Spectral problems with mixed Dirichlet-Neumann boundary conditions: isospectrality and beyond*

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September 9, 2004

Abstract

Consider a bounded domain with the Dirichlet condition on a part of the boundary and the Neumann condition on its complement. Does the spectrum of the Laplacian determine uniquely which condition is imposed on which part? We present some results, conjectures and problems related to this variation on the isospectral theme.

Keywords: Laplacian, mixed Dirichlet-Neumann problem, isospectrality.

2000 Mathematics Subject Classification: 58J53, 58J50, 35P15.

1   Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, its boundary being decomposed as $\partial\Omega = \overline{\partial_1\Omega \cup \partial_2\Omega}$, where $\partial_1\Omega$, $\partial_2\Omega$ are finite unions of open segments of $\partial\Omega$ and $\partial_1\Omega \cap \partial_2\Omega = \emptyset$. Suppose that there are no isometries of $\mathbb{R}^2$ exchanging $\partial_1\Omega$ and $\partial_2\Omega$. We call such a decomposition of the boundary *nontrivial*. Consider a Laplace operator on Ω and assume that on one part of the boundary we have the Dirichlet condition and on the other part the Neumann condition. Does the spectrum of the Laplacian determine uniquely which condition is imposed on which part?

We recall the classical question of Mark Kac, *"Can one hear the shape of a drum?"* [Kac] related to the (Dirichlet) Laplacian on the plane, which still remains open for smooth (as well as for convex) domains. For arbitrary planar domains it was answered negatively in [GWW] using an algebraic construction of [Sun], see also reviews and extensions [BCDS, Bro, Bus] and references therein.

We may reformulate our question in a similar way. Consider two drums with drumheads which are partially attached to them. The drumhead of the first drum is attached exactly where the drumhead of the second drum is free and vice versa. Can one distinguish between the two drums by hearing them?

Similarly to the question of Kac, the answer to our question is in general negative. In this note we construct a family of domains, each of them having a nontrivial isospectral (with respect to exchanging the Neumann and Dirichlet boundary conditions) boundary decomposition. We say that such domains *admit Dirichlet-Neumann isospectrality*.

Our main example is a half-disk which is considered in sections 2.1–2.4. In sections 3.1 and 3.2 we construct generalizations of the main example, in particular to non-planar domains. Section 3.3 provides a simple necessary condition for a boundary decomposition to be Dirichlet-Neumann isospectral. In section 3.4 we discuss if there exist domains not admitting Dirichlet-Neumann isospectrality and conjecture that a disk should be one of them. Some numerical evidence in favour of this conjecture is also presented.

Our motivation for the study of Dirichlet-Neumann isospectrality, rather surprisingly, comes from a seemingly unrelated problem of obtaining a sharp upper bound for the first eigenvalue on a surface of genus two. We discuss it, as well as some other relevant eigenvalue inequalities, in sections 4.1–4.2.

2   Domains isospectral with respect to the Dirichlet-Neumann swap

2.1   Main Example

Our principle example is constructed using the half-disk. Let $\Omega := \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$ be an upper half of a disc centered at the point O (here and

further on we shall often identify the real plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and shall use the complex variable $z = x + \mathrm{i}y$ instead of real coordinates (x, y) on the plane). Consider the following boundary decomposition:

(2.1.1)
$$\partial_1 \Omega = \{\operatorname{Re} z \in (-1, 0),\ \operatorname{Im} z = 0\} \cup \{|z| = 1,\ |\arg z - \pi/2| < \pi/4\}\,,$$

(2.1.2)
$$\partial_2 \Omega = \{\operatorname{Re} z \in (0, 1),\ \operatorname{Im} z = 0\} \cup \{|z| = 1,\ \pi/4 < |\arg z - \pi/2| < \pi/2\}\,.$$

Obviously, such a decomposition is non-trivial.

Consider the following boundary value spectral problems on Ω:

$$\textbf{Problem I:}\ -\Delta u = \lambda u \quad \text{in } \Omega\,, \quad u|_{\partial_1 \Omega} = 0\,, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial_2 \Omega} = 0\,,$$

and

$$\textbf{Problem II:}\ -\Delta v = \lambda v \quad \text{in } \Omega\,, \quad v|_{\partial_2 \Omega} = 0\,, \quad \left.\frac{\partial v}{\partial n}\right|_{\partial_1 \Omega} = 0\,;$$

(see Figure 1). Hereinafter $\frac{\partial}{\partial n}$ is the normal derivative and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator.
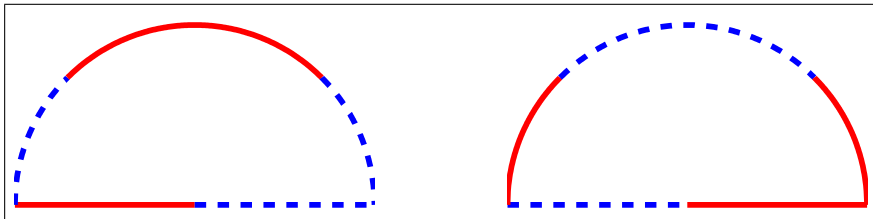

Figure 1: Problems I and II on the half-disk. Here and further on, red solid line denotes the Dirichlet boundary conditions and blue dashed line the Neumann ones.

Let σ_{I} denote the spectrum of Problem I and σ_{II} — the spectrum of Problem II. Both spectra are discrete and positive. Our main claim is the following

Theorem 2.1.3. *With account of multiplicities, $\sigma_{\mathrm{I}} \equiv \sigma_{\mathrm{II}}$.*

We give three different proofs of Theorem 2.1.3, each of them, in our opinion, instructive in its own right, and generalize this Theorem later on for a certain class of examples.

2.2 Proof of Theorem 2.1.3 by transplantation

This proof uses the transplantation trick similar to [Ber, BCDS]. Let $u(z) = u(r, \phi)$ be an eigenfunction of Problem I corresponding to an eigenvalue λ;

here (r, ϕ) denote the usual polar coordinates. Let us introduce a mapping $T : u \mapsto v$, where

$$v(z) = (Tu)(z)$$

$$:= \frac{1}{\sqrt{2}} \begin{cases} u(r, \frac{\pi}{2} - \phi) - u(r, \frac{\pi}{2} + \phi) & \text{if } \phi = \arg z \in (0, \frac{\pi}{2}], \\ u(r, \frac{3\pi}{2} - \phi) + u(r, \phi - \frac{\pi}{2}) & \text{if } \phi = \arg z \in [\frac{\pi}{2}, \pi). \end{cases}$$

Then it is easily checked that $v(z)$ is an eigenfunction of Problem II: it satisfies the equation and the boundary conditions as well as the matching conditions for the trace of the function and the trace of the normal derivative on the central symmetry line $r \in (0, 1)$, $\phi = \pi/2$.

Similarly, if $v(z)$ is an eigenfunction of Problem II, in order to construct an eigenfunction $u(z)$ of Problem I we use an inverse mapping T^{-1} (one may check that $T^8 = \mathrm{Id}$ and hence $T^{-1} = T^7$):

$$u(z) = (T^{-1}v)(z)$$

$$:= \frac{1}{\sqrt{2}} \begin{cases} v(r, \frac{\pi}{2} - \phi) + v(r, \frac{\pi}{2} + \phi) & \text{if } \phi = \arg z \in (0, \frac{\pi}{2}], \\ v(r, \frac{3\pi}{2} - \phi) - v(r, \phi - \frac{\pi}{2}) & \text{if } \phi = \arg z \in [\frac{\pi}{2}, \pi). \end{cases}$$

This proves that the sets σ_I and σ_2 coincide. The equality of multiplicities for each eigenvalue follows immediately from the linearity of the map T. $\qquad\square$
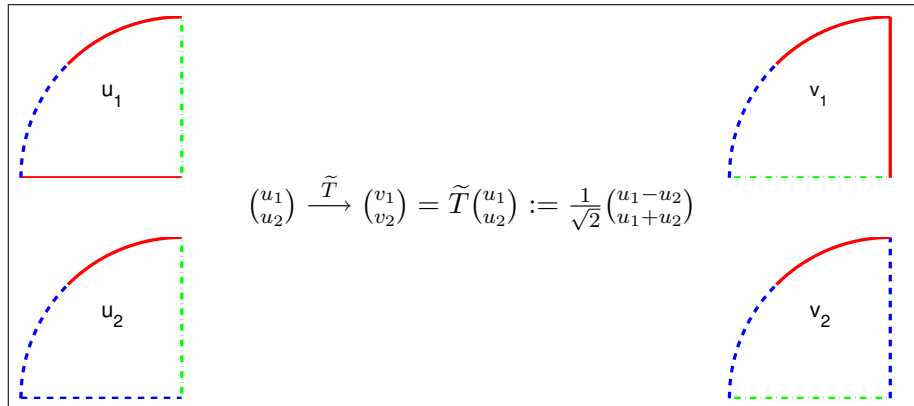

Figure 2: Problems $\widetilde{\mathrm{I}}$ and $\widetilde{\mathrm{II}}$ on the quarter-disk and the map $\widetilde{T}$. Here and further on, green dash-dotted line denotes the matching conditions.

It is easy to visualise the mapping T in the following way. Cutting a half-disk along the symmetry line, we can rewrite Problem I as a system of two boundary value problems with respect to functions u_1, u_2 on a quarter-disk $\Upsilon := \{z \in \mathbb{C} : |z| < 1 \ \mathrm{Im}\, z > 0, \mathrm{Re}\, z < 0\}$; u_1, u_2 should satisfy the matching conditions ($u_1 = u_2$, $\frac{\partial u_1}{\partial n} = -\frac{\partial u_2}{\partial n}$) on the line $\arg z = \pi/2$, see Figure 2. We call this equivalent statement **Problem $\widetilde{\mathrm{I}}$**. The map $\widetilde{T}$ shown

in Figure 2 maps the eigenfunction (u_1, u_2) of Problem $\widetilde{\text{I}}$ into an eigenfunction (v_1, v_2) of **Problem** $\widetilde{\text{II}}$, which is in turn equivalent to Problem II (it is obtained by rotating a half-disk in problem II clockwise by $\pi/2$ and then cutting along the symmetry line).

The possibility of transplanting the eigenfunctions indicates that our problem on a half-disk has a "hidden" symmetry. An attempt to unveil it is presented in the next section.

2.3 Proof of Theorem 2.1.3 using a branched double covering of the disk

Consider an auxiliary spectral problem for the Laplacian on the branched double covering $\mathbb{D}$ of the unit disk D, with alternating Dirichlet and Neumann boundary conditions on quarter-circle arcs, see Figure 3. Let (r, ϕ) and (r, θ) denote the polar coordinates on $\mathbb{D}$ and D, respectively, with $r \in (0, 1]$, $\phi \in [0, 4\pi)$, $\theta = \phi \pmod{2\pi} \in [0, 2\pi)$. The metric on $\mathbb{D} \setminus \{O\}$ is a pull-back of the Euclidean metric from D, where $O = (0, 0)$ is the branch point of the covering. Though $\mathbb{D}$ has a conical singularity at O, eigenvalues and eigenfunctions on $\mathbb{D}$ are well defined using the variational formulation. Consider three symmetries
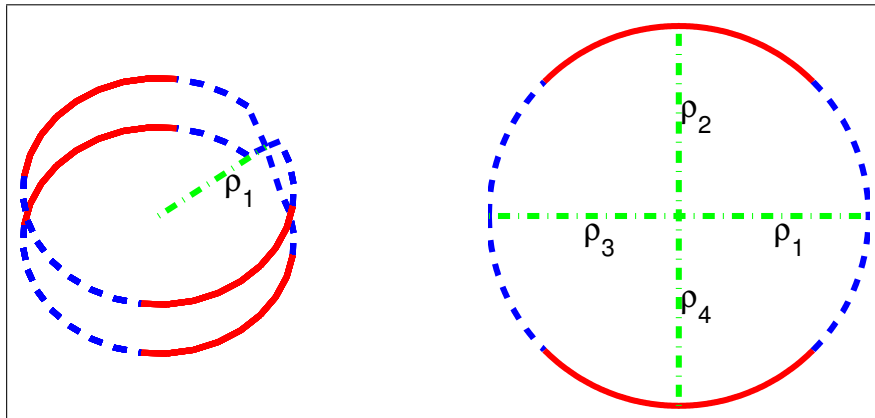

Figure 3: The problem on the double covering $\mathbb{D}$ (left) of the disk D (right), and the radii on D which lift to the lines of symmetry for this problem on $\mathbb{D}$.

of $\mathbb{D}$: $U : (r, \phi) \rightarrow (r, 4\pi - \phi)$, $T : (r, \phi) \rightarrow (r, (\phi + 2\pi) \pmod{4\pi})$ and $V : (r, \phi) \rightarrow (r, (2\pi - \phi) \pmod{4\pi})$. These symmetries are involutions, they commute with each other, and satisify $V = U \circ T$. Symmetries U and V are axial symmetries, and T is an intertwining of sheets of $\mathbb{D}$. By the spectral theorem we find a basis of eigenfunctions that are either even or odd with respect to T, U and V. Consider a space E_- of eigenfunctions on $\mathbb{D}$ that are *odd* with respect to T and the corresponding spectrum $\sigma_-(\mathbb{D})$. We have $E_- = E_-^{+,-} \cup E_-^{-,+}$, where $E_-^{+,-}$ is a subspace of eigenfunctions that are even with respect to U and odd

with respect to V, and $E_-^{-,+}$ is a subspace of eigenfunctions that are odd with respect to U and even with respect to V. Denote $F_U = \{\phi = 0\} \cup \{\phi = 2\pi\}$ and $F_V = \{\phi = \pi\} \cup \{\phi = 3\pi\}$ the fixed point sets of U and V. Any $f \in E_-^{+,-}$ (respectively, $f \in E_-^{-,+}$) satisfies Neumann (respectively, Dirichlet) condition on F_U and Dirichlet (respectively, Neumann) condition on F_V.

Choose a coordinate system on $\mathbb{D}$ in such a way that $\theta = 0$ corresponds to the radius $\rho_1 \subset D$ on Figure 3. For any eigenfunction $f \in E_-^{+,-}$ consider its restriction on the "upper" part $\widetilde{\mathbb{D}} = \{(r, \phi) | 0 < r < 1, \ 0 \le \phi < 2\pi\}$. Then $f|_{\widetilde{\mathbb{D}}}$ projects to an eigenfunction of our boundary problem on a disk D *with a cut* along a diameter $\rho_1 \cup \rho_3$: on ρ_1 it satisfies the Neumann condition and on ρ_3 it satisfies the Dirichlet condition. Similarly, any eigenfunction $f \in E_-^{+,-}$ can be projected from $\widetilde{\mathbb{D}}$ to an eigenfunction of our boundary problem on a disk D with the same cut, but now it satisfies Dirichlet condition on ρ_1 and Neumann condition on ρ_3. In either case, we obtain an eigenfunction of the Problem I. Hence, $\sigma_-(\mathbb{D})$ equals σ_{II} with doubled multiplicities.

Now, let us choose the coordinate system differently so that $\theta = 0$ corresponds to the radius ρ_2. Arguing in exactly the same way as above we obtain that $\sigma_-(\mathbb{D})$ equals σ_{II} with doubled multiplicities. Therefore, $\sigma_{\mathrm{I}} = \sigma_{\mathrm{II}}$ with account of multiplicities which completes the proof of the theorem. $\qquad \square$

Remark. The construction of a "common" covering for Problems I and II described above can be viewed as an application of Sunada's approach ([Sun]) to mixed Dirichlet-Neumann problems.

2.4 Proof of Theorem 2.1.3 by Dirichlet-to-Neumann type mappings

For those who prefer operator theory to geometric constructions we sketch yet another proof of the main theorem. Consider the following auxiliary problem. Let Υ be a quarter-disk introduced in section 2.2. Denote $\partial_1 \Upsilon = \{|z| = 1, \ \arg z \in (\pi/2, 3\pi/4)\}$, $\partial_2 \Upsilon = \{|z| = 1, \ \arg z \in (3\pi/4, \pi)\}$, $\partial_3 \Upsilon = \{\mathrm{Re}\, z \in (-1, 0), \ \mathrm{Im}\, z = 0\}$, $\partial_4 \Upsilon = \{\mathrm{Re}\, z = 0, \ \mathrm{Im}\, z \in (0, 1)\}$, so that $\partial \Upsilon = \overline{\partial_1 \Upsilon \cup \partial_2 \Upsilon \cup \partial_3 \Upsilon \cup \partial_4 \Upsilon}$, see Figure 4.

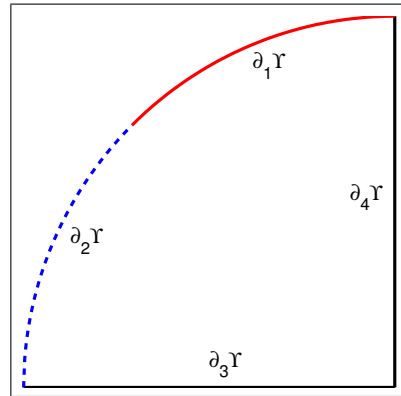Let, for a given $\lambda \in \mathbb{R}$, $w(z)$ satisfy the equation

$$(2.4.1) \qquad -\Delta w = \lambda w \quad \text{in } \widetilde{\Omega},$$

and the boundary conditions

$$(2.4.2) \qquad w|_{\partial_1 \Upsilon} = 0, \qquad \left. \frac{\partial w}{\partial n} \right|_{\partial_2 \Upsilon} = 0$$

(we do not impose at the moment any boundary conditions on w on $\partial_{3,4} \Upsilon$). Denote $\xi = w|_{\partial_4 \Upsilon}$, $\eta = \frac{\partial w}{\partial n}\big|_{\partial_4 \Upsilon}$, $p = w|_{\partial_3 \Upsilon}$, $q = \frac{\partial w}{\partial n}\big|_{\partial_3 \Upsilon}$, Consider four linear operators which depend on λ as a parameter:

$$(2.4.3) \qquad (\mathcal{DD})_\lambda : \xi \mapsto p, \quad \text{subject to} \quad q = 0,$$

Figure 4: The quarter-disk Υ.

(2.4.4) $\qquad (\mathcal{DN})_\lambda : \xi \mapsto q, \quad$ subject to $\quad p = 0\,,$

(2.4.5) $\qquad (\mathcal{ND})_\lambda : \eta \mapsto p, \quad$ subject to $\quad q = 0\,,$

(2.4.6) $\qquad (\mathcal{NN})_\lambda : \eta \mapsto q, \quad$ subject to $\quad p = 0\,.$

These operators acting on the radius $\partial_4 \Upsilon$ are well defined as long as λ does not belong to the spectra of any of the four homogeneous boundary value problems (2.4.1), (2.4.2) with Dirichlet or Neumann boundary conditions imposed on $\partial_4 \Upsilon$ and $\partial_3 \Upsilon$ (cf. [Fr]). Consider an operator $\mathcal{C}_\lambda = (\mathcal{DD})_\lambda^{-1}(\mathcal{ND})_\lambda(\mathcal{NN})_\lambda^{-1}(\mathcal{DN})_\lambda$. Theorem 2.1.3 then follows from

Proposition 2.4.7.

$$\lambda \in \sigma_{\mathrm{I}} \iff \mu = -1 \text{ is an eigenvalue of } \mathcal{C}_\lambda \iff \lambda \in \sigma_{\mathrm{II}}\,.$$

Proposition 2.4.7 is obtained by re-writing the Problems $\widetilde{\mathrm{I}}$ and $\widetilde{\mathrm{II}}$ in terms of operators (2.4.3)–(2.4.6). We leave the details of the proof to an interested reader. $\qquad\qquad\square$

3 Extensions, generalizations, open questions

3.1 From half-disks to quarter-spheres

Consider two quarter-spheres with the boundary conditions as shown in Figure 5. To prove that they are isospectral one can use the same trick as shown on Figure 2. In general, analogous argument works for half-disks endowed with an arbitrary radial metric $ds^2 = f(|z|)dz d\bar{z}$ (note that the matching conditions on Figure 5 are imposed along the radii), quarter-spheres being a special case for a metric $ds^2 = \dfrac{4 dz d\bar{z}}{(1 + |z|^2)^2}$. Example on Figure 5 was in fact the first non-trivial isospectral boundary decomposition that we observed, and it motivated our study, see section 4.1.
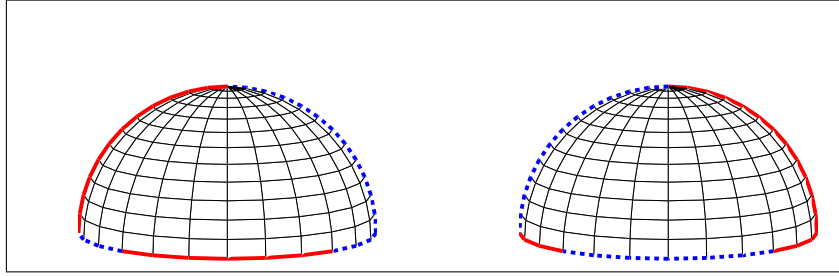
Figure 5: Dirichlet-Neumann isospectral problems on quarter-spheres. The upper semicircles are divided into two equal parts, the lower semicircles are divided in proportion 1:2:1.

3.2 Domains built from sectorial blocks

Example of section 2.1 can be also generalized to a class of domains constructed by gluing together four copies of a sectorial block, i.e. a domain bounded by the sides of an acute angle and an arbitrary continuous curve (without self-intersections) inside it. Namely, let $0 < \alpha < \pi/2$, and choose any points $z_1, z_2 \neq 0$ such that $\arg z_1 = 0$ and $\arg z_2 = \alpha$. Now, let Γ_1 be a piecewise smooth non-self-intersecting curve with end-points z_1, z_2 which lies in the sector $\{0 < \arg z < \alpha\}$, and let K_1 denote an open set bounded by the radii $[0, z_1]$, $[0, z_2]$, and the curve Γ_1.

Let now $\mathcal{S}_\beta : (r, \phi) \mapsto (r, 2\beta - \phi)$ be a map which sends a point into its mirror image with respect to the axis $\{\arg z = \beta\}$. Let

(3.2.1) $$\Gamma_2 := \mathcal{S}_\alpha \Gamma_1 \,, \quad \Gamma_3 := \mathcal{S}_{2\alpha} \Gamma_2 \,, \quad \Gamma_4 := \mathcal{S}_{2\alpha} \Gamma_1$$

and

$$K_2 := \mathcal{S}_\alpha K_1 \,, \quad K_3 := \mathcal{S}_{2\alpha} K_2 \,, \quad K_4 := \mathcal{S}_{2\alpha} K_1 \,,$$

and let K be the interior of $\overline{K_1 \cup K_2 \cup K_3 \cup K_4}$. The domain K is bounded by the radii $[0, z_1]$, $[0, \mathcal{S}_{2\alpha} z_2]$ and the curve $\overline{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}$

We construct a family of pairwise Dirichlet-Neumann isospectral boundary value problems on Ω in the following way. Suppose Γ_1 is decomposed into a union of two non-intersecting sets $\Gamma_{1,1}$ and $\Gamma_{1,2}$ (one of which may be empty). We define the sets $\Gamma_{j,m}$, $j = 1, 2, 3, 4$; $m = 1, 2$ similarly to (3.2.1). We now set

$$\partial_1 K := \Gamma_{1,1} \cup \Gamma_{2,2} \cup \Gamma_{3,2} \cup \Gamma_{4,1} \cup [0, \mathcal{S}_{2\alpha} z_2]$$

and

$$\partial_2 K := [0, z_1] \cup \Gamma_{1,2} \cup \Gamma_{2,1} \cup \Gamma_{3,1} \cup \Gamma_{4,2}$$

(see Figure 6).

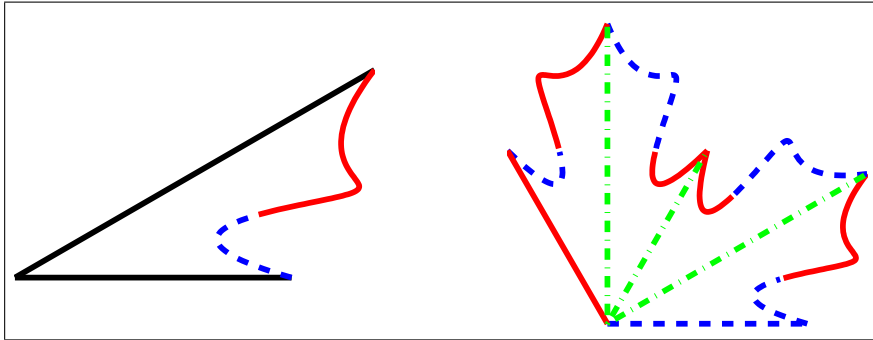The following result generalises Theorem 2.1.3.

Figure 6: A sectorial block K_1 and the resulting domain K. The spectral problem on K with boundary conditions as shown is Dirichlet-Neumann isospectral.

Theorem 3.2.2. *With the above notation, the problem*

$$-\Delta u = \lambda u \quad in\ K\,, \quad u|_{\partial_1 K} = 0\,, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial_2 K} = 0\,,$$

is isospectral with respect to exchanging the Dirichlet and Neumann boundary conditions.

The first proof of Theorem 2.1.3 (see section 2.2) is straightforwardly adapted for Theorem 3.2.2. Note that to obtain Theorem 2.1.3 we just set $z_1 = 1$, $z_2 = e^{i\pi/4}$, $\Gamma_1 = \Gamma_{1,2} = \{e^{it},\ t \in (0,\pi/4)\}$, $\Gamma_{1,2} = \emptyset$ in Theorem 3.2.2.

Other simple examples are illustrated in Figure 7.

Remark. All our examples of domains admitting Dirichlet-Neumann isospectrality are constructed using essentially the same principle. Are there other examples of such domains? For instance, all our domains have one axis of symmetry. Do there exist non-symmetric domains that admit Dirichlet-Neumann isospectrality? In general, can one characterize in geometric terms domains admitting Dirichlet-Neumann isospectrality?

3.3 A necessary condition for Dirichlet-Neumann isospectrality

After presenting various examples of Dirichlet-Neumann isospectrality it would be natural to ask about restrictions. Intuitively, isospectral decompositions should occur rarely. A simple necessary condition for a boundary decomposition to be isospectral is given by

Proposition 3.3.1. *If a boundary decomposition $\partial\Omega = \overline{\partial_1\Omega \cap \partial_2\Omega}$ of a bounded planar domain is isospectral with respect to the Dirichlet-Neumann swap then the total lengths of the parts are equal: $|\partial_1\Omega| = |\partial_2\Omega|$.*
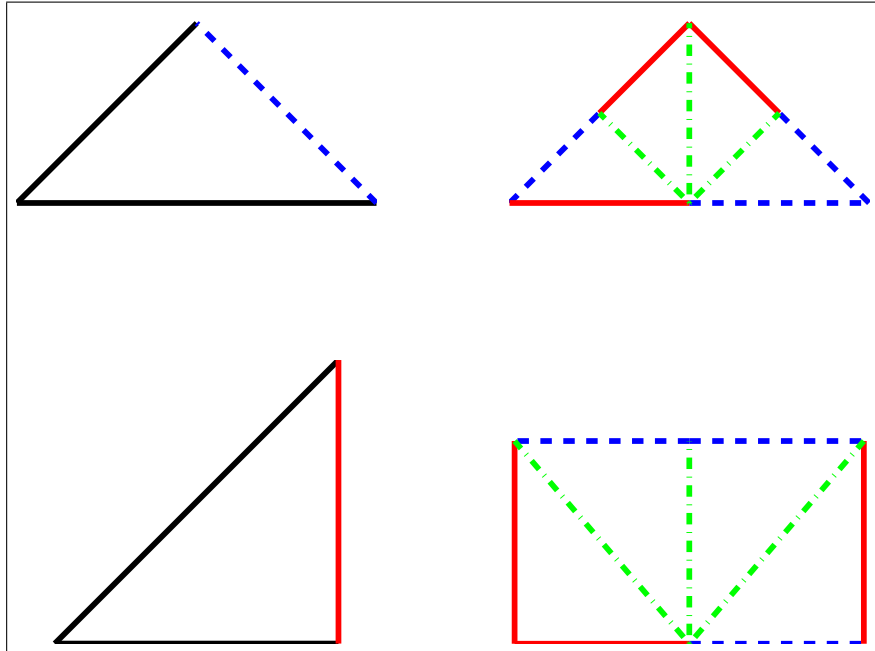

Figure 7: Two more examples built using sectorial blocks. In the first example, $z_1 = 1$, $z_2 = 1/2 + i/2$, $\Gamma_1 = \Gamma_{1,2} = [z_1, z_2]$, $\Gamma_{1,1} = \emptyset$; the resulting set K is a triangle. In the second example, $z_1 = 1$, $z_2 = 1 + i$, $\Gamma_1 = \Gamma_{1,1} = [z_1, z_2]$, $\Gamma_{1,2} = \emptyset$; the resulting set K is a 2×1 rectangle.

Proof. We use asymptotics of the heat trace for a domain with mixed boundary conditions (see [BGKV], [DG]). The first heat invariant a_1 is equal (up to a multiplicative constant) to $|\partial_N \Omega| - |\partial_D \Omega|$, $\partial_N \Omega$ and $\partial_D \Omega$ being Neumann and Dirichlet parts of the boundary respectively. This immediately implies the proposition. □

One can probably deduce more sophisticated necessary conditions for Dirichlet-Neumann isospectrality using higher heat invariants.

3.4 Are there domains not admitting Dirichlet-Neumann isospectrality?

Though in general the question of Kac has a negative answer, there exist domains that are determined by their Dirichlet spectrum (see [Z]), for example, a disk. It would be natural to ask if there are domains not admitting non-trivial Dirichlet-Neumann isospectral decompositions of their boundaries.

Conjecture 3.4.1. A disk does not admit Dirichlet-Neumann isospectrality.

We conducted a simple numerical experiment providing some evidence for this conjecture, by considering boundary decompositions of a unit disk such

that $\partial_1 \Omega$ and $\partial_2 \Omega$ are unions of two segments each, $|\partial_1 \Omega| = |\partial_2 \Omega| = \pi$ by Proposition 3.3.1. Each partition is parametrized by a pair (α, β), where $\alpha, \pi - \alpha$ are lengths of segments in $\partial_1 \Omega$, and $\beta, \pi - \beta$ are lengths of segments in $\partial_2 \Omega$. For every pair $(k\pi/24, n\pi/24)$, $0 < k \leq n < 12$ we compute numerically using FEMLAB [FEM] the L^2-norm $\nu(k, n)$ of a vector $(\lambda_1^{\mathsf{I}} - \lambda_1^{\mathsf{II}}, \lambda_2^{\mathsf{I}} - \lambda_2^{\mathsf{II}}, \lambda_3^{\mathsf{I}} - \lambda_3^{\mathsf{II}})$. Here λ_i^{I} are the eigenvalues of the mixed problem with Dirichlet conditions on $\partial_1 \Omega$ and Neumann conditions on $\partial_2 \Omega$, and λ_i^{II} are the eigenvalues of the problem with the conditions swapped. We observe that for trivial decompositions ($n = k$) the norm $\nu(k, n)$ is by at least an order of magnitude smaller than for any non-trivial decomposition. For example, in a trivially isospectral case $\nu(12, 12) = 0.0012$, and in a non-isospectral case $\nu(11, 12) = 0.0725$ (this value is in fact the minimal one achieved among all non-trivial combinations).

4   Dirichlet-Neumann isospectrality and eigenvalue inequalities

4.1   Genus 2: where did Dirichlet-Neumann isospectrality come from

In this section we briefly describe our motivation to study Dirichlet-Neumann isospectrality. It comes, quite unexpectedly, from a problem to obtain a sharp upper bound for the first positive eigenvalue λ_1 of the Laplacian on a surface of genus 2. It is known ([YY], [N]) that on a surface M of genus p

$$(4.1.1) \qquad\qquad \lambda_1 \mathrm{Area}(M) \leq 8\pi \left[\frac{p+3}{2} \right].$$

On a surface $\mathcal{P}$ of genus 2 this implies

$$(4.1.2) \qquad\qquad \lambda_1 \mathrm{Area}(\mathcal{P}) \leq 16\pi.$$

In general (4.1.1) is not sharp, for example for $\gamma = 1$ ([N]). In [JLNP] we work towards proving the following

Conjecture 4.1.3. There exists a metric on a surface of genus 2 that attains the upper bound in (4.1.2).

The candidate for the extremal metric is a singular metric of constant curvature $+1$ that is lifted from a sphere $\mathbb{S}^2$. The surface $\mathcal{P}$ here is viewed as a branched double covering over a sphere with 6 branching points. The branching points are chosen to be the intersections of $\mathbb{S}^2$ with the coordinate axes in $\mathbb{R}^3$. The punctured sphere has an octahedral symmetry group, and the corresponding hyperelliptic cover corresponds to *Bolza's surface* $w^2 = z^5 - z$ (known also as the *Burnside curve*), and has a symmetry group with 96 elements (a central extension by $\mathbb{Z}_2$ of an octahedral group), the largest possible symmetry group for surfaces of genus 2, see e.g. [I, KW].

Note that $\mathrm{Area}(\mathcal{P}) = 2\mathrm{Area}(\mathbb{S}^2) = 8\pi$ and, therefore it suffices to show that

(4.1.4) $$\lambda_1(\mathcal{P}) = \lambda_1(\mathbb{S}^2) = 2.$$

It remains to be proved that there exists a first eigenfunction on $\mathcal{P}$ that projects to $\mathbb{S}^2$, i.e. which is even with respect to the hyperelliptic involution τ intertwining the sheets of the double cover. We conjecture (see Conjecture 4.2.2) that a first eigenfunction on $\mathcal{P}$ can not be odd with respect to τ. The symmetry group of $\mathcal{P}$ contains many commuting involutions, and this allows us to exploit the ideas of section 2.3. Consider an odd eigenfunction (with respect to τ) and symmetrize it with respect to those involutions. On their fixed point sets we get either Dirichlet or Neumann conditions. Applying the projection $\mathcal{P} \to \mathbb{S}^2$ we obtain an eigenfunction of a spectral problem on a sphere *with cuts* along certain arcs of great circles, where Dirichlet or Neumann conditions are imposed. In particular, in this way we obtain mixed boundary problems shown on Figure 5. These problems are isospectral, since the spectrum of each problem coincides with the odd (with respect to τ) part of the spectrum of $\mathcal{P}$.

All the details of this argument will appear in [JLNP].

4.2 Bounds on the first eigenvalue of mixed boundary problems

Dirichlet-Neumann isospectrality can be viewed as a special case of the following question. Consider a mixed Dirichlet-Neumann problem on a domain with a boundary of length l, where the Dirichlet and the Neumann conditions are specified on parts of the boundary of total length $l/2$ each. For a given domain, how does the geometry of the boundary decomposition affect the spectrum? We discuss this question in relation to the first eigenvalue λ_1.

It is natural to ask how large and how small can λ_1 be. Extremal boundary decompositions for the first eigenvalue of a mixed Dirichlet-Neumann problem are studied in [Den]. In particular, it is proved that λ_1 can get arbirarily close to the first eigenvalue of the pure Dirichlet problem (which is hence a sharp upper bound for λ_1): it is achieved in the limit as Dirichlet and Neumann conditions get uniformly distributed on the boundary. It is also shown that a decomposition minimizing λ_1 always exists for bounded Lipschitz domains. However, an explicit minimizer is found only for a disk, where Dirichlet and Neumann conditions have to be imposed on half-circles [Den].

The problem of comparing the first eigenvalues for different boundary decompositions seems to be rather transcendental in general. Below we present a result, communicated to us by Brian Davies and Leonid Parnovski, that applies to a special case of axisymmetric/centrally symmetric decompositions.

Let Ξ_a be a simply connected planar domain, which is symmetric with respect to an axis d. We consider a mixed boundary value spectral problem for the Laplacian on Ξ_a with some combination of Dirichlet and Neumann boundary

conditions on $\partial\Xi_a$ which is also symmetric with respect to d. Denote the first eigenvalue of this problem by $\lambda_1(\Xi_a)$.

Let $\Xi_{1,2}$ denote two halves of Ξ_a lying on either side of d, and let $\widetilde{\Xi}_2$ be an image of Ξ_1 under the central symmetry with respect to the midpoint O of the interval $d_\Xi := \Xi_a \cap d$. Consider a centrally symmetric domain Ξ_c which is the interior of $\overline{\Xi_1 \cup \widetilde{\Xi}_2}$, and the spectral mixed boundary value problem on Ξ_c with boundary conditions on $\partial\widetilde{\Xi}_2$ centrally symmetric to the ones on $\partial\Xi_1$, see Figure 8. Denote the first eigenvalue of this problem by $\lambda_1(\Xi_c)$.
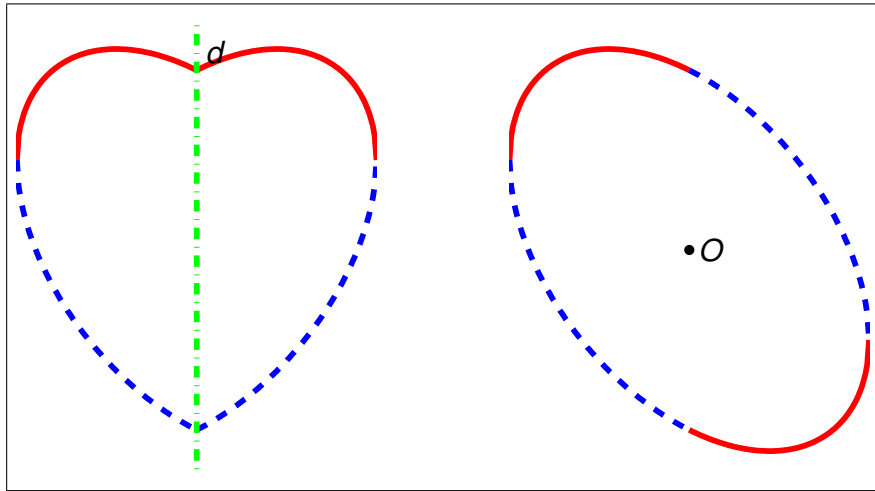

Figure 8: Axisymmetric domain Ξ_a and centrally symmetric domain Ξ_c.

Theorem 4.2.1. ([DP]) $\lambda_1(\Xi_c) \geq \lambda_1(\Xi_a)$.

Proof. Consider an auxiliary boundary value problem for the Laplacain on Ξ_1 obtained by keeping the given boundary conditions on $\partial\Xi_1 \setminus d_\Xi$ and imposing the Neumann condition on d_Ξ. Denote the first eigenvalue of this auxiliary problem by $\lambda_1(\Xi_1)$. By the variational principle and Dirichlet-Neumann bracketing argument, $\lambda_1(\Xi_c) \geq \lambda_1(\Xi_1)$. On the other hand, as the first eigenfunction of the symmetric problem (corresponding to the eigenvalue $\lambda_1(\Xi_a)$) should be symmetric with respect to d and therefore should satisfy the Neumann condition on d_Ξ, we have $\lambda_1(\Xi_a) = \lambda_1(\Xi_1)$, thus implying the result. Note that the equality $\lambda_1(\Xi_c) = \lambda_1(\Xi_1)$ (and therefore the equality $\lambda_1(\Xi_c) = \lambda_1(\Xi_a)$) can be attained if and only if Ξ_1 has an additional line of symmetry d_1 perpendicular to d and passing through the midpoint O of d_Ξ, with the boundary conditions being imposed on $\partial\Xi_1$ symmetrically with respect to d_1. □

Theorem 4.2.1 can be used for obtaining estimates of eigenvalues of boundary value problems on domains with two lines of symmetry. For example, the boundary of a quarter-sphere has a natural decomposition into two halves of

great circles. If we impose the mixed Dirichlet-Neumann boundary conditions on the halves of these great circles as shown in Figure 9, we immediately obtain that the first eigenvalue in any of the axisymmetric cases is smaller than the first eigenvalue in the centrally symmetric case.
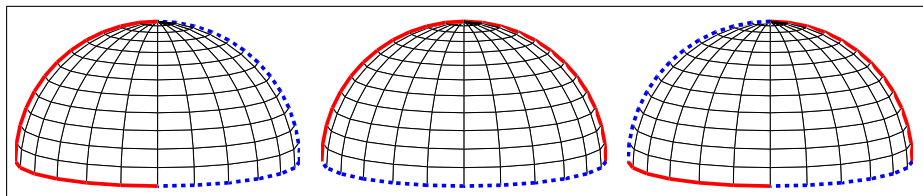

Figure 9: Axisymmetric (domains $Q_{a,1}$, left, and $Q_{a,2}$, centre) and centrally symmetric (domain Q_c, right) positioning of Dirichlet and Neumann boundary conditions on the quarter-sphere. The first eigenvalue of the Laplacian is larger in the centrally symmetric case: $\lambda_1(Q_c) > \lambda_1(Q_{a,j})$, $j = 1, 2$.

We would like to conclude with another inequality on the first eigenvalue for quarter-spheres that we need to check in order to complete the proof of sharpness of (4.1.2) in [JLNP]. Let Q be any of the two isospectral problems on a quarter-sphere shown on Figure 5, and remind that $Q_{a,2}$ is the problem shown in the middle of Figure 9 (with the Dirichlet condition imposed on one half of the big circle and the Neumann condition on another).

Conjecture 4.2.2. $\lambda_1(Q) > \lambda_1(Q_{a,2})$

One can immediately check that $\lambda_1(Q_{a,2}) = \lambda_1(\mathbb{S}^2) = 2$. An affirmative solution of Conjecture 4.2.2 excludes the possibility that the first eigenfunction on $\mathcal{P}$ is odd with respect to the intertwining of sheets (see section 4.1), and hence we have

Theorem 4.2.3. ([JLNP]) *Conjecture 4.2.2 implies Conjecture 4.1.3.*

Using FEMLAB [FEM] one can verify Conjecture 4.2.2 numerically: $\lambda_1(Q) \approx 2.28 > 2$. Our current project is to find a rigorous (possibly, computer-assisted) proof of this conjecture.

Acknowledgements. We are grateful to Leonid Parnovski for many useful suggestions, in particular those providing the proof in section 2.3. We also thank Leonid Polterovich for helpful remarks on a preliminary version of this paper.

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