INTRODUCTION TO GRAPH THEORY

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1. Definitions

- A *graph* G consists of vertices $\{v_1, v_2, \ldots, v_n\}$ and *edges* $\{e_1, e_2, \ldots, e_m\}$ connecting pairs of vertices. An edge $e = (uv)$ is *incident* with the vertices u and v. The vertices u, v connected by an edge are called *adjacent*. An edge (u, u) connecting the vertex u to itself is called a *loop*. Example: v_2 is adjacent to v_1, v_3, v_6 in Figure 1.

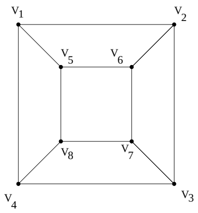

Figure 1: The cube.

- A *degree* $\deg(u)$ of a vertex u is the number of edges incident to u, e.g. every vertex of a cube has degree 3 (such graphs are called *cubic* graphs). Let V be the number of vertices of G, and E be the number of edges. Then
$$\sum_{v \in G} \deg(v) = 2 \cdot E$$
It follows that G has an *even* number of vertices of *odd* degree.
- A *walk* is a sequence of vertices $v_0, v_1, v_2, \ldots, v_k$ where v_i and v_{i+1} are adjacent for all i, i.e. (v_i, v_{i+1}) is an edge. If $v_k = v_0$, the walk is *closed*. Example: $v_1, v_2, v_6, v_7, v_3, v_2, v_1$.
- If all the edges in the walk are distinct, the walk is called a *path*. G is *connected* if every 2 vertices of G are connected by a path. A closed walk that is also a path is called a *closed path*. Example: $v_1, v_2, v_6, v_7, v_8, v_5, v_1$.

2. Eulerian paths and circuits

- An *Eulerian path* visits every edge exactly once. A closed Eulerian path is called an *Eulerian circuit*. Example: remove snow from all streets in a neighborhood, passing every street exactly once.
- **Theorem:** A connected graph G has an *Eulerian circuit* if and only if all vertices of G have even degrees. The circuit can start at any vertex.
  Such graphs are called *Eulerian* or *unicursal*.
- G has an Eulerian path if it has exactly *two* vertices of odd degree. An Eulerian path must start at one of these vertices, and must end at another one.

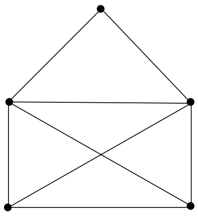

Figure 2: G has exactly 2 vertices of odd degree.

- These notions were studied by Leonard Euler in a 1736 paper, considered to be the first paper on graph theory. Euler considered the following graph:

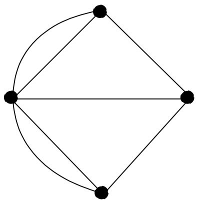

Figure 3: "Seven bridges of Königsberg" graph.

Sketch of the proof of the theorem about Eulerian circuits.

- One direction is easy: let an Eulerian circuit C start and end at v_0. Every time a vertex $v_j \neq v_0$ is visited, we "go in" along one edge, and "go out" along another edge, hence *two edges* are traversed during each visit. Since C contains every edge, the degrees of all $v_j \neq v_0$ are even.
- For v_0, the argument is similar: C starts and ends at v_0 (thus two edges are traversed), and all other visits to v_0 are similar to the visits to $v_j \neq v_0$.
- Now, suppose G is connected and all vertices have even degrees. Assume for contradiction that G doesn't have an Eulerian circuit; choose such G with as few edges as possible. One can show that G contains a closed path (**Exercise!**). Let C be such a path of *maximal* length.
- By assumption, C is not an Euler circuit, so $G \setminus E(C)$ (G with edges of C removed) has a component G_1 with at least one edge. Also, since C is Eulerian, all its vertices have even degrees, therefore all vertices of G_1 also have even degrees.
- Since G_1 has fewer edges than G, G_1 is Eulerian; let C_1 be an Eulerian circuit in G_1. C_1 shares a common vertex v_0 with C; assume (without loss of generality) that C_1 and C both start and end at v_0. Then C followed by C_1 is a closed path in G with more edges than C, which contradicts our

assumption that C has the maximal number of edges. The contradiction shows that G is Eulerian.

3. Planar graphs and Euler characteristic

Let G be a connected *planar* graph (can be drawn in the plane or on the surface of the 2-sphere so that edges don't intersect except at vertices). Then the edges of G bound (open) regions that can be mapped bijectively onto the interior of the disc; in the plane $\mathbf{R}^2$ there will be a single *unbounded* region. These regions are called *faces*.

- Draw graphs of 2 or 3 of your favorite polyhedra (e.g. tetrahedron, cube, octahedron, prism, pyramid), and count the number V of vertices, the number E of edges, and the number F of faces for each graph. Try adding and subtracting these 3 numbers; is there a linear combination that is the same for all graphs that you drew?
- You have just discovered *Euler's formula*:

$$V - E + F = 2.$$

  This formula holds for *all* connected planar graphs!
- Outline of the proof: Any connected graph on S^2 can be obtained from a trivial graph consisting of a single point by a finite sequence of the following operations:
- (1) Add a new vertex connected by an edge to one of the old vertices. Then

$$V_{new} = V_{old} + 1, E_{new} = E_{old} + 1, F_{new} = F_{old},$$

  so $V_{new} - E_{new} + F_{new} = V_{old} - E_{old} + F_{old}$.
- (2) Add an edge connecting 2 vertices that were not connected before. Then

$$V_{new} = V_{old}, E_{new} = E_{old} + 1, F_{new} = F_{old} + 1,$$

  so $V_{new} - E_{new} + F_{new} = V_{old} - E_{old} + F_{old}$.
- (3) Add a new vertex in the middle of existing edge (subdivide). Then

$$V_{new} = V_{old} + 1, E_{new} = E_{old} + 1, F_{new} = F_{old},$$

  so $V_{new} - E_{new} + F_{new} = V_{old} - E_{old} + F_{old}$.

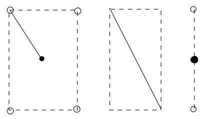

Figure 4: Operations 1, 2, 3.

- Since for the trivial graph $V = F = 1, F = 0$, we get $V - E + F = 2$ for that graph. Since the linear combination does not change under operations (1), (2) or (3), we find that $V - E + F = 2$ for any connected graph on the sphere.

Dual graph: let G be a graph in the plane or on S^2 (*planar graph*). A *dual graph* $\widetilde{G}$ is obtained as follows: *faces* of G become *vertices* of $\widetilde{G}$; two vertices of $\widetilde{G}$ are adjacent (connected by an edge) if the two corresponding faces of G have a common edge. It is easy to see that there is a bijection between *faces* of $\widetilde{G}$ and *vertices* of G. Example: tetrahedron is dual to itself, and cube is dual to octahedron. A dual of $\widetilde{G}$ is isomorphic to G.

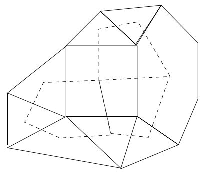

Figure 5: Dual graphs.

We have

$$V(\widetilde{G}) = F(G), F(\widetilde{G}) = V(G), E(\widetilde{G}) = E(G).$$

4. Regular Polyhedra

- We shall apply Euler's formula to describe all regular polyhedra (platonic solids).
- Let G be a planar graph corresponding to a regular polyhedron, with V vertices, E edges and F faces. Let all vertices of G have degree a, and let all of the faces have b edges.
- Homework exercise: Prove that

$$a \cdot V = 2 \cdot E = b \cdot F.$$

  Hint: every edge belongs to the boundary of exactly 2 faces.
- It follows that $V = 2E/a, F = 2E/b$.
- Substituting into Euler's formula, we get $E(2/a - 1 + 2/b) = 2$, or

$$1/a + 1/b - 1/2 = 1/E > 0.$$

- We cannot have $a \geq 4, b \geq 4$, since then $1/a + 1/b - 1/2 \leq 0$, so either $a \leq 3$ or $b \leq 3$. Also, $a \geq 3, b \geq 3$.
- Now it is easy to check that only the following combinations of a, b, E are possible:
- $a = b = 3, E = 6, V = F = 4$: tetrahedron;

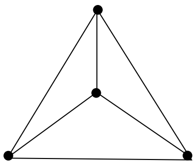

Figure 6: Tetrahedron.

- $a = 3, b = 4, E = 12, V = 8, F = 6$: cube;
- $a = 4, b = 3, E = 12, V = 6, F = 8$: octahedron;

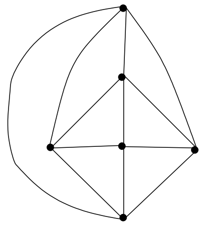

Figure 7: Octahedron.

- $a = 3, b = 5, E = 30, V = 20, F = 12$: dodecahedron;

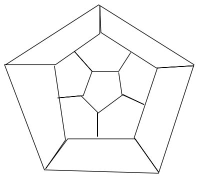

Figure 8: Dodecahedron.

- $a = 5, b = 3, E = 30, V = 12, F = 20$: icosahedron.

5. Problems

- 1. Prove that the *complete graph* K_5 (5 vertices all adjacent to each other) is not planar.

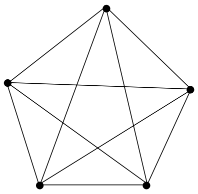

Figure 9: The graph K_5.

- 2. Consider a *complete bipartite graph* $K_{3,3}$ (each of the vertices $\{u_1, u_2, u_3\}$ is adjacent to each of the vertices $\{v_1, v_2, v_3\}$, but u_i is not adjacent to u_j, and v_i is not adjacent to v_j for $i \neq j$). Sometimes it is called "3 houses, 3 utilities graph." Prove that $K_{3,3}$ is not planar.

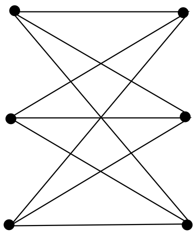

Figure 10: The graph $K_{3,3}$.

- An important *Kuratowski's theorem* states that any graph that is not planar contains a subgraph that can be obtained from either K_5 or $K_{3,3}$ by repeatedly subdividing the edges. That theorem is not easy, so I am not asking you to prove it.