_h defined in (1).

Differentiating the eigenfunction equation

$$(-\Delta + c_n R)\psi = \lambda\psi,$$

we find that

$$\dot{\lambda}\psi = (-\Delta - \lambda + c_n R)\dot{\psi} + (c_n \dot{R} - \dot{\Delta})\psi.$$

It suffices to show that there exists a metric deformation g_t real-analytic in t such that $\dot{\lambda} \neq 0$.

Take the inner product (with respect to dV_{g_0}) of both sides with ψ. Since $(-\Delta - \lambda + c_n R)$ is self-adjoint, we find that

$$((-\Delta - \lambda + c_n R)\dot{\psi}, \psi) = (\dot{\psi}, (-\Delta - \lambda + c_n R)\psi) = 0.$$

Then

$$\dot{\lambda}(\psi, \psi) = ((c_n \dot{R} - \dot{\Delta})\psi, \psi).$$

We assume for contradiction that $\dot{\lambda} = 0$ for *any* perturbation g_t, i.e. that (1) is identically zero for any g_t.

We next give the expressions for $\dot{\Delta}$ and $\dot{R}$. We need to recall some notation. Let $C^\infty(\otimes^r T^*M)$ be the space of $(r,0)$-tensors on M, and $C^\infty(M) = C^\infty(\otimes^0 T^*M)$. We consider the covariant derivative

$$\nabla : C^\infty(\otimes^r T^*M) \to C^\infty(\otimes^{r+1} T^*M),$$

which in local coordinates is given by

$$\nabla\alpha = \sum_i \nabla_i \alpha \otimes dx_i.$$

Notice that $d = \nabla : C^\infty(M) \to C^\infty(T^*M)$. We denote the formal adjoint of ∇ by

$$\delta : C^\infty(\otimes^{r+1} T^*M) \to C^\infty(\otimes^r T^*M),$$

i.e. for every $\alpha \in C^\infty(\otimes^r T^*M)$ and $\beta \in C^\infty(\otimes^{r+1} T^*M)$, $(\nabla\alpha, \beta) = (\alpha, \delta\beta)$. Here, $(\cdot,\cdot) = \int_M \langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the pointwise inner product. We can now recall the expressions for $\dot{\Delta}$ and $\dot{R}$ computed in [DWW05, (2.5) and (2.6)], see also [Ber70, Ber73]. They are,

$$\dot{R} = -\langle h, \mathrm{Ric}\rangle + \delta^2 h - \Delta \mathrm{tr} h,$$

and

$$\dot{\Delta} f = -\langle h, \nabla^2 f\rangle + \langle \delta h + \frac{1}{2} d\mathrm{tr} h, df\rangle,$$

where $\nabla^2 f$ is the Hessian of f, and δ^2 is the formal adjoint of the Hessian. Recall the pointwise inner product on $C^\infty(\otimes^2 T^*M)$ is

$$\langle \alpha, \beta\rangle = \sum_{i,j} \alpha^{ij}\beta_{ij};$$

in particular, $\mathrm{tr}_g h = \langle g, h\rangle$. We know *apriori* that the only metric deformations that will change the eigenvalue 0, are *transverse traceless* deformations h of g. However, we shall only insist that h is *traceless*, and so $\mathrm{tr}_g h \equiv 0$. Then the previous expressions simplify to

$$\dot{R} = -\langle h, \mathrm{Ric}\rangle + \delta^2 h$$

and

$$\dot{\Delta} f = -\langle h, \nabla^2 f\rangle + \langle \delta h, df\rangle.$$

Let $A := \dot{\lambda}(\psi, \psi)$. Combining the above expressions, we find that

$$\begin{aligned}
A &= (\langle h, \nabla^2\psi - c_n\psi\mathrm{Ric}\rangle - \langle \delta h, d\psi\rangle + c_n\psi\delta^2 h, \psi) \\
&= (h, \psi\nabla^2\psi) - c_n(h, \psi^2\mathrm{Ric}) - (h, \nabla(\psi d\psi)) + c_n(\psi\delta^2 h, \psi) \\
&= (h, \psi\nabla^2\psi) - c_n(h, \psi^2\mathrm{Ric}) - (h, \psi\nabla^2\psi) - (h, d\psi\otimes d\psi) + c_n(\psi\delta^2 h, \psi) \\
&= -(h, c_n\psi^2\mathrm{Ric}) - (h, d\psi\otimes d\psi) + (\delta^2 h, c_n\psi^2) \\
&= \int_M \langle h, c_n(\nabla^2\psi^2 - \psi^2\mathrm{Ric}) - d\psi\otimes d\psi\rangle \\
&= \int_M \langle h, c_n\psi(2\nabla^2\psi - \psi\mathrm{Ric}) + (2c_n - 1)d\psi\otimes d\psi\rangle.
\end{aligned}$$

To get the last equality, we use the identity $\nabla^2\psi^2 = 2(\psi\nabla^2\psi + d\psi\otimes d\psi)$. Using the assumption that $\mathrm{tr}_g h = \langle g, h\rangle = 0$, we find that

$$A = \int_M \langle h, c_n\psi(2\psi\mathring{\nabla}^2\psi - \psi^2\mathring{\mathrm{Ric}}) + (2c_n - 1)(d\psi\otimes d\psi)^o\rangle,$$

where $\mathring{V} = V - \frac{1}{n}\mathrm{tr}_g V g$ is the traceless part of the corresponding expression V.

Putting $h = c_n\psi(2\psi\mathring{\nabla}^2\psi - \psi^2\mathring{\mathrm{Ric}}) + (2c_n - 1)(d\psi\otimes d\psi)^o$ (which is symmetric and traceless), we find that $A = 0$ if and only if

$$c_n\psi(2\psi\mathring{\nabla}^2\psi - \psi^2\mathring{\mathrm{Ric}}) + (2c_n - 1)(d\psi\otimes d\psi)^o \equiv 0. \tag{3}$$

We next remark that equation (3) has no non-trivial solutions. Indeed, take g_0 to be the Yamabe metric; recall we assume that $R_{g_0} \equiv -1$. By assumption ψ_{g_0} is an eigenfunction of Δ_{g_0} with eigenvalue c_n, hence it is L^2-orthogonal to the constant function, and changes sign on M. Let $\mathcal{N}(\psi)$ denote the nodal set of ψ. The term $c_n\psi(2\psi\mathring{\nabla}^2\psi - \psi^2\mathring{\mathrm{Ric}})$ vanishes on $\mathcal{N}(\psi)$, so it follows from (3) that

$$(d\psi\otimes d\psi)^o \equiv 0,$$

on $\mathcal{N}(\psi)$. This is equivalent to

$$d\psi\otimes d\psi = \frac{1}{n}|d\psi|_g^2\cdot g$$

Now, the right-hand side has rank n whenever $|d\psi|_g \neq 0$. On the other hand, the left-hand side has rank ≤ 1. The only way the equality is possible if both side are identically zero on $\mathcal{N}(\psi)$, i.e. if

$$d\psi\,\big|_{\mathcal{N}(\psi)} \equiv 0. \tag{4}$$

However, it is well-known (see e.g. [Che76, Han94, HHL98, HHHN99]) that the intersection of the nodal and critical sets of ψ has locally finite Hausdorff $(n-2)$-dimensional measure, and so (4) is impossible for non-zero ψ. This contradiction finishes the proof of Proposition 4.1.

□

An immediate consequence of the proof of Proposition 4.1 is the following corollary.

Corollary 4.2. *Let g_0 be a Yamabe metric on M, $g_0 \in \mathcal{M}_{0,m}(M) \cap \mathcal{M}_{0,m+1}(M)^c$. Let $h = c_n\psi(2\psi\mathring{\nabla}^2\psi - \psi^2\mathring{\mathrm{Ric}}) + (2c_n - 1)(d\psi \otimes d\psi)^o$, where $\psi = \psi_{g_0}$ is a nonzero eigenfunction of Y_{g_0} with eigenvalue 0. Consider the perturbation $g(t) = g_0 + th$. Then for every $\epsilon > 0$, there exists $|t| \leq \epsilon$ such that $g(t) \in \mathcal{M}^c_{0,m}$.*

To complete the proof of Theorem 2.3, we need to prove the following

Claim 4.3. Proposition 4.1 implies Theorem 2.3.

Proof of Claim 4.3. Since $\mathcal{T}_{0,1}(M)$ is a *closed* subspace of $\mathcal{T}(M)$, its complement $\mathcal{T}_{0,1}(M)^c$ is clearly open in $\mathcal{T}(M)$. We thus need to show that $\mathcal{T}_{0,1}(M)^c$ is *dense* in $\mathcal{T}(M)$. We shall show that $\mathcal{M}_{0,1}(M)^c$ is dense in $\mathcal{M}(M)$. Let g_0 be a metric on M. It suffices to show that $\mathcal{M}_{0,1}(M)^c$ is dense in some neighbourhood U of g_0 in $\mathcal{M}(M)$. If $g_0 \in \mathcal{M}_{0,1}(M)^c$, we are done, so we can assume that $g_0 \in \mathcal{M}_{0,1}(M)$.

The proof proceeds by induction on the dimension m of E_0. We note that m is finite for any g_0, and that Proposition 4.1 was proved for *arbitrary* m. First, let $m = 1$, meaning 0 is a simple eigenvalue of Y_{g_0}. By Corollary 4.2, we know that one can choose a curve of metrics $g(t)$, real analytic in t, such that $g(0) = g_0$ and $g(t) \notin \mathcal{M}_{0,1}$ for arbitrary small t.[2] Hence, the proof in case $m = 1$ is complete.

Next, assume that we have shown that $\mathcal{M}_{0,1}(M)^c$ is dense in any neighborhood U of any metric $g_0 \in \mathcal{M}_{0,1}(M)$ such that zero is an eigenvalue of Y_{g_0} with the multiplicity at most $m-1$; we would like to prove the corresponding statement for a metric g_0 such that 0 is an eigenvalue of Y_{g_0} with multiplicity *exactly* m. By Corollary 4.2, there exists a small perturbation that *decreases* the multiplicity m of 0 as an eigenvalue of Y_g. By the inductive hypothesis, it follows that for any neighborhood U of g_0, there exists a metric $g_1 \in U$, such that 0 is an eigenvalue of Y_{g_1} with multiplicity $\leq m-1$, and in a suitable neighborhood V of g_1 (which can be chosen to satisfy $V \subset U$), have a nonempty intersection with $\mathcal{M}_{0,1}(M)^c$. This completes the proof of the Claim 4.3, and hence also of Theorems 2.3 and 1.1.

□

[2]It will then follow that (in the notation of [Tey99]), the space of conformal structures corresponding to metrics in $\mathcal{M}_{0,1}$ is of *meager codimension* 1 in the space of all conformal structures; we leave the details of the argument to the reader.

5. Negative eigenvalues of the conformal Laplacian

In [CGJP14, CGJP13, ES14] the authors showed that the number of negative eigenvalues of the conformal Laplacian cannot be unifromly bounded above on any compact manifold M of dimension $n \geq 3$. Accordingly, it seems interesting to consider sequences of metrics g_k on M where the number of negative eigenvalues of $Y_{g_k} = -\Delta_{g_k} + c_n R_{g_k}$ is growing.

It is known that the set of metrics g_k on a manifold M of dimension $n \geq 3$ is *pre-compact* in Gromov-Hausdorff topology if it satifies either condition 5.1 or condition 5.2 below:

Condition 5.1. *The volume* $\mathrm{Vol}(M, g_k) \leq V < \infty$ *is bounded above; the injectivity radius* $\mathrm{inj}(M, g_k) \geq r > 0$ *is bounded from below; the Ricci curvature* $\mathrm{Ric}(M, g_k) \geq -a^2$ *is bounded from below.*

Condition 5.2. *The diameter* $\mathrm{diam}(M, g_k) \leq D < \infty$ *is bounded above; the Ricci curvature* $\mathrm{Ric}(M, g_k) \geq -a^2$ *is bounded from below.*

Consider a sequence of metrics $\tilde{g}_k$ on a fixed Riemannian manifold such that the number of negative eigenvalues of the conformal Laplacian $Y_{\tilde{g}_k}$ goes to infinity. It is natural to choose a unique Yamabe representative g_k in the conformal class $[\tilde{g}_k]$; the scalar curvature of g_k is constant and equal to -1; the number of negative eigenvalues of Y_{g_k} and $Y_{\tilde{g}_k}$ are equal.

Proposition 5.3. *The sequence g_k cannot satisfy the pre-compactness condition 5.1; nor can it satisfy the condition 5.2.*

Proof of Proposition 5.3: The result follows from [Bus82, Thm. 6.2] and [Gro99, Appendix C]. Indeded, since g_k is Yamabe, the number of negative eigenvalues of Y_{g_k} is equal to the number $N(\frac{n-2}{4(n-1)}; g_k)$ of eigenvalues of the Laplacian $-\Delta_{g_k}$ that are less than $(n-2)/(4(n-1))$. Assuming g_k satisfies 5.1, it follows from [Bus82, Thm. 6.2] that $N(\frac{n-2}{4(n-1)}; g_k) \leq C_1 < \infty$ where the constant C_1 only depends on V, r, n, δ. Similarly, assuming g_k satisfies Condition 5.2, it follows from Gromov's result in [Gro99, Appendix C] that $N(\frac{n-2}{4(n-1)}; g_k) \leq C_2 < \infty$ where the constant C_2 only depends on D, n, a. These contradict the assumption on the number of negative eigenvalues of Y_{g_k}. $\square$

Proposition 5.3 shows that sequences of metrics with increasing number of negative eigenvalues of Y_{g_k} cannot stay in the "thick" part of $\mathcal{M}$ satisfying natural pre-compactness conditions 5.1 or 5.2, and thus we cannot use those conditions to choose a convergent subsequence of metrics. On the other hand, we remark that on certain high-dimensional manifolds (cf. [GL80]) there exist infinitely many connected components of the set of metrics with *positive* scalar curvature. Accordingly, the sequence of metrics can diverge but the number of negative eigenvalues of Y can stay equal to 0.

5.1. **Example: product of a surface with another manifold.**

We consider (a slight modification of) one of the examples discussed in [CGJP14, §4]. Let M be a manifold of dimension $d \geq 2$, and let Σ be a Riemann surface of genus $\gamma \geq 2$. Assume that M admits a metric with positive scalar curvature, and fix a Yamabe metric G on M with scalar curvature $R_G > 0$. Fix $\epsilon > 0$. By a result of Buser [Bus77, Theorem 4], for every $k \geq 1$, there exists a hyperbolic metric h_k on Σ such that the hyperbolic Laplacian $-\Delta_{h_k}$ has at least k eigenvalues in the interval $(1/4, 1/4+\epsilon)$. Choose k of those eigenvalues and denote them by $1/4 < \lambda_{k,1} \le \lambda_{k,2} \le \ldots \le \lambda_{k,k} < 1/4+\epsilon$. Denote the corresponding eigenfunctions by $u_{k,j}, 1 \le j \le k$.

Consider the product metric $g_k := (G \otimes t^{-1}h_k)$ on $M \times \Sigma$, where t is a positive constant to be chosen later. It is easy to show that the scalar curvature of g_k is equal to $R_G - 2t$ for all k (the Gauss curvature of (Σ, h_k) is equal to -1). If we choose

$$t > R_G/2, \tag{5}$$

then the scalar curvature of g_k will be negative.

Denote the coordinates on $M \times \Sigma$ by (x, y). Then the conformal Laplacian is given by

$$Y_{g_k} = -\Delta_{G,x} - t\Delta_{h_k,y} + \frac{d}{4(d+1)}(R_G - 2t).$$

It follows that

$$Y_{g_k} u_{j,k} = \left(t\lambda_{j,k} + \frac{d(R_G - 2t)}{4(d+1)}\right) u_{j,k}.$$

We would like to choose t so that the eigenvalues $t\lambda_{j,k} + \frac{d(R_G-2t)}{4(d+1)}$ are all negative. Since $\lambda_{j,k} < 1/4+\epsilon$ by assumption on h_k, it suffices to choose t so that

$$\frac{d(2t - R_G)}{4t(d+1)} > \frac{1}{4} + \epsilon.$$

This can be rewritten as

$$\left(\frac{d}{d+1}\right)(1 - R_G/2t) > 1/2 + 2\epsilon.$$

A straightforward calculation shows that this is equivalent to choosing

$$t < R_G \cdot \frac{d}{d - 1 - 4\epsilon(d+1)}. \tag{6}$$

The inequalities (5) and (6) are compatible provided $d/(d-1-4\epsilon(d+1)) > 1/2$, which is easy to achieve by choosing ϵ small enough. It follows that the functions $u_{j,k}(y)$ will be eigenfunctions of Y_{g_k} with negative eigenvalues.

After rescaling g_k by $(2t - R_G)$, we can make the scalar curvature $R_{g_k} \equiv -1$. Note that the rescaling does not depend on k. It is well-known that as the number of eigenvalues of $-\Delta_{h_k}$ in $(1/4, 1/4+\epsilon)$ increases, the injectivity radius of the metric h_k goes to 0, and h_k leaves the "thick" part of the moduli space $\mathcal{M}_\gamma$ of the hyperbolic metrics on Σ.[3] Accordingly, the injectivity radius of $(M \times \Sigma, g_k/(2t-R_G))$ also goes to 0. The moduli space $\mathcal{M}_\gamma$ can be compactified by adding surfaces with cusps; the sequence h_k will then have a convergent subsequence in $\overline{\mathcal{M}_\gamma}$, and the sequence $(M \times \Sigma, g_k/(2t - R_G))$ will also have a convergent subsequence.

It seems interesting to better understand under what circumstances a sequence of metrics g_k, with increasing number of negative eigenvalues of Y_{g_k}, can be made to converge in a suitable completion of the moduli space $\mathcal{R}(M)$ of conformal structures on M.

[3] See e.g. [Bus92]; for finer asymptotics of small eigenvalues we refer to [Bat98] and references therein.

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