CONFORMALLY COVARIANT OPERATORS AND CONFORMAL INVARIANTS ON WEIGHTED GRAPHS

DMITRY JAKOBSON, THOMAS NG, MATTHEW STEVENSON, AND MASHBAT SUZUKI

Abstract. Let G be a finite connected simple graph. We define the moduli space of conformal structures on G. We propose a definition of conformally covariant operators on graphs, motivated by [23]. We provide examples of conformally covariant operators, which include the edge Laplacian and the adjacency matrix on graphs. In the case where such an operator has a nontrivial kernel, we construct conformal invariants, providing discrete counterparts of several results in [11, 12] established for Riemannian manifolds. In particular, we show that the nodal sets and nodal domains of null eigenvectors are conformal invariants.

1. Introduction: conformally covariant operators

Conformal transformations in Riemannian geometry preserve angles between tangent vectors at every point x on a Riemannian manifold M. A Riemannian metric g_1 is conformally equivalent to a metric g_0 if

$$(g_1)_{ij}(x) = e^{\omega(x)}(g_0)_{ij}(x), \tag{1.1}$$

where $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$ defines the metric g in local coordinates, and where $e^{\omega(x)}$ is a positive function on M called a *conformal factor*. A *conformal class* $[g_0]$ of a metric g_0 is the set of all metrics of the form $\{e^{\omega(x)}g_0(x) : \omega(x) \in C^\infty(M)\}$. The Uniformization theorem for compact Riemann surfaces says that on such a surface, in every conformal class there exists a metric of constant Gauss curvature; the corresponding statement in dimension $n \geq 3$ (solution of the Yamabe problem) stipulates that in every conformal class there exists a metric of constant *scalar* curvature.

Conformally covariant differential operators include the Laplacian in dimension two, as well as the conformal Laplacian, Paneitz operator, and the operators constructed in [23] on manifolds of dimension $n \geq 3$. Their defining property is the transformation law under a conformal change of metric: there exist $a, b \in \mathbb{R}$ such that if g_1 and g_0 are related as in (1.1), then

$$P_{g_1} = e^{a\omega} P_{g_0} e^{b\omega}. \tag{1.2}$$

Date: March 13, 2014.

2010 *Mathematics Subject Classification.* Primary: 05C22. Secondary: 05C50, 53A30, 53A55, 58D27, 58J50.

Key words and phrases. Weighted graph, conformal structure, moduli space, conformally covariant operator, conformal invariant, adjacency matrix, incidence matrix, edge Laplacian, kernel, signature, nodal set.

D. J. was supported by NSERC and FQRNT. T. Ng was supported by ISM and CRM. M. St. was supported by NSERC. M. Suz. was supported by McGill Faculty of Science.

It follows easily that $\ker P_{g_1} = e^{-b\omega} \ker P_{g_0}$. Based on this observation, the authors of the papers [11, 12] constructed several conformal invariants related to the nodal sets of eigenfunctions in $\ker P_g$ (that change sign). In the current paper, the authors initiate the development of the theory of conformally covariant operators on graphs, giving several examples of such operators and providing discrete counterparts to several results in [11, 12].

1.1. **Differential operators on graphs.** Let $G = (V, E)$ be a finite simple graph, i.e. it has a finite vertex set and no loops or multiple edges. A *weighted graph* is a pair (G, w) where $w \colon E \to \mathbb{R}_+$ is a weight function.

A *differential operator* on G is a linear endomorphism on either $\mathrm{Hom}(V, \mathbb{R})$ or $\mathrm{Hom}(E, \mathbb{R})$. These vector spaces are equipped with the L^2-norms

$$\langle f, g\rangle_V = \sum_{v\in V} f(v)g(v), \quad \text{and} \quad \langle \tilde{f}, \tilde{g}\rangle_E = \sum_{e\in E} \tilde{f}(e)\tilde{g}(e). \tag{1.3}$$

This extends to locally-finite graphs with countable vertex sets, where the function spaces are replaced by those functions with finite L^2-norms.

Example 1.1. The *adjacency matrix* A_w of the weighted graph (G, w) is the $|V|\times|V|$ matrix given by

$$[A_w]_{ij} = \begin{cases} w(v_i, v_j) & : (v_i, v_j) \in E, \\ 0 & : \text{otherwise.} \end{cases} \tag{1.4}$$

The *degree matrix* D_w is the the diagonal matrix with $[D_w]_{ii} = \sum_{j=1}^n [A_w]_{ij}$, and the *vertex Laplacian* is $\Delta_w = D_w - A_w$. The vertex Laplacian is an example of an elliptic Schrödinger operator in the sense of [17].

2. Conformal changes of metric

Let $\mathcal{W}(G)$ be the space of all weight functions on the graph G. Inspired by the notion of conformal equivalence of Riemannian metrics on a Riemannian manifold, we define below the notion of conformal equivalence of weights as in [8, 14, 22, 30].

Definition 2.1. Two weight functions $w, \tilde{w} \in \mathcal{W}(G)$ are *conformally equivalent* if there exists a function $u \in \mathrm{Hom}(V, \mathbb{R})$ such that

$$\tilde{w}(v_i, v_j) = w(v_i, v_j)e^{u(v_i)+u(v_j)}. \tag{2.1}$$

We say that u is the *conformal factor* relating w and $\tilde{w}$. This equivalence relation allows us to partition the space of all weights $\mathcal{W}(G)$ on the graph G into conformal equivalence classes. Given $w \in \mathcal{W}(G)$, we will denote its conformal class by $[w]$.

If $\sim_c$ denotes conformal equivalence, then let $\mathcal{W}(G)/{\sim_c}$ be the space of conformal classes of weights on G. We will refer to $\mathcal{W}(G)/{\sim_c}$ as the *(conformal) moduli space of the graph* G. In § 3, we study the structure of the moduli space and characterize it explicitly for connected graphs.

3. The space of conformal classes

If $G = (V, E)$ is a finite simple graph, recall that $\mathcal{W}(G)$ is the space of weights on G. If $\sim_c$ denotes conformal equivalence, then let $\mathcal{M} := \mathcal{W}(G)/{\sim_c}$ be the space of conformal classes of weights on G. We will refer to $\mathcal{M}$ as the *(conformal) moduli space of* G.

We remark that if $|V| > |E|$ (i.e. G is a tree), then $\mathcal{M}$ is just a point. If $|V| = |E|$, there are different possibilities: an odd-length cycle C_{2n+1} has only one conformal class, but an even-length cycle C_{2n} has infinitely many. If $|E| > |V|$, then $\mathcal{M}$ is generally nontrivial.

Enumerate the vertex set as $V = \{v_1, \ldots, v_n\}$ and the edge set $E = \{e_1, \ldots, e_m\}$. The (unsigned) edge-vertex incidence matrix $B = (B_{ij})$ of G

$$B_{ij} = \begin{cases} 1 & : \text{the edge } e_j \text{ is adjacent to the vertex } v_i \\ 0 & : \text{otherwise} \end{cases} \tag{3.1}$$

Notice that B is determined only by the topology (combinatorics) of the graph G; it does not depend on the weight. We will be mostly interested in calculating the rank of B. The following is a result of Grossman, Kulkarni, and Schochetman (see [25]) to that effect.

Theorem 3.1. ([25, Thm 5.2]) *For any graph $G = (V, E)$, let ω_0 be the number of bipartite components.*[1] *Then, rank$(B) = |V| - \omega_0$.*

Fix a reference weight $w_0 \in \mathcal{W}(G)$ and consider $w \in [w_0]$ with conformal factor $u \in \mathrm{Hom}(V, \mathbb{R})$. Then we have a system of $|E|$ linear equations given by

$$u(v_i) + u(v_j) = \ln w(v_i, v_j) - \ln w_0(v_i, v_j) \text{ for } (v_i, v_j) \in E. \tag{3.2}$$

Consider the transpose of B as an $\mathbb{R}$-linear operator[2] $B^t \colon \mathrm{Hom}(V, \mathbb{R}) \to [w_0]$ that sends $u \in \mathrm{Hom}(V, \mathbb{R})$ to the weight $w \in [w_0]$ defined by Eq. (3.2). This map is by definition surjective, so $\dim([w_0]) = \mathrm{rank}(B^T) = \mathrm{rank}(B)$.

Let [1] denote the conformal class of the combinatorial weight, then we can identify $\mathcal{M} = \mathcal{W}(G)/\!\sim_c$ with the $\mathbb{R}$-vector space $\mathcal{W}(G)/[1]$. The above considerations then imply that

$$\dim(\mathcal{M}) = \dim(\mathcal{W}(G)) - \dim([1]) = |E| - \mathrm{rank}(B).$$

This discussion therefore yields the following conclusion.

Theorem 3.2. *If ω_0 is the number of bipartite components of G, then*

$$\mathcal{M} = W(G)/\!\sim_c \simeq (\mathbb{R}_+)^{|E|-|V|+\omega_0}. \tag{3.3}$$

Remark 3.3. Let $G_1, \ldots, G_k$ denote the connected components of G, then

$$\mathcal{W}(G)/\!\sim_c = \mathcal{W}(G_1)/\!\sim_c \times \ldots \times \mathcal{W}(G_k)/\!\sim_c.$$

As a consequence, we can reduce to the case where G is connected; in particular, Theorem 3.1 tells us that when G is connected,

(1) If G is bipartite, $\dim(\mathcal{M}) = |E| - |V| + 1$.
(2) If G is not bipartite, $\dim(\mathcal{M}) = |E| - |V|$.

The following proposition specifies a manner of choosing a canonical representative of each conformal class.

---

[1] A bipartite component is a connected component that is also bipartite. Equivalently, ω_0 is the number of connected components of G that do not contain an odd cycle.

[2] By an abuse of notation, we consider $\mathcal{W}(G)$ to be an $\mathbb{R}$-vector space of dimension $|E|$ by identifying a weight $w = (w_e)_{e \in E}$ with the vector $(\ln w_e)_{e \in E} \in \mathbb{R}^{|E|}$. In this way, [1] is a linear subspace and the other conformal classes are the equivalences classes in the quotient $\mathcal{W}(G)/[1]$.

Proposition 3.4. *In each conformal class $[w] \in \mathcal{M}$, there exists a unique representative $\overline{w} \in [w]$ such that for any $v_i \in V$,*

$$\prod_{e \sim v_i} \overline{w}(e) = 1, \tag{3.4}$$

where $e \sim v_i$ denotes that the edge e has the vertex v_i as an endpoint.

Remark 3.5. The equation (3.4) is a very useful normalization condition, simplifying computations in examples of section 5.2. In the continuous setting ([11, 12]), a convenient normalization condition was choosing a metric with constant scalar curvature in each conformal class; such metrics exist by the Uniformization theorem in dimension 2, and by solution of the Yamabe problem in higher dimensions.

Proof. The statement is equivalent to finding $\overline{w} \in [w]$ such that $\sum_{e\sim v_i} \ln \overline{w}(e) = 0$ for all $v_i \in V$. Equivalently, $\ln \overline{w} \in \operatorname{Ker}(B)$ i.e. $[w] \cap \operatorname{Ker}(B) = \{\overline{w}\}$. Notice that we have the orthogonal decomposition

$$\ln \mathcal{W}(G) = \operatorname{Ker}(B) \oplus \operatorname{Im}(B^T), \tag{3.5}$$

since $(\operatorname{Ker}(B))^\perp = \operatorname{Im}(B^T)$. Recall that we can identify $\ln[1]$ with $\operatorname{Im}(B^T)$ inside $\ln \mathcal{W}(G)$, so when we pass to the quotient, we find that

$$\ln \mathcal{W}(G)/\ln[1] = \operatorname{Ker}(B).$$

Therefore, the conformal classes are in bijection with the elements of $\operatorname{Ker}(B)$; in particular, each $[w]$ intersects $\operatorname{Ker}(B)$ at exactly one point. $\square$

Example 3.6. Pick an edge in the even cycle C_{2n} and assign to it the weight $a \in (0, \infty)$, then assign each adjacent edge the weight $\frac{1}{a}$. The following two edges will be assigned the weight a, and if we continue this process, we have define a weight w_a on C_{2n}. By construction, $\prod_{e\sim v_i} w_a(e) = 1$ for each $v_i \in C_{2n}$. Conversely, given an arbitrary weight $w \in \mathcal{W}(C_{2n})$, we may compute its canonical representative as follows: put a cyclic orientation $\{e_1, \ldots, e_m\}$ on E, then

$$a = \left(\prod_{i:\ \mathrm{odd}} w(e_i) \right) \Big/ \left(\prod_{i:\ \mathrm{even}} w(e_i) \right). \tag{3.6}$$

Thus, we have explicitly the isomorphism $\mathcal{M}(C_{2n}) \xrightarrow{\sim} (0, \infty)$.

Remark 3.7. It is well-known that on compact Riemann surfaces, in each conformal class there exists a unique metric of constant Gauss curvature. The space of such metrics is called the *moduli space* of surfaces. Riemannian geometry of moduli spaces has been extensively studied; in particular, the *Weil-Petersson* metric on the moduli space. In [31], the authors propose and study analogous notions for weighted graphs. It seems quite interesting to relate the results in our paper to those in [31]. The authors intend to consider this in a future paper.

4. Conformally covariant operators

Motivated by the transformation law (1.2) for conformally covariant operators on manifolds, we define discrete conformally covariant operators below, and provide several examples of such operators. The "continuous" transformation law (1.2) involves pre- and post-multiplication by positive functions (powers of the conformal weight); on a graph, the multiplication operator by a positive function $f : V \to \mathbb{R}$ corresponds to multiplication by the diagonal matrix $\mathrm{d}iag(f(v_1), \ldots, f(v_n))$.

Definition 4.1. Fix a graph G and let $\{S_w\}$ be a collection of differential operators, indexed by $w \in \mathcal{W}(G)$. We say that S_w is *conformally covariant* if for any weight $\tilde{w} \in [w]$, there exist two invertible diagonal matrices D_α, D_β with positive entries (those entries should only depend on the conformal factors in (2.1)), such that

$$S_{\tilde{w}} = D_\alpha S_w D_\beta. \tag{4.1}$$

In this case, we say that $S_{\tilde{w}}$ is the *conformal transformation* of S_w.[3]

Example 4.2. Write $e = (e^+, e^-)$ to denote the head and tail of an edge, and fix $F \subset E$. In [31], the authors employ a $|E| \times |E|$ matrix, which we denote by $A_0(F, w)$, to study a Weil-Petersson type metric on $\mathcal{M}$. This matrix is given by

$$[A_0(F,w)]_{ij} := \begin{cases} w(e_i, e_j) & e_i, e_j \in F \text{ and } e_j^- = e_i^+ \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Let $\tilde{w} \in [w]$ with conformal factor $u \in \mathrm{Hom}(V, \mathbb{R})$ and let D_u be the diagonal matrix given by $[D_u]_{ii} = e^{u(e_i^+)+u(e_i^-)}$, then $A_0(F, \tilde{w}) = A_0(F, w)D_u$. Therefore, $A_0(F, w)$ is a conformally covariant operator on $\mathrm{Hom}(E, \mathbb{R})$.

In § 5 and § 6, we describe other classes of operators satisfying the transformation law (4.1), which include the adjacency matrix and the edge Laplacian.

4.1. **Signature is a conformal invariant.** Let (G, w) be a fixed finite simple weighted graph throughout this section.

Theorem 4.3. *Let S_w be a conformally covariant operator on Hom$(E, \mathbb{R})$, then up to isomorphism ker(S_w) is conformally invariant. A similar results holds for conformally covariant operators on Hom$(V, \mathbb{R})$.*

Proof. If $\tilde{w} \in [w]$, then there exists an invertible diagonal matrices D_α, D_β such that $S_{\tilde{w}} = D_\alpha S_w D_\beta$. Let $f \in \mathrm{Hom}(E, \mathbb{R})$, then $S_{\tilde{w}} f = 0$ iff $S_w(D_\beta f) = 0$. It follows that $\ker(S_{\tilde{w}}) = D_\beta^{-1} \cdot \ker(S_w)$. □

The first assertion of Theorem 4.3 implies that the dimension of $\ker(S_w)$ is a numerical invariant of the conformal class $[w]$ (as the graph is finite, the rank of S_w must also then be a conformal invariant). As the multiplicity of zero eigenvalues of a matrix is the dimension of its nullspace, notice that

Corollary 4.4. *The multiplicity of the zero eigenvalue of S_w is a conformal invariant.*

Theorem 4.5. *Let S_w be a conformally covariant operator on Hom$(E, \mathbb{R})$ or Hom$(V, \mathbb{R})$. Then, the number of positive and negative eigenvalues, and the multiplicity of the zero eigenvalues of S_w are conformal invariants.*

Proof. The proof is similar to the proof of the corresponding result in [11, 12]. Assume for contradiction that there exist two weights $w_1 \in [w_0]$ such that the signatures of S_{w_1} and S_{w_0} are different. The conformal class $[w_0]$ is a path connected space, hence there exists a curve $w_t, t \in [0, 1]$ starting at w_0 and ending at w_1. The eigenvalues of A_{w_t} depend continuously on t, and the multiplicity of 0 is constant

[3]The manner in which these differential operators transform under a conformal change of weight is analogous to how the GMJS operators transform under a conformal change of Riemannian metric in [11, 12].

by Corollary 4.4. That means that the number of positive and negative eigenvalues of A_{w_t} remains constant, which is a contradiction. $\square$

Let M be a matrix, then the *signature* $\mathrm{sig}(M)$ is the triple $(N_+(M), N_0(M), N_-(M))$, where $N_+(M)$, $N_0(M)$, and $N_-(M)$ are the number of positive, zero, and negative eigenvalues of M respectively. In this setting, Theorem 4.5 says that the signature $\mathrm{sig}(S_w)$ is a conformal invariant, when S_w is a conformally covariant operator.

A special case of that result is *Sylvester's Law of Inertia*, which states that given a symmetric matrix M and an invertible matrix N, the matrix NMN^T has the same number of positive, negative, and zero eigenvalues as M. Large classes of operators, as described in § 4, satisfy the above hypothesis on the transformation matrices.

Let S_w be a conformally covariant operator on $\mathrm{Hom}(E, \mathbb{R})$ (the analogues still hold if S_w acts on $\mathrm{Hom}(V, \mathbb{R})$). Now, order the eigenvalues of S_w as

$$\lambda_1(S_w) \leq \lambda_2(S_w) \leq \ldots \leq \lambda_{|E|}(S_w).$$

Our previous considerations imply the following about the sign of $\lambda_1(S_w)$:

(1) $\lambda_1(S_w) < 0$ iff the number of negative eigenvalues of S_w is greater or equal to 1.
(2) $\lambda_1(S_w) = 0$ iff the number of zero eigenvalues is $\eta > 0$ and the number of positive eigenvalues is $|E| - \eta$.
(3) $\lambda_1(S_w) > 0$ iff the number of positive eigenvalues of S_w is equal to $|E|$.

It follows that

Corollary 4.6. *The sign of $\lambda_1(S_w)$ is a conformal invariant.*

Remark 4.7. As a consequence of Proposition 6.4 and Proposition 6.6, the above statements are vacuous for the edge Laplacian $\Delta(E, w)$ and $\Delta(\emptyset, w)$, as the dimensions of their kernels are independent of the weight w. Thus, the dimension of the cycle subspace $\ker(\Delta(E, w))$ is also independent of w.

5. Adjacency matrices

Recall the adjacency matrix A_w associated to the weighted graph (G, w), as in Eq. (1.4). For any subset $F \subset E$, the *generalized adjacency matrix* $A(F, w)$ is the $|V| \times |V|$ matrix where the sign of the entries in A_w has been changed for the edges $e \in F$; it is given by

$$A(F, w)_{ij} = \begin{cases} -[A_w]_{ij}, & (v_i, v_j) \in F, \\ [A_w]_{ij}, & \text{otherwise.} \end{cases} \tag{5.1}$$

Theorem 5.1. *For each $F \subset E$, $A(F, w)$ is a conformally covariant operator.*

This statement remains true even in the more general case when we allow the graph to have loops.

Proof. Let $\tilde{w}(v_i, v_j) = e^{u(v_i)+u(v_j)} w(v_i, v_j)$ for $(v_i, v_j) \in E$ and $u \in \mathrm{Hom}(V, \mathbb{R})$. Then, $A(F, \tilde{w}) = D_u A(F, w) D_u$ where $D_u = \mathrm{diag}(e^{u(v_1)}, \ldots, e^{u(v_n)})$. $\square$

Notice that the case $F = \emptyset$ gives that $A(F, w) = A_w$, so the adjacency matrix is also conformally covariant. Furthermore, it follows that the "random walk" matrix $M_w = (D_w)^{-1} A_w$, which consists of the probability of travelling from one vertex to another along a random walk, is also conformally covariant.

Remark 5.2. The vertex Laplacian is not in general conformally covariant.

Remark 5.3. A generalized adjacency matrix $A(F, w)$ remains conformally covariant if we allow the graph G to contain loops. The same holds for the matrices $A_0(F, w)$ defined in Eq. (4.2).

5.1. **Ranks of generalized adjacency matrices.** Let G be finite simple graph, which in this section we assume to be connected.

Definition 5.4. The *maximal* rank $\mathrm{maxrank}(G)$ is equal to $\sup_{F,w} \mathrm{rank}\, A(F, w)$. The *minimal* rank $\mathrm{minrank}(G)$ is equal to $\inf_{F,w} \mathrm{rank}\, A(F, w)$.

Equivalently, this is the largest (respectively, the smallest) rank of a symmetric matrix with zero diagonal, whose off-diagonal entries are nonzero off the corresponding edges are neighbours in G (we do not allow zero edge weights).

Example 5.5. Let S_k be a *star graph* with $k + 1$ vertices; a elementary calculation shows that $\mathrm{maxrank}(S_k) = \mathrm{minrank}(S_k) = 2$. The same holds for $G = K_{a,b}$, the complete bipartite graph.

Related questions are discussed e.g. in [20], and the nullity of combinatorial graphs was discussed e.g. in [9].

It was shown in [32] that adjacency matrices of *cographs* have full rank. Recall that G is a cograph, or complement-reducible graph, iff G *does not* have the path P_4 on 4 vertices as an induced subgraph.

The maximal size of a graph whose adjacency matrix has a given rank was studied in [28, 26].

The rank of adjacency matrices of Erdös-Renyi random graphs was studied in [18]; it was shown in [21, Prop. 3.2.2] that adjacency matrices of certain families of random graphs have full rank with probability going to 1.

5.2. **Partition of the space of conformal classes.** We next discuss a partition of the set of conformal classes defined by the signature of a (generalized) adjacency matrix.

5.2.1. *The case* $\mathrm{maxrank}(G) = V(G)$. Assume that $\mathrm{maxrank}(G) = V(G) = n$. Then there exists a choice of F such that for all weights w in an open subset of $\mathcal{W}$, 0 is not an eigenvalue of the adjacency matrix $A(F, w)$; a similar statement holds on an open subset of the conformal moduli space $\mathcal{M}$. In this section, we restrict our attention to F satisfying $\sup_w \mathrm{rank}\, A_F(w) = \mathrm{maxrank}(G)$.

Definition 5.6. Fix a subset $F \subset E$ of the set of edges of a graph G satisfying $\mathrm{maxrank}(G) = n$. A *discriminant hypersurface* $\mathcal{D}(F)_{\mathcal{W}}$ in the weight space $w \in \mathcal{W}$ is the set of all weights such that the generalized adjacency matrix $A(F, w)$ has eigenvalue 0. Since the multiplicity of zero of $A(F, w)$ is conformally invariant, this defines a hypersurface $\mathcal{D}(F)_{\mathcal{M}}$ in the conformal moduli space $\mathcal{M}$.

Example 5.7. Let C_5 be the 5-cycle with vertices $\{v_1, \ldots, v_5\}$. Let $G_{5,2}$ be the non-bipartite graph obtained from C_5 by adding the edges (v_1, v_3), and (v_1, v_4). Remark that $\dim \mathcal{M}(G_{5,2}) = 2$. The discriminant hypersurface $D(\emptyset)_{\mathcal{M}}$ associated to the standard adjacency matrix $A(\emptyset, w)$ is the subset

$$D(\emptyset)_{\mathcal{M}} = \{(a, b) \in \mathcal{M}(G_{5,2}) \colon (ab)^4 = a^3 + b^3\}.$$

The curve $D(\emptyset)_{\mathcal{M}}$ in $\mathcal{M}(G_{5,2})$ is depicted in Fig. 1.

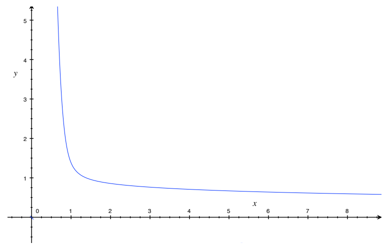

FIGURE 1. The discriminant hypersurface partitions $\mathcal{M}(G_{5,2})$ into 2 components.

Consider a curve $w(t) \in \mathcal{W}, t \in [0,1]$. It is clear that to change the signature of $A(F, w(t))$ the curve $w(t)$ has to cross $\mathcal{D}(F)_{\mathcal{W}}$. We conclude that

Proposition 5.8. *The signature of $A(F,w)$ does not change on connected components of $\mathcal{W} \setminus \mathcal{D}(F)_{\mathcal{W}}$ and $\mathcal{M} \setminus \mathcal{D}(F)_{\mathcal{M}}$.*

Since $\operatorname{tr} A(F,w) = 0$ for any choice of $F \subset E$ and $w \in \mathcal{W}$, we find that $A(F,w)$ always has at least one positive eigenvalue, and at least one negative eigenvalue. We let $N_+(A)$ be the number of *positive* eigenvalues of A; and $N_-(A)$ be the number of *negative* eigenvalues of A.

Definition 5.9. We let $Pos_F(G) \geq 1$ to be $\sup_w N_+(A(F,w))$. Similarly, we let $Neg_F(G) \geq 1$ to be $\sup_w N_-(A(F,w))$.

The signature of $A(F,w)$ ranges between $(Pos_F(G), n - Pos_F(G))$ and $(n - Neg_F(G), Neg_F(G))$.

Definition 5.10. The *signature list* $\mathrm{List}_F(G)$ is the list of all possible signatures of $A(F,w)$, for different w.

It seems interesting to study the *number, topology* and *geometry* of connected components of $\mathcal{M} \setminus \mathcal{D}(F)_{\mathcal{M}}$.

Example 5.11. Let C_6 be the 6-cycle with vertices $\{v_1, \ldots, v_6\}$. Let $G_{6,3}$ be the non-bipartite graph obtained from C_6 by adding the edges (v_1, v_5), (v_2, v_4), and (v_3, v_6). Remark that $\dim \mathcal{M}(G_{6,3}) = 3$. Let $F = \{(v_1, v_2)\}$, then the discriminant hypersurface $\mathcal{D}(F)_{\mathcal{M}}$ associated to the generalized adjacency matrix $A(F,w)$ is the curve seen in Fig. 2, which is cut out by the equation

$$x^8y^2z^2 - 4(xyz)^6 + x^5(2y^2z^5 - 2y^5z^2) + 2x^4yz + (xyz)^2(y^3 - z^3)^2 + x(2yz^4 - 2y^4z) + 1 = 0.$$

The signature of $A(F,w)$ is described by the following list:

(1) If $w \in \mathcal{M}$ is in the component whose boundary contains the origin, then the signature of $A(F,w)$ is $(3,0,3)$.
(2) If $w \in \mathcal{D}(F)_{\mathcal{M}}$, then the signature of $A(F,w)$ is $(3,1,2)$.
(3) Otherwise, the signature of $A(F,w)$ is $(4,0,2)$.

We remark that the spectrum of unweighted adjacency matrix $A(G)$ has been studied extensively. In particular, Graham and Pollack showed in [24] that *biclique partition number* $bp(G)$ satisfies $bp(G) \geq \max\{N_+(A(G)), N_-(A(G))\}$.

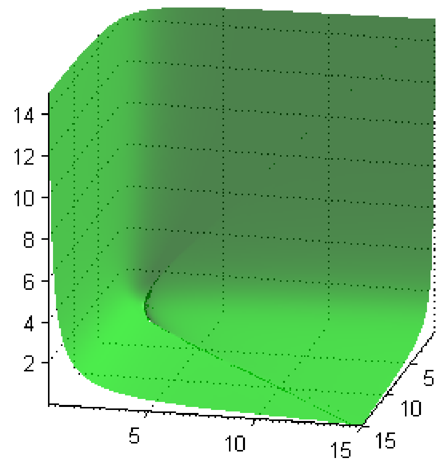

FIGURE 2. The discriminant hypersurface partitions $\mathcal{M}(G_{6,3})$ into 2 components.

5.2.2. *The case* $\operatorname{maxrank}(G) < V(G)$. In this section we consider graphs with $\operatorname{maxrank}(G) < V(G) = n$. In addition, we consider graphs with $\operatorname{maxrank}(G) = n$ and subsets $F \subset E(G)$ satisfying $\sup_w \operatorname{rank} A(F,w)) < \operatorname{maxrank}(G)$. In those cases, 0 is an eigenvalue of $A_F(w)$ for all $w \in \mathcal{W}$.

We adjust the definition of the discriminant hypersurface:

Definition 5.12. A *discriminant hypersurface* $\mathcal{D}(F)_{\mathcal{W}}$ in the weight space $w \in \mathcal{W}$ is the set of all weights w such that the generalized adjacency matrix $A(F,w)$ satisfies $\operatorname{rank}(A(F,w)) < \operatorname{maxrank}(G)$. Since the multiplicity of zero of $A(F,w)$ is conformally invariant, this defines a hypersurface $\mathcal{D}(F)_{\mathcal{M}}$ in the conformal moduli space $\mathcal{M}$.

Proposition 5.8 easily extends. Consequently, this allows one to extend the Definitions 5.9 and 5.10.

5.3. **Bipartite graphs.** It is well-known that $\operatorname{tr}(A(F,w)^k) = \sum_{|\gamma|=k} \prod_{e\in\gamma} w(e)$; here the sum is taken over all closed paths of length k in G, and the edges in $F \subset E$ have negative weights. In bipartite graphs, all closed paths have even length. Accordingly, for any odd $k \geq 1$ we have

$$\operatorname{tr}(A(F,w)^k) = 0.$$

It follows that the set of eigenvalues of $A(F,w)$ is symmetric around 0. Accordingly,

Proposition 5.13. *Let G be a bipartite simple connected graph. Then the signature of $A(F,w)$ is always of the form k,k for any $F \subset E$ and $w \in \mathcal{W}$. If $|V(G)|$ is* even *(resp.* odd*), then the multiplicity of* 0 *as an eigenvalue of $A(F,w)$ is also* even *(resp.* odd*). In particular, if $|V(G)|$ is odd, then* $\ker A(F,w)$ *is always nontrivial.*

6. Incidence matrices

We next describe a class of conformally covariant operators constructed using the (weighted) incidence matrix, or related matrices defined below.

Let $G = (V, E)$ be a finite simple graph. Let $F \subset E$ be a subset of edges which we shall *orient:* for each $e \in F$ we shall choose a *head vertex* $e^+ \in V$ and a *tail vertex* $e^- \in V$. We next define a variant of a well-known *incidence matrix* as follows. Enumerate the vertex set $V = \{v_1, \dots, v_n\}$ and the edge set $\vec{E} = \{e_1, \dots, e_m\}$. The signed weighted vertex-edge incidence matrix $M(F, w)$ (see [6, 16] for related constructions) is an $n \times m$ matrix given by

$$M(F,w)_{ij} = \begin{cases} \sqrt{w(e_j)} & : \text{if } e_j \in F,\ v_i = e_j^+, \\ -\sqrt{w(e_j)} & : \text{if } e_j \in F,\ v_i = e_j^-, \\ \sqrt{w(e_j)} & : \text{if } e_j \notin F, v_i \sim e_j, \\ 0 & : \text{if } v_i \not\sim e_j. \end{cases} \tag{6.1}$$

Let $\tilde{w} \in [w]$ be a different weight in $[w]$ given by the function $u \in \mathrm{Hom}(V, \mathbb{R})$. Then it is easy to show that

$$M(F, \tilde{w}) = M(F, w) D_u, \tag{6.2}$$

where D_u is an invertible diagonal matrix given by

$$(D_u)_{ii} = \begin{cases} e^{\frac{1}{2}(u(e_i^+)+u(e_i^-))} & : \text{if } e_i \in F, \quad e_i = (e_i^-, e_i^+), \\ e^{\frac{1}{2}(u(v_1(i))+u(v_2(i)))} & : \text{if } e_i \notin F, \quad e_i = (v_1(i), v_2(i)). \end{cases} \tag{6.3}$$

In other words,

Theorem 6.1. *For each $F \subset E$, $M(F, w)$ is a conformally covariant operator.*

The signed incidence matrix (the discrete gradient) arises from this construction when $F = E$ and the unsigned incidence matrix is $M(\emptyset, w)$. Consequently, these operators are conformally covariant.

In addition, define the following generalization of the edge Laplacian:

$$\Delta(F, w) := M(F, w)^T \cdot M(F, w). \tag{6.4}$$

It follows immediately that

Theorem 6.2. *Let $\tilde{w} \in [w]$ be a different weight in $[w]$ given by the function $u \in \mathcal{H}_G$. Then*

$$\Delta(F, \tilde{w}) = D_u \cdot \Delta(F, w) \cdot D_u$$

Accordingly, $\Delta(F, w)$ is a conformally covariant operator in the sense of (4.1) *for any choice of $F \subset E$.*

We next discuss important special cases of Theorem 6.2 corresponding to different choices of $F \subset E$.

6.1. **The edge Laplacian.** Consider first the case $F = E$; the operator $\Delta(E, w)$ corresponds to the weighted edge Laplacian of [6], so

Corollary 6.3. *The edge Laplacian is a conformally covariant operator.*

Proposition 6.4. *Given a connected weighted graph (G, w), $\dim\ker(\Delta(E, w)) = |E| - |V| + 1$. In particular, $\dim\ker(\Delta(E, w))$ is independent of w.*

Proof. Clearly, $\ker(\Delta(E, w)) = \ker(M(E, w))$. Now, the cycle space $\ker(M(E, w))$ is isomorphic to the real homology $H_1(G, \mathbb{R})$. For a connected graph, $\dim H_1(G, \mathbb{R}) = |E| - |E(T)|$, where $|E(T)| = |V| - 1$ is the number of edges in a minimal spanning tree T of the combinatorial graph. $\square$

Remark 6.5. The results in this section seem to be related to the results in [10]. The authors intend to study this relation further in a future paper.

6.2. **The Case $F = \emptyset$.** The other extreme example is where $F = \emptyset$: Theorem 6.2 again implies that $\Delta(\emptyset, w)$ is a conformally covariant operator. Note that $M(\emptyset, w)$ is the unsigned weighted vertex-edge incidence matrix, and it is the weighted analogue of the matrix B from § 3. Indeed, they have the same rank, as one is obtained from the other by scaling the columns. It follows from Theorem 3.1 that

Proposition 6.6. *Given a weighted graph (G, w), $\dim\ker(\Delta(\emptyset, w)) = |E| - |V| + \omega_0$, where ω_0 denotes the number of bipartite components of G. In particular, $\dim\ker(\Delta(\emptyset, w))$ is independent of w.*

6.3. **Generalized incidence matrices: left and right kernels.** A generalized incidence matrix $M(F, w)$ will in general be a rectangular matrix; accordingly, we shall consider separately the *left kernel* of M: $\ker_{left} M = \{U : U \cdot M = 0\}$; and the *right kernel* of M: $\ker_{right} M = \{V : M \cdot V = 0\}$.

In this setting, we have the 'rectangular' analogue of Theorem 4.3: this Proposition below follows easily from (6.2).

Proposition 6.7. *Let G be a simple connected graph, $F \subset E(G)$; let also $w \in \mathcal{W}$ and $\widetilde{w} \in [w]$. Then $\ker_{left} M(F, \widetilde{w}) = \ker_{left} M(F, w)$; also, $\ker_{right} M(F, \widetilde{w}) = D_u^{-1} \cdot \ker_{right} M(F, w)$*

6.4. **Additional variants of the edge Laplacian.** We first prove the following elementary lemma:

Lemma 6.8. *Let the graph G have m edges. Let $J \subset \{1, 2, \dots, m\}$. Denote by $M_J(F, w)$ the signed incidence matrix defined by* (6.1) *with the* columns *indexed by J omitted; we ignore the* edges *labelled by the elements of J. Then the operator*

$$\Delta_J(F, w) := M_J(F, w)^t \cdot M_J(F, w) \tag{6.5}$$

is also conformally covariant.

Proof. We showed in Theorem 6.2 that $\Delta(F, w) := M(F, w)^t \cdot M(F, w)$ is conformally covariant. The new matrix $M_J(F, w)^t \cdot M_J(F, w)$ (of order $(m-|J|) \cdot (m-|J|)$) is the minor $\Delta(F, w)$, with rows and columns indexed by J omitted. Let $D_J(u)$ be the diagonal matrix that appears in Theorem 6.2 with entries corresponding to the edges labelled by J omitted. Then it follows easily from the definition that

$$\Delta_J(F, \widetilde{w}) = D_J(u) \cdot \Delta_J(F, w) \cdot D_J(u), \tag{6.6}$$

finishing the proof. □

Let $I = \{1, \dots, |E| - |J|\}$, then for each pair $(i_1, i_2) \in I \times I$, let $\Lambda_{i_1,i_2}(J, F, w)$ be the matrix obtained from $\Delta_J(F, w)$ by removing the i_1st row and i_2nd column. Then,

Proposition 6.9. *For each $(i_1, i_2) \in I \times I$, $\Lambda_{i_1,i_2}(J, F, w)$ is conformally covariant.*

Proof. Let $D_J^{i_1}(u)$ be the matrix obtained from $D_J(u)$ by removing the i_1st row and i_1st column, and let $D_J^{i_2}(u)$ be the matrix obtained from $D_J(u)$ by removing the i_2nd row and i_2nd column. Then, it follows from Lemma 6.8 that

$$\Lambda_{i_1,i_2}(J, F, \tilde{w}) = D_J^{i_1}(u) \cdot \Lambda_{i_1,i_2}(J, F, w) \cdot D_J^{i_2}(u).$$

□

The same procedure can be applied for any square subset of $I \times I$ in order to get further conformally covariant operators.

7. Conformal invariants from Schrödinger operators

Let (G, w) be a finite simple weighted graph and let $H : V(G) \to \mathbb{R}$ be a function on vertices.

Definition 7.1. The *nodal set* $\mathcal{N}(H)$ is the set of all edges $e = (u, v) \in E(G)$ such that $H(u)H(v) < 0$, i.e. such that H changes sign across e; together with the set of all vertices v such that $H(v) = 0$. A *strong nodal domain* U of H is a connected subgraph of G such that F had constant sign on all the vertices of U.

Now, let S be a conformally covariant operator (satisfying 4.1). Let $w \in \mathcal{W}$, and let $\widetilde{w} \in [w]$. Then it follows from (4.1) that

$$\ker S_{w_1} = D_\beta^{-1} \ker S_w. \tag{7.1}$$

This defines a canonical bijection between $\ker S_w$ and $\ker S_{\widetilde{w}}$.

Since the entries of D_β are all positive, the following Proposition is immediate:

Proposition 7.2. *Assume S_w is conformally covariant, and $\ker S_w \neq 0$. Let $H \in \ker S_w$. Then the nodal set $\mathcal{N}(H)$ and strong nodal domains of H are invariant under* (7.1)*.*

It also follows follows easily from (7.1) that

Proposition 7.3. *If $\dim ker(S_w) \geq 2$, then the nonempty intersection of nodal sets of $H_1, H_2 \in ker(S_w)$ and of their complements are invariant under* (7.1)*.*

One can define nodal sets and strong nodal domains for functions in $\mathrm{Hom}(E, \mathbb{R})$, and prove analogues of Propositions 7.2 and 7.3 for conformally covariant operators on $\mathrm{Hom}(E, \mathbb{R})$.

We can say more in the special case $S_w = A(F, w)$.

Theorem 7.4. *Let $A(F, w)$ be a generalized adjacency matrix as in* (5.1)*. Assume that $\ker A(F, w) \neq 0$, and let $H \in \ker A(F, w)$. Consider the map $\Psi_H : E(G) \to \mathbb{R}$ defined for an edge $e = (v_1, v_2)$ by*

$$\Psi_H(e) := H_w(v_1)H_w(v_2)w(e) \tag{7.2}$$

Then, Ψ_H is invariant under (7.1)*.*

Proof. Let $\tilde{w} \in [w]$. Then $H_{\tilde{w}}(v_1) = e^{-f(v_1)}H_w(v_1)$, and $H_{\tilde{w}}(v_2) = e^{-f(v_2)}H_w(v_2)$. Also, $\widetilde{w}(e) = e^{f(v_1)+f(v_2)}w(e)$. The result follows. □

Example 7.5. Let $G_{5,2}$ be as in Example 5.7, then along the discriminant hypersurface $\mathcal{D}(\emptyset)_{\mathcal{M}}$ of the standard adjacency matrix, we have that $A(\emptyset, w)$ has a simple zero eigenvalue. Identifying the canonical representative of a conformal class with a pair $(a, b) \in (0, \infty)^2$, we get a 'canonical' basis vector $H_{(a,b)} \in \ker(A(\emptyset, (a, b)))$, which is given by

$$H_{(a,b)} = \left(\frac{a^2}{b^2}, a^5 - \frac{a}{b}, -a^2, -1, 1 \right).$$

For fixed $e \in E(G_{5,2})$, we consider the range of $\psi_{H_{(a,b)}}(e)$ as we vary the conformal class along the discriminant hypersurface. Namely, consider the set

$$X_e = \{(a, b, \psi_{H_{(a,b)}}(e) \in (\mathbb{R}_+)^3 \colon (a, b) \in \mathcal{D}(\emptyset)_{\mathcal{M}}\}.$$

Projections of some X_e's are pictured in Fig. 3.

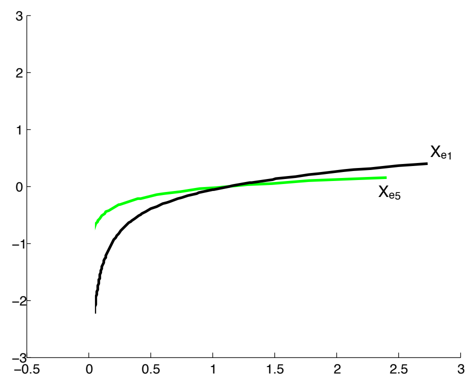

FIGURE 3. The horizontal axis corresponds to points $a+b$ for $(a,b) \in \mathcal{D}(\emptyset)_{\mathcal{M}}$ and the curves give the value $\psi_{H_{(a,b)}}(e_1)$ and $\psi_{H_{(a,b)}}(e_5)$ respectively. In this sense, the two curves are projections of X_{e_1} and X_{e_5} respectively onto the plane.

Remark 7.6. As a consequence of Proposition 6.4 and Proposition 6.6, the above statements are vacuous for the edge Laplacian $\Delta(E,w)$ and $\Delta(\emptyset,w)$, as the dimensions of their kernels are independent of the weight w. Thus, the dimension of the cycle subspace $\ker(\Delta(E,w))$ is also independent of w.

Moreover, let $f_1,\ldots,f_m$ be a basis of $\ker(S_w)$ and let $X_w = \cap_{i=1}^m f_i^{-1}(0)$. Define the map $\Phi_w\colon E\backslash X_w \to \mathbb{RP}^{m-1}$ by

$$\Phi_w(e) = (f_1(e) : \ldots : f_m(e)) \quad \forall e \in E\backslash X_w. \tag{7.3}$$

Take $\tilde{w} \in [w]$, and let D_β be the invertible diagonal matrix as in 4.1. Then, $D_\beta^{-1}f_1,\ldots,D_\beta^{-1}f_m$ is a basis of $\ker(S_{\tilde{w}})$ and Proposition 7.3 implies that $X_{\tilde{w}} = X_w$; in particular, the domain of $\Phi_{\tilde{w}}$ is equal to the domain of Φ_w. Finally, for $e \in E\backslash X_{\tilde{w}}$,

$$\Phi_{\tilde{w}}(e) = (D_\beta^{-1}(e)f_1(e) : \ldots : D_\beta^{-1}(e)f_m(e)) = (f_1(e) : \ldots : f_m(e)) = \Phi_w(e),$$

as the diagonal entries $D_\beta^{-1}(e)$ are nonzero. It immediately follows that

Proposition 7.7. *For fixed S_w, the map Φ_w is a conformal invariant.*

Remark 7.8. The results of this section hold in particular for the class of elliptic Schrödinger operators in the sense of [17]. The authors have partial results on classifying its intersection with the collection of conformally covariant operators, and will consider this problem further in a future paper. In particular, given an elliptic Schrödinger operator of the form $\Delta_w + P_w$, where Δ_w is the vertex Laplacian and P_w is any diagonal matrix, then P_w transforms as

$$[P_{\tilde{w}}]_{ii} + \sum_{v_j\sim v_i} \tilde{w}(v_i,v_j) = e^{2u(v_i)}\left([P_w]_{ii} + \sum_{v_j\sim v_i} w(v_i,v_j)\right), \tag{7.4}$$

where $\tilde{w} \in [w]$ is related to w by the conformal factor $u \in \mathrm{Hom}(V,\mathbb{R})$.

8. *Determinants* and *Permanents* of conformally covariant operators

There exist polynomial graph invariants that can be constructed as determinants of certain operators on graphs. For example, consider the *tree polynomial*

$$\mathcal{T}(G,w) := \sum_T \prod_{e\in T} w(e), \tag{8.1}$$

where the sum is taken over all spanning trees of G.

Let $F = E$ (the edge set of G), and let $M_i(E,w)$ be the matrix defined by the formula (6.1) with the i-th row (corresponding to the vertex v_i of G) omitted. It can be shown ([29, p. 135]) that $\mathcal{T}(G,w) = \det(M_i(E,w)\cdot M_i(E,w)^t)$. Omitting several rows in $M(E,w)$ gives the generating polynomial for *rooted spanning forests of* G, see e.g. [13, (3)]. These results motivate the study below.

Let $A(F,w)$ be the generalized adjacency matrix of *Eq.* (5.1). We next define two multivariate polynomials, with the variables given by the edge weights.

Definition 8.1. Let $P_1(F,w) := \operatorname{Perm} A(F,w)$ and let $P_2(F,w) := \det A(F,w)$.

Recall that $\operatorname{Perm} X$ of a square matrix X is defined by the same formula as $\det X$, except that the product corresponding to every permutation appears with the sign $+1$. By the multiplicativity of the determinant, it follows that $\det(D\cdot X\cdot D) = \det(D)^2\cdot\det(X)$. It follows easily from the definition of the permanent that for any diagonal matrix D, we have $\operatorname{Perm}(D\cdot X\cdot D) = \det(D)^2\cdot\operatorname{Perm}(X)$.

More generally, let $\chi_1,\dots,\chi_k$ be the irreducible characters of $S_{|E|-|J|}$, then by replacing sgn with χ_j in the determinant formula we get the immanant polynomials of the matrix $A(F,w)$, denoted $\operatorname{Imma}_{\chi_j}(A(F,w))$. Define the following sequence of polynomials in $w = (w(e))_{e\in E}$:

$$P_j(F,w) := \operatorname{Imma}_{\chi_j}(A(F,w)) = \sum_{\sigma\in S_{|V|}} \chi_j(\sigma)\prod_{i=1}^{|V|}[A(F,w)]_{i,\sigma(i)}. \tag{8.2}$$

Take χ_1 to be the trivial representation and χ_2 to be the sign representation, then the first two polynomials defined by Eq. (8.2) coincide with those given in Definition 8.1.

In general, the immanant is not a multiplicative function; however, as $D(u)$ is a diagonal matrix, a simple calculation reveals that

$$\operatorname{Imma}_{\chi_j}(D(u)\cdot A(F,w)\cdot D(u)) = \det(D(u))^2\operatorname{Imma}_{\chi_j}(A(F,w)).$$

It follows that

Theorem 8.2. *For each $j = 1,\dots,k$, the polynomials $P_j(F,w)$ satisfy*

$$P_j(F,\widetilde{w}) = \det(D(u))^2 P_j(F,w),$$

where $D(u)$ is the invertible diagonal matrix such that $A(F,\tilde{w}) = D(u)A(F,w)D(u)$.

To each polynomial $P_j(F,w)$, associate a vector $x(j,F,w)\in\mathbb{RP}^d$ for some $d<\infty$ where the components of $x(j,F,w)$ are the coefficients of the monomials and the coefficient of the constant. As $\det(D(u))^2$ is a strictly positive real number, it follows from Theorem 8.2 that

Corollary 8.3. *Fix $j\in\{1,\dots,k\}$ and $F\subset E$. Then, the vector $x(j,F,w)\in\mathbb{RP}^d$ associated to $P_j(F,w)$ is a conformal invariant.*

The polynomials $P_j(F, w)$ for $j = 1, \dots, k$ determine a subset $Z_j(F; w)$ of $(\mathbb{R}^+)^m$ as follows:

$$Z_j(F; w) := \{w \in (\mathbb{R}^+)^m : P_j(F; w) = 0\}. \tag{8.3}$$

As $D(u)$ is an invertible diagonal matrix, Theorem 8.2 implies the following:

Corollary 8.4. *For $j = 1, \dots, k$,*

$$Z_j(F; \widetilde{w}) = Z_j(F; w).$$

That is, the zero set $Z_j(F; w)$ is a conformal invariant.

Example 8.5. Let $G = C_n$ where $4|n$, and enumerate $E(G) = \{e_1, \dots, e_n\}$. A calculation from [5] gives that

$$\det(A(E, w)) = \left(\prod_{i\colon \text{even}} w(e_i) - \prod_{i\colon \text{odd}} w(e_i) \right)^2, \tag{8.4}$$

where $A(E, w)$ is the usual adjacency matrix. It is then clear that the polynomial $P_2(E, w)$ has a nonempty zero locus $Z_2(E; w)$, which is a proper subset of $\mathcal{W}(C_n)$.

Example 8.6. Assume G has an even number n of vertices. Take $F = E$, then the generalized adjacency matrix $A(E, w)$ is skew-symmetric. Then, the *Pfaffian* of a skew-symmetric matrix A is given by

$$\mathrm{pf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1), \sigma(2i)}. \tag{8.5}$$

It is clear that $\mathrm{pf}(A(E, \tilde{w})) = \det(D(u)) \mathrm{pf}(A(E, w))$. Consequently, we can again associate to $\mathrm{pf}(A(E, w))$ a conformally invariant projective vector as above, and the zero locus is also conformally invariant.

Remark 8.7. In principle, one can consider the polynomials of Eq. (8.2) and the zero set of Eq. (8.3) that arise from any conformally covariant operator. However, for certain classes of examples (in particular, the variants of the edge Laplacian given in Eq. (6.5)), these invariants are known to be trivial.

9. Open problems

9.1. **Classification of conformally covariant operators.** In the present paper, we proposed a definition of conformally covariant operators on graphs, and provided several examples of such operators. Motivated by [23], it seems interesting to classify all conformally covariant operators on graphs (in the sense of [17]). On manifolds, a very important role in the study of conformally covariant operators is played by the ambient space construction of C. Fefferman; can this construction be extended to graphs?

9.2. **Conformal moduli space.** It is well-known that on compact Riemann surfaces, in every conformal class there exists a unique metric with constant Gauss curvature (up to scaling and the action of the diffeomorphism group). For surfaces of genus ≥ 2, we get the moduli space of hyperbolic metrics; its quotient by the mapping class group is the Teichmuller space, whose geometry and topology has been studied extensively. If the graph has nontrivial group Γ automorphisms, it seems natural to consider the quotient $\mathcal{M}/\Gamma$ of the conformal moduli space $\mathcal{M}$; for many graphs G, $\Gamma(G)$ is trivial. What is a natural analogue of the Teichmuller space for graphs?

There exist several natural metrics on moduli spaces of surfaces, including the Weil-Petersson metric and the Teichmuller metric. Related problems for graphs have been studied in [31]. It seems interesting to consider related metrics on $\mathcal{M}$. The boundary of $\mathcal{M}$ naturally corresponds to weights on G that are 0 on one or more edges; it seems interesting to describe the geometry of that boundary with respect to different metrics.

Finally, some natural operations on graphs that preserve degree sequence (e.g. edge switches) can be realized geometrically by letting the weights of several edges decrease from 1 to 0, then letting the weights of several *other* edges increase from 0 to 1. It seems that this realization would allow to "glue" the corresponding spaces of weights $\mathcal{W}$ for the two graphs along a common boundary; it could be interesting to extend this construction to conformal moduli spaces $\mathcal{M}$. The authors hope that this will provide some intuition for related problems on manifolds of metrics.

9.3. **Graph Jacobians.** In the papers [1, 2, 3] and related articles, the authors developed discrete counterpart of the theory of Riemann surfaces, and explored connections to tropical geometry. Conformal maps play an important role in the theory of Riemann surfaces; it seems interesting to explore connections between the papers cited above and the present paper.

9.4. **Discretization, and higher-dimensional complexes.** In the paper [19], the authors proved that spectra of discretized Laplacian on manifolds converge to the spectrum of the manifold Laplacian, for suitable choices of discretized operators. In [8, 14, 22, 30] and many other papers, connections between discrete and continuous conformal geometry were investigated. In [33] the author showed that for a triangulated Riemann surface, and a suitable choice of inner product, the combinatorial period matrix converges to the (conformal) Riemann period matrix. It seems interesting to develop a theory of conformally covariant operators on higher-dimensional simplicial complexes, and provide discrete counterparts to the results in [10] and related papers.

9.5. **Other transformation laws.** The transformation law (4.1), motivated by (1.2), preserves the signature of an operator, leads to a simple transformation law for the kernel, and preserves the nodal set of nullvectors. However, it follows from Sylvester's theorem that signature is preserved under more general transformations. It could be interesting to construct operators satisfying more general transformation laws, to study their properties, and to possibly construct continuous analogues.

References

[1] R. Bacher, P. de la Harpe and T. Nagnibeda. *The lattice of integral flows and the lattice of integral cuts on a finite graph.* Bull. Soc. Math. France 125 (1997), no. 2, 167–198.

[2] M. Baker and X. Faber. *Metrized graphs, Laplacian operators, and electrical networks.* Quantum graphs and their applications, 15–33, Contemp. Math., 415, AMS, Providence, RI, 2006.

[3] M. Baker and S. Norine. *Riemann-Roch and Abel-Jacobi theory on a finite graph.* Adv. Math. 215 (2) (2007) 766–788.

[4] G. Berkolaiko, H. Raz, and U. Smilansky. *Stability of nodal structures in graph eigenfunctions and its relation to the nodal domain count.* J. Phys. A 45 (2012), no. 16, 165203, 16 pp.

[5] A. Bién. *On the Determinant of Hexagonal Grids $H_{k,n}$.* arXiv:1309.0087v2.

[6] N. Biggs. *Algebraic Graph Theory.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, Second Edition, 1993.
[7] N. Biggs. *Algebraic potential theory on graphs.* Bull. London Math. Soc. 29(6) (1997), 641–682.
[8] A. Bobenko, U. Pinkall and B. Matthew. *Discrete conformal maps and ideal hyperbolic polyhedra.* arXiv:1005.2698
[9] B. Borovicanin and I. Gutman. *Nullity of graphs: an updated survey.* Zb. Rad. (Beogr.) 14(22) (2011), Selected topics on applications of graph spectra, 137–154.
[10] T. Branson and R. Gover. *The conformal deformation detour complex for the obstruction tensor.* Proc. AMS 135 (2007), no. 9, 2961–2965.
[11] Y. Canzani, R. Gover, D. Jakobson and R. Ponge. *Conformal invariants from nodal sets.* arxiv:1208.3040. To appear in IMRN. Advance access version, doi:10.1093/imrn/rns295.
[12] Y. Canzani, R. Gover, D. Jakobson and R. Ponge. *Nullspaces of conformally invariant operators. Applications to Q_k-curvature.* Electron. Res. Announc. Math. Sci. 20 (2013), 43–50.
[13] S. Caracciolo, C. De Grandi and A. Sportiello. *Renormalization flow for unrooted forests on a triangular lattice.* Nuclear Phys. B 787 (2007), no. 3, 260–282.
[14] D. Champion, A. Marchese, J. Miller, and A. Young. Constant scalar curvature metrics on boundary complexes of cyclic polytopes. arXiv:1009.3061
[15] F. Chapon. Interlacing property of zeros of eigenvectors of Schrödinger operators on trees. arXiv:0904.1336
[16] F.R.K. Chung. Spectral Graph Theory. CBMS Regional Regional Conference Series in Mathematics, 92. AMS, 1997.
[17] Y. Colin de Verdière. *Spectre d'opérateurs différentiels sur les graphes.* Random walks and discrete potential theory (Cortona, 1997), 139–164, Sympos. Math., XXXIX, Cambridge Univ. Press, Cambridge, 1999.
[18] K. Costello and V. Vu. *The rank of random graphs.* Random Structures and Algorithms, 33(3) (2008), 269–285.
[19] J. Dodziuk and V.K. Patodi. *Riemannian structures and triangulations of manifolds.* J. Indian Math. Soc. (N.S.) 40 (1976), no. 1-4, 1–52 (1977).
[20] S. Fallat and L. Hogben. *The minimum rank of symmetric matrices described by a graph: A survey.* Linear Algebra and its Applications 426 (2007), 558–582.
[21] L. Fernandez. *The rank of symmetric random matrices via a graph process.* M. Sc. Thesis, McGill University, 2012.
[22] D. Glickenstein. *Discrete conformal variations and scalar curvature on piecewise flat two- and three-dimensional manifolds.* J. Differential Geom. 87 (2011), no. 2, 201–237.
[23] C.R. Graham, R. Jenne, L.J. Mason and G.A. Sparling. *Conformally invariant powers of the Laplacian, I: Existence.* J. Lond. Math. Soc. **46** (1992), 557–565.
[24] R.L. Graham, H.O. Pollak. *On the addressing problem for loop switching.* Bell System Tech. J. 1 (1971) 2495–2519.
[25] J. Grossman, D. Kulkarni, M. Devadatta, I. Schochetman. *Algebraic graph theory without orientation.* Proceedings of the 3rd ILAS Conference (Pensacola, FL, 1993). Linear Algebra Appl. 212/213 (1994), 289–307.
[26] W. Haemers and M. Peeters. *The maximum order of adjacency matrices of graphs with a given rank.* Des. Codes Cryptogr. 65 (2012), 223–232.
[27] D. Jakobson and I. Rivin. *Extremal metrics on graphs I.* Forum Math. 14(1), (2002), 147–163.
[28] A. Kotlov and L. Lovasz. *The rank and size of graphs.* J. Graph Theory 23 (1996), 185–189.
[29] J. Kung. *Preface: Old and New Perspectives on the Tutte Polynomial.* Annals of Combinatorics 12 (2008), 133–137.
[30] F. Luo. *Combinatorial Yamabe flow on surfaces.* Commun. Contemp. Math. 6(5): (2004), 765–780.
[31] M. Pollicott and R. Sharp. *A Weil-Petersson type metric on spaces of metric graphs.* To appear in Geometria Dedicata.
[32] G. Royle. *The Rank of a Cograph.* The Electronic Journal of Combinatorics 10 (2003), No. 11.
[33] S. Wilson. *Conformal cochains.* Trans. AMS 360 (2008), no. 10, 5247–5264.

E-mail address: jakobson@math.mcgill.ca

E-mail address: thomas.ng2@mail.mcgill.ca

E-mail address: matthew.stevenson2@mail.mcgill.ca

E-mail address: mashbat.suzuki@mail.mcgill.ca

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Str. West, Montréal QC H3A 0B9, Canada.