ON DISTRIBUTION OF ZEROS OF HEINE-STIELTJES POLYNOMIALS

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INTRODUCTION We study nodal statistics of polynomial solutions of a generalized Lamé differential equation:

$$A(x)y''(x) + 2B(x)y'(x) + C(x)y(x) = 0, \tag{1}$$

where $A(x), B(x), C(x)$ are polynomials of degree $N+1, N, N-1$ respectively, and $y(x)$ is a polynomial of degree K.

Definition. A *solution* of (1) is a pair $(y(x), C(x))$. Here $y(x)$ is called *Heine-Stieltjes polynomial* and $C(x)$ is called *Van Vleck polynomial.*

We shall assume that A has distinct roots (real or complex),

$$A(x) = (x-\alpha_0)(x-\alpha_1)\dots(x-\alpha_N)$$

and that

$$\frac{B(x)}{A(x)} = \sum_{i=0}^{N} \frac{r_i}{x-\alpha_i},$$

where $r_i > 0$ are positive real numbers, e.g. $B(x) = A'(x)$.

A very important special case is when α_i are all real, say

$$\alpha_0 < \alpha_1 < \dots < \alpha_N. \tag{2}$$

In that case the condition $r_i > 0$ is equivalent to saying that the roots β_i of $B(x)$ *interlace* the roots of $A(x)$, i.e.

$$\alpha_0 < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \dots < \beta_N < \alpha_N.$$

Heine (cf. [H]) proved that given $A(x), B(x)$ there exists at most

$$\sigma(N,K) = \frac{(N+K-1)!}{K!(N-1)!}$$

polynomial solutions (y, C) where y has degree K.

Stieltjes (cf. [S]) showed that in case (2) there exist exactly $\sigma(N,K)$ solutions. In that case, Lamé DE (1) arises when one separates variables for spherical harmonics of degree $2K$ on S^N in elliptic-spherical coordinates (cf. [T1, T2, WW]). Those are coordinates

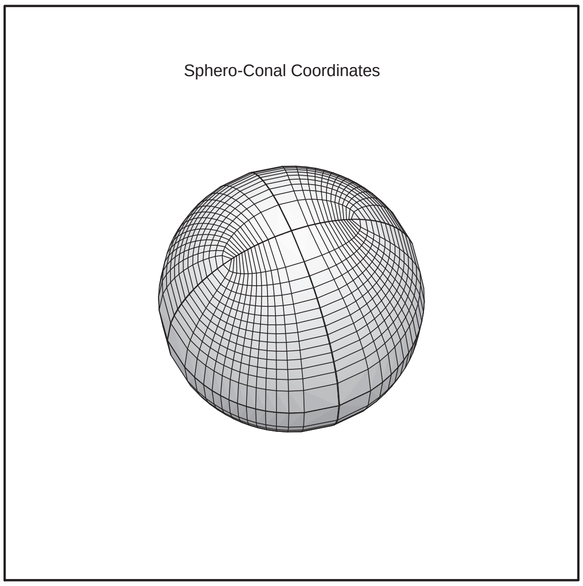
$$\{(u_1, \dots, u_N) : \alpha_{i-1} < u_i < \alpha_i\}.$$

Given a point $(t_0, \dots, t_N) \in S^N$, the coordinates u_j are defined as the roots the equation

$$\sum_{\nu=0}^{N} \frac{t_\nu^2}{u-\alpha_\nu} = 0.$$

They are related to $(t_0, \dots, t_N)$ by the formula

$$t_\nu^2 = \frac{\prod_{i=1}^{N}(\alpha_\nu - u_i)}{\prod_{j=0, j\neq\nu}^{N}(\alpha_j - \alpha_\nu)}.$$



The *Lamé harmonic* ϕ in that coordinate system has the form

$$\phi(u_1, \dots, u_N) = \prod_{i=1}^{N} y(u_i),$$

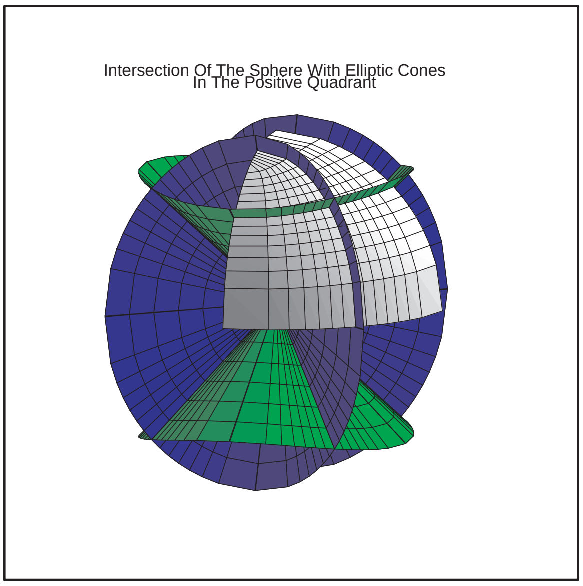
where $y(x)$ is a Heine-Stieltjes polynomial. It is an eigenfunction of N commuting operators $P_m, 0 \le m \le N-1$, given by

$$P_m = \sum_{i<j} S_m^{i,j}(\alpha_0, \dots, \alpha_N) \left(t_i \frac{\partial}{\partial t_j} - t_j \frac{\partial}{\partial t_i} \right)^2,$$

where $S_m^{i,j}$ is the m-th symmetric polynomial in $\{t_0, \dots, t_N\} - \{t_i, t_j\}$. P_0 is the Laplacian on S^N.

The zeros of the Heine-Stieltjes polynomial y correspond to nodal surfaces of ϕ that are intersections of S^N with elliptic cones

$$\sum_{\nu=0}^{N} \frac{\xi_\nu^2}{u_i - \alpha_\nu} = 0.$$



ELECTROSTATIC INTERPRETATION OF ZEROS AND THE WORK OF STIELTJES.

Let $\{x_1, \dots, x_K\}$ be the zeros of a Heine-Stieltjes polynomial $y(x)$; one can show ([Sz]) that x_i-s are all distinct from α_ν-s and from each other. The equation (1) evaluated at x_i becomes $Ay'' + 2By' = 0$ or $y''(x_i)/(2y'(x_i)) + B(x_i)/A(x_i) = 0$ which can be rewritten as the so-called *Niven equation*

$$\sum_{j=1, j\neq i}^{K} \frac{1}{x_i - x_j} + \sum_{\nu=0}^{N} \frac{r_\nu}{x_i - \alpha_\nu} = 0 \tag{3}$$

that is valid for all i. It follows easily from (3) that all x_i-s must lie in the *convex hull* $CH(\alpha_0, \dots, \alpha_N)$. The same is true for the roots of Van Vleck polynomials $C(x)$.

In case $\alpha_0 < \dots < \alpha_N$, we see that all x_i must lie in one of the intervals $]\alpha_{j-1}, \alpha_j[, 1 \leq j \leq N$. Denote by l_j the number of interval containing the root x_j:

$$x_j \in]\alpha_{l_j-1}, \alpha_{l_j}[, \quad 1 \leq j \leq K. \tag{4}$$

We denote by $\mathbf{l}$ the K-tuple $(l_1, l_2, \dots, l_K)$ It is easy to see that the total number of k-tuples with $\mathbf{l} = (l_1, \dots, l_K)$ satisfying $1 \leq l_1 \leq l_2 \leq \dots \leq l_K \leq N$ is equal to $\sigma(N, K)$. Each such K-tuple defines a *configuration* describing the position of H-S zeros in $]\alpha_0, \alpha_N[$. We denote the set of all configurations by $\Sigma(N, K)$.

The following theorem is due to Stieltjes ([S]):

Theorem 1. *Given a configuration* $\mathbf{l} \in \Sigma(N, K)$*, there exists a unique solution* $(y(x), C(x))$ *of* (1) *whose zeros satisfy* (4).

Proof. Let $\mathbf{x} = (x_1, \dots, x_K), x_1 < \dots < x_K$. Given a configuration $\mathbf{l} = (l_1, \dots, l_K)$, define $\Omega = \Omega(\mathbf{l})$ to be the set of x_j-s satisfying (4).

We regard x_i-s as *moving* particles with unit charge, and α_ν-s as *fixed* particles with charge $r_\nu > 0$. All the charges interact according to the following *potential* $V = V(x_1, \dots, x_K, \alpha_0, \dots, \alpha_N)$:

$$V(\mathbf{x}, \mathbf{a}) := \sum_{1 \leq i < j \leq K} \log\left(\frac{1}{|x_i - x_j|}\right) + \sum_{\nu=0}^{N} \sum_{i=1}^{K} \log\left(\frac{r_\nu}{|x_i - \alpha_\nu|}\right). \tag{5}$$

It is easy to see that solutions of Niven equations (3) correspond to critical points of (5).

We next see that V is a smooth function defined on a connected open set Ω of x_j-s (cf. (4)) and approaching infinity at the boundary (the set where one of x_j-s coincides with another or with one of α_ν-s). Accordingly, V must attain a minimum on Ω which must be a solution of (3). It is also easy to show uniqueness. $\square$.

REGIMES

We assume that $\alpha_0 < \ldots < \alpha_N$ are all real and study the nodal statistics of Heine-Stieltjes polynomials as $N+K \to \infty$ in the following regimes:

- Thermodynamic regime: $N \to \infty$ (dimension of the sphere increases); will be discussed by John Toth.
- Semiclassical regime: N fixed, $K \to \infty$ (degree of the spherical harmonic increases); discussed in the rest of the talk.

We remark that if $A(x)$ and $B(x)/\beta$ are monic then $C(x)$ has highest coefficient equal to $-K(K-1+\beta)$. So, in the semiclassical limit the highest coefficient of $-C_K(x)/K^2$ is asymptotic to 1.

REAL SEMI-CLASSICAL REGIME: RESULTS OF MARTINEZ-FINKELSTEIN AND SAFF.

We next summarize results due to Fedoryuk, Martinez-Finkelstein and Saff ([MFS]) about the limiting density of H-S zeros in the semi-classical limit ($A(x), B(x)$ fixed). Let $\alpha_0, \alpha_1 < \ldots < \alpha_n$ be the roots of A. Consider a sequence of solutions (y_K, C_K) of (1), and let $x_1(K) < x_2(K) < \ldots < x_K(K)$ be the roots of y_K s.t. the sequence of measures

$$d\mu_K = \frac{1}{K}\sum_{i=1}^{K}\delta(x_i(K)) \to d\mu$$

as $K \to \infty$. Let θ_i denote the limiting proportion of roots in $]\alpha_{i-1}, \alpha_i[$:

$$\theta_i = \mu(]\alpha_{i-1}, \alpha_i[) = \lim_{K\to\infty}\frac{\#\{j : x_j(K) \in]\alpha_{i-1}, \alpha_i[\}}{K}.$$

Then there exists a "limiting" VV polynomial

$$\lim_{K\to\infty}\frac{-C_K(x)}{K^2} = C(x) := \prod_{j=1}^{N-1}(x-\gamma_j)$$

The limiting measure μ is equal to

$$\frac{1}{\pi}\sqrt{-\frac{C(x)}{A(x)}}dx$$

and is supported on the set where $\Gamma = \{x : C(x)/A(x) < 0\}$ which consists of N intervals

$$I_j \subset [\alpha_{j-1}, \alpha_j], \qquad 1 \le j \le N.$$

with endpoints from the set

$$\{\alpha_0, \ldots, \alpha_N, \gamma_1, \ldots, \gamma_{N-1}\}.$$

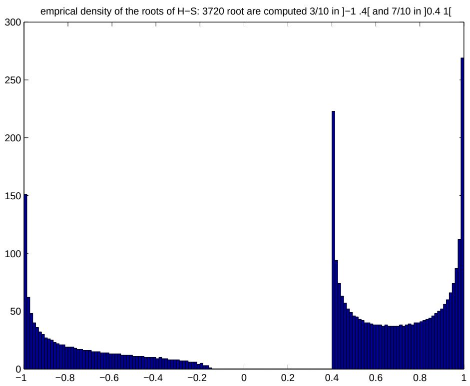
The γ_j-s are determined by equations

$$\frac{1}{\pi}\left|\mathrm{Im}\int_{\alpha_{i-1}}^{\alpha_i}\sqrt{\frac{C(x)}{A(x)}}\,dx\right| = \theta_i.$$

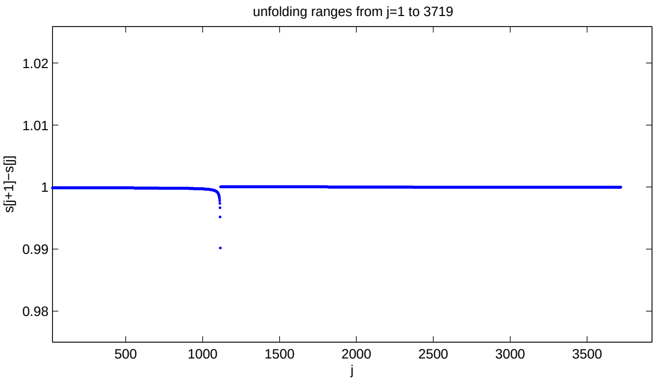
The spectral density found in [MFS] can be integrated over the set $\{(\theta_1, \theta_2, \ldots, \theta_N) : \sum\theta_j = 1\}$ to produce the nodal set density for Lamé harmonics on S^N averaged over all $\sigma(N,K)$ configurations. One obtains a certain conformal multiple of the surface measure on S^N.

Here is an example for $N = 2$: the limiting VV is a linear monic polynomial $x - \gamma$, and the H-S density is equal to

$$\frac{1}{\pi}\sqrt{\frac{-(x-\gamma)}{(x-\alpha_0)(x-\alpha_1)(x-\alpha_2)}}$$



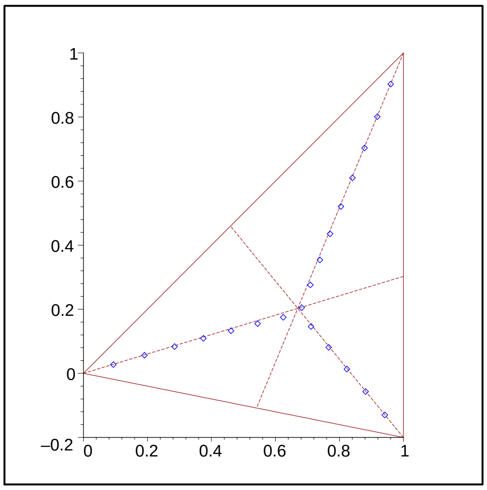
E. Kritchevski [K] has numerically unfolded the H-S zeros (for $K = 3720, \theta_1 = 3/7$) with respect to that density and looked at the nearest neighbor spacing distribution. He found that it was converging to the delta-measure at 1 (one can regard this as a generalization to H-S polynomials of a known result for orthogonal polynomials, [Sz]).



COMPLEX SEMI-CLASSICAL REGIME: $N = 3$

We finally discuss the case of *complex* α-s. In general, it is probably true that for Lebesgue-a.a. α-s there exist exactly $\sigma(N, K)$ solutions of (1) of degree K. However, for $N = 2$ (so VV polynomial $C(x)$ has degree 1), the roots of C turn out to be equal to the eigenvalues of a certain explicit tridiagonal (non-self-adjoint) $(K+1)\times(K+1)$ complex matrix. We henceforth restrict ourselves to the case $N = 2$.

Here is an example of all VV roots for $K = 20$, [K].

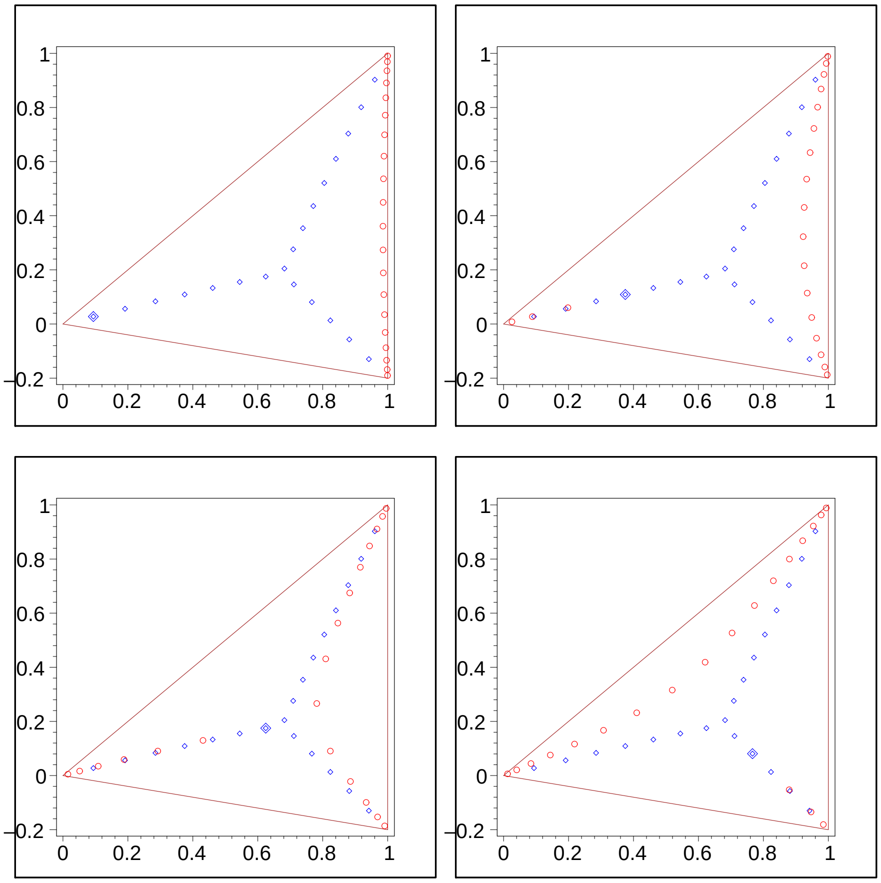


It seems reasonable to make the following

Assumption 1. *The roots of VV polynomial concentrate on a branched curve Γ with 3 branches starting at $\alpha_0, \alpha_1, \alpha_2$.*

Remark 2. *We also note that the branches seem to be well approximated by bisectors of the angles of the triangle $\alpha_0\alpha_1\alpha_2$.*

The corresponding H-S zeros are plotted below, [K].



Suppose the limiting VV polynomial is $x - \gamma, \gamma \in \Gamma$. Let Γ lie on a branch starting at α_0. It seems reasonable to make the following

Assumption 2. *The roots of the corresponding H-S polynomials concentrate on a union of two disjoint curves $S_1 \cup S_2$, where S_1 has endpoints at α_0 and γ, and S_2 has endpoints at α_1 and α_2.*

Making those two assumptions, E. Kritchevski [K] has derived a Taylor expansion at α_0 of the branch of Γ starting at α_0 (by carefully extending the argument of [MFS] to the complex case and using a very nice trick). Let $\theta(\gamma)$ denote the proportion of H-S roots on the curve S_1 ending at γ. We assume that θ changes monotonically with γ, and so we can use θ to parametrize Γ:

$$\gamma = \gamma(\theta), \alpha_0 = \gamma(0).$$

Let $\theta = \Theta(\gamma)$ be the inverse function.

Proposition 3. *Under Assumptions 1 and 2, the m-th derivative*

$$\Theta^{(m)}(\gamma)|_{\gamma=\alpha_0}$$

is equal to

$$\frac{-(2m-1)!!}{(m-1)!2^m}(d^{m-1}/dz^{m-1})_{z=\alpha_0}\left(\frac{1}{\sqrt{(z-\alpha_1)(z-\alpha_2)}}\right).$$

In particular, for $m=1$ we get

$$\gamma'(0) = 1/\theta'(\alpha_0) = -2\sqrt{(\alpha_0-\alpha_1)(\alpha_0-\alpha_2)},$$

confirming Remark 2.

Finally, it seems reasonable (cf. [EF]) to make the following

Conjecture 4. *Under Assumptions 1 and 2, zeros of H-S polynomials corresponding to the VV root γ concentrate on the Stokes lines of the 2nd order equation*

$$y''(x) = \frac{(x-\gamma)}{(x-\alpha_0)(x-\alpha_1)(x-\alpha_2)}\, y(x).$$

We also expect suitable analogues of Proposition 3 and Conjecture 4 to hold in the *general* semi-classical complex case (more than three α-s).

Explanation: Let $h_K = (y_K)'/y_K$ be the logarithmic derivative of a solution y_K of (1). Then h satisfies Ricatti equation

$$A(h_K' + h_K^2) + Bh_K + C_K = 0. \tag{6}$$

However,

$$\frac{h_K(z)}{K} = \int_{\mathrm{supp}\mu_K} \frac{d\mu_K(w)}{z-w},$$

is the Borel transform of the measure μ_K of H-S zeros, and thus $h_K/K \to H = H(\mu)$, the Borel transform of the limiting measure μ.

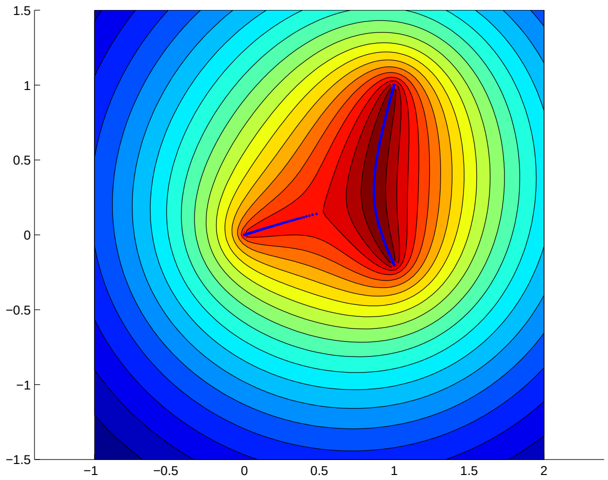
Divide the Ricatti equation (6) by K^2 to get

$$A\left[\frac{h_K}{K}\right]^2 + \frac{1}{K}\left(\frac{Ah_K'}{K} + \frac{Bh_K}{K}\right) = -\frac{C_K}{K^2}$$

The middle term is divided by K and so goes to zero, hence the LHS converges to $A(z)H(z)^2$ as $K \to \infty$. Accordingly, the RHS approaches a limiting (monic!) polynomial $C(z)$, and

$$H(\mu)^2 = C/A.$$

After this, one can use Cauchy's integral formula to recover μ from $H(\mu)$, and Assumptions 1 and 2 guarantee existence of suitable contours.



References

[B] A. Bourget. *Nodal statistics for the Van Vleck polynomials.* Comm. Math. Phys. 230 (2002), no. 3, 503–516.

[BJMT] A. Bourget, D. Jakobson, M.Min-Oo and J. Toth. *A law of large numbers for the zeroes of Heine-Stieltjes polynomials.* Lett. Math. Phys. 64 (2003), 105-118.

[BT] A. Bourget and J. Toth. *Asymptotic statistics of zeroes for the Lamé ensemble.* Comm. Math. Phys. 222 (2001), no. 3, 475–493.

[EF] M. Evgrafov and M. Fedoryuk. *Asymptotic behavior of solutions of the equation $w''(z) - p(z, \lambda)w(z) = 0$ as $\lambda \to \infty$ in the complex z-plane.* Russian Math. Surveys **21** (1966) 1–48.

[H] M. Heine, Handbuch der Kugelfunctionen, Bd. I (2nd ed.), G. Reimer, Berlin 1878.

[K] E. Kritchevski. *Nodal Statistics of Heine-Stieltjes and Van Vleck polynomials.* Master's thesis, McGill, 2004.

[M] M. Marden. *Geometry of Polynomials,* 2nd ed. Mathematical Surveys, Vol. 3, AMS, 1966.

[MFS] A. Martnez-Finkelshtein and E. Saff. *Asymptotic properties of Heine-Stieltjes and Van Vleck polynomials.* J. Approx. Theory 118 (2002), no. 1, 131–151.

[S] T.J. Stieltjes, Sur certains polynômes que vérifient une équation différentielle linéaire du second ordre et sur la théorie des fonctions de Lamé, Acta Math. (6) (1885), 321-326.

[Sz] G. Szegö, Orthogonal Polynomials, Vol. 23 (third edition), Amer. Math. Soc., Providence, RI, 1967.

[T1] J.A. Toth, The quantum C. Neumann problem, Internat. Math. Res. Notices (5) (1993), 137-139.

[T2] J.A. Toth, Various quantum mechanical aspects of quadratic forms, J. Funct. Anal. (130) (1995), 1-42.

[WW] E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, (fourth edition), Camb. Univ. Press, Cambridge U.K., 1963.