Gauss curvature of random metrics

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• \((M, g)\) is a compact surface with a Riemannian metric \(g\).

• Goal: study Gauss curvature \(K\) of random Riemannian metrics on \(M\).

• Gauss curvature: Geometric meaning: as \(r \to 0\),

\[
\text{vol}(B_M(x_0, r)) = \pi r^2 \left[ 1 - \frac{K(x_0) r^2}{12} + O(r^4) \right].
\]

\(K > 0 \Rightarrow \) surface in \(\mathbb{R}^3\) is convex; volume grows slower than in \(\mathbb{R}^2\).

\(K < 0 \Rightarrow \) surface in \(\mathbb{R}^3\) is concave; volume grows faster than in \(\mathbb{R}^2\).
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$K > 0 \Rightarrow$ surface in $\mathbb{R}^3$ is convex; volume grows slower than in $\mathbb{R}^2$.
$K < 0 \Rightarrow$ surface in $\mathbb{R}^3$ is concave; volume grows faster than in $\mathbb{R}^2$. 
• **Conformal class:** Metric $g_1$ is *conformally equivalent* to $g_0$ if for all $x \in M$ and $U, V \in T_x M$,

$$g_1(x)(U, V) = F(x) \cdot g_0(x)(U, V), \quad F(x) > 0.$$  

The set of all such metrics is called a *conformal class* $[g_0]$ of $g_0$.

• **Uniformization theorem:** in every conformal class, there exists a unique metric of constant Gauss curvature $K_0$. $K_0 > 0$ for $M = S^2$, $K_0 = 0$ for $M = T^2$, and $K_0 < 0$ for surfaces of genus $\gamma \geq 2$.

• **Gauss-Bonnet theorem:** $\int_M K dA = 2\pi \chi(M)$, where $\chi$ is the *Euler characteristic*, $\chi(\text{sphere with } \gamma \text{ handles}) = 2 - 2\gamma$, e.g. $\chi(S^2) = 2$, $\chi(T^2) = 0$ etc.
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• **Questions:** Assume $M \neq T^2$, and $g_0$ has non-vanishing curvature $K_0$. What is the probability that a random metric $g_1$ in the conformal class $[g_0]$ also has non-vanishing curvature $K_1$?

• Use Laplacian to define random metrics in a conformal class and to estimate that probability.

• Techniques: differential geometry; spectral theory of Laplacian; Gaussian random fields on manifolds (Borell, Tsirelson-Ibragimov-Sudakov, Adler-Taylor).
• **Questions:** Assume $M \neq T^2$, and $g_0$ has *non-vanishing* curvature $K_0$. What is the *probability* that a random metric $g_1$ in the conformal class $[g_0]$ also has non-vanishing curvature $K_1$?

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• $g_0$ - reference metric on $M$. Conformal class of $g_0$: 
\{g_1 = e^f \cdot g_0\}; $f$ is a random (suitably regular) function on $M$.

• $\Delta_0$ - Laplacian of $g_0$. Spectrum: 
$\Delta_0 \phi_j + \lambda_j \phi_j = 0, \ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$. Define $f$ by

$$f(x) = - \sum_{j=1}^{\infty} a_j c_j \phi_j(x), \quad (1)$$

where $a_j \sim N(0, 1)$ are i.i.d standard Gaussians, $c_j = F(\lambda_j) \rightarrow 0$ are decreasing.
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where $a_j \sim \mathcal{N}(0, 1)$ are i.i.d standard Gaussians, $c_j = F(\lambda_j) \rightarrow 0$ are decreasing.
- Functions on $\mathbf{T}^2$: $h(x) = \sum_m c_m e^{i(x,m)}$, $m \in \mathbb{Z}^2$. Sobolev norm: $(\|f\|_{H^s})^2 = \sum_m |c_m|^2 (1 + \|m\|^2)^s$.

**General surface:** $f(x) = \sum_j c_j \phi_j(x)$.

$\|f\|_{H^s}^2 = \sum_j c_j^2 (1 + \lambda_j)^s$.

**Sobolev embedding theorem:** If $s > k + 1$, and $\|f\|_{H^s} < \infty$, then $f \in C^k(M)$.

**Weyl’s law:** $\lambda_j \asymp \text{const} \cdot j$.

- **Random functions:** $f$ as in (1), then

$$
\mathbb{E}(\|f\|_{H^s}^2) = \sum_j c_j^2 (1 + \lambda_j)^s.
$$

- **Proposition 1:** If $c_j < C/\lambda_j^s$, $s > 1$, then $f \in C^0(M)$ a.s.; if $c_j < C/\lambda_j^s$, $s > 2$, then $f \in C^2(M)$ a.s.
• Functions on $\mathbf{T}^2$: $h(x) = \sum_m c_m e^{i(x,m)}, m \in \mathbb{Z}^2$. Sobolev norm: $(\|f\|_{H^s})^2 = \sum_m |c_m|^2 (1 + \|m\|^2)^s$.

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$$\mathbb{E}(\|f\|_{H^s}^2) = \sum_j c_j^2 (1 + \lambda_j)^s.$$
• The covariance function
\[ r_f(x, y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)\phi_j(y), \text{ for } x, y \in M. \]

• For \( x \in M \), \( f(x) \) is mean zero Gaussian of variance
\[ r_f(x, x) = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)^2. \]

• Area change: Let \( A_0 = \text{area}(M, g_0) \). If \( g_1 := g_1(a) = e^{2af} g_0 \), then \( dA_1 = e^{2af} dA_0 \). One can show that \( \lim_{a \to 0} \mathbb{E}[\text{area}(M, g_1(a))] = A_0 \).
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• Let $g_1 = e^{2af}g_0$. Then

$$K_1 = e^{-2af}[K_0 - a\Delta_0 f] \quad (2)$$

$M \neq T^2$. Estimate the probability of

$$\{\text{Sgn}(K_1) = \text{Sgn}(K_0)\}$$

• **Observation:** If $K_0 \neq 0$, then

$$\text{Sgn}(K_1) = \text{Sgn}(K_0)\text{Sgn}(1 - a\Delta_0 f/(K_0)).$$

• Let $P(a) := \text{Prob}\{\exists x : \text{Sgn}K_1(x) \neq \text{Sgn}K_0\}$, or

$$P(a) = \text{Prob}\{\exists x \in M : 1 - a(\Delta_0 f)(x)/K_0(x) < 0\}. \text{ Then}$$

$$P(a) = \text{Prob}\{\sup_{x \in M}(\Delta_0 f)(x)/K_0(x) > 1/a\},$$

Consider the random field $v = (\Delta_0 f)/K_0$. Then

$$r_v(x, y) = \frac{\sum_j(c_j\lambda_j)^2\phi_j(x)\phi_j(y)}{K_0(x)K_0(y)}.$$
• Let \( g_1 = e^{2af} g_0 \). Then

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\[
r_v(x, y) = \frac{\sum_j (c_j \lambda_j)^2 \phi_j(x) \phi_j(y)}{K_0(x)K_0(y)}.
\]
• We shall estimate $P(a)$ in the limit $a \to 0$. Geometrically, this implies that a.s. $g_1(a) \to g_0$, so $P(a) \to 0$. We want to estimate the rate.

• First use Proposition 2 (Borell, TIS, 1975-76): Let $v$ be a centered Gaussian process, a.s. bounded on $M$, and

\[
\sigma_v^2 := \sup_{x \in M} \mathbb{E}[v(x)^2].
\]

Let $||v|| := \sup_{x \in M} v(x)$; then $E\{||v||\} < \infty$, and $\exists \alpha$ so that for $\tau > E\{||v||\}$ we have

\[
\text{Prob}\{||v|| > \tau\} \leq e^{\alpha \tau - \tau^2/(2\sigma_v^2)}.
\]

• Assume that $K_0 \in C^0$, $s > 2$, then $v \in C^0(M)$ a.s. and Proposition 2 applies. In our situation, $\tau = (1/a) \to \infty$ as $a \to 0$, so $P(a) \leq \exp[C_2/a - 1/(2a^2\sigma_v^2)]$. 
• We shall estimate $P(a)$ in the limit $a \to 0$. Geometrically, this implies that a.s. $g_1(a) \to g_0$, so $P(a) \to 0$. We want to estimate the rate.

• First use **Proposition 2** (Borell, TIS, 1975-76): Let $\nu$ be a centered Gaussian process, a.s. bounded on $M$, and $\sigma^2_\nu := \sup_{x \in M} \mathbb{E}[\nu(x)^2]$. Let $||\nu|| := \sup_{x \in M} \nu(x)$; then $E\{||\nu||\} < \infty$, and $\exists \alpha$ so that for $\tau > E\{||\nu||\}$ we have

$$\text{Prob}\{||\nu|| > \tau\} \leq e^{\alpha \tau - \tau^2/(2\sigma^2_\nu)}.$$ 

• Assume that $K_0 \in C^0$, $s > 2$, then $\nu \in C^0(M)$ a.s. and Proposition 2 applies. In our situation, $\tau = (1/a) \to \infty$ as $a \to 0$, so $P(a) \leq \exp[C_2/a - 1/(2a^2\sigma^2_\nu)].$
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• Assume that $K_0 \in C^0$, $s > 2$, then $v \in C^0(M)$ a.s. and Proposition 2 applies. In our situation, $\tau = (1/a) \to \infty$ as $a \to 0$, so $P(a) \leq \exp[C_2/a - 1/(2a^2\sigma_v^2)]$. 
• To estimate $P(a)$ \textit{from below} choose $x_0 \in M$ where the variance $r_\nu(x, x)$ attains its supremum $\sigma^2_\nu$. Clearly, 
\[ \text{Prob}(\|\nu\| > 1/a) \geq \text{Prob}(\nu(x_0) > 1/a) = \frac{1}{\sqrt{2\pi}} \int_1^{\infty} e^{-t^2/2} dt. \]
Combining the estimates:

• \textbf{Theorem 3:} Assume that $R_0 \in C^0, c_j = O(\lambda_j^{-s}), s > 2$. Then $\exists C_1 > 0, C_2 > 0$ such that
\[ (C_1 a) e^{-1/(2a^2\sigma^2_\nu)} \leq P(a) \leq e^{C_2/a-1/(2a^2\sigma^2_\nu)}, \]
as $a \to 0$. In particular \[ \lim_{a \to 0} a^2 \ln P(a) = \frac{-1}{2\sigma^2_\nu}. \]
• To estimate $P(a)$ from below choose $x_0 \in M$ where the variance $r_v(x, x)$ attains its supremum $\sigma_v^2$. Clearly,
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\text{Prob}(|v| > 1/a) \geq \text{Prob}(v(x_0) > 1/a) = \frac{1}{\sqrt{2\pi}} \int_1^{\infty} \frac{1}{(a \sigma_v)} e^{-t^2/2} dt.\]
Combine the estimates:

• **Theorem 3:** Assume that $R_0 \in C^0$, $c_j = O(\lambda_j^{-s})$, $s > 2$. Then $\exists C_1 > 0$, $C_2 > 0$ such that

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(C_1 a)e^{-1/(2a^2\sigma_v^2)} \leq P(a) \leq e^{C_2/a - 1/(2a^2\sigma_v^2)},
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as $a \to 0$. In particular $\lim_{a \to 0} a^2 \ln P(a) = \frac{-1}{2\sigma_v^2}$. 
• Random real-analytic metrics. Choose the coefficients $c_j = e^{-\lambda_j T/2}/\lambda_j$. Then

$$r_v(x, x, T) = e^*(x, x, T)/(K_0(x))^2.$$ 

where $e^*(x, x, T)$ is the heat kernel, without the constant term.

• Small $T$ asymptotics of $e^*(x, x, T)$ imply that as $T \to 0^+$,

$$\sigma_v^2 \sim \frac{1}{4\pi T \inf_{x \in M}(K_0(x))^2}.$$
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• **Theorem 4.** $M \neq T^2$. Let $g_0$ and $g_1$ have equal areas, $R_0$ and $R_1$ have constant sign, $K_0 \equiv \text{const}$ and $K_1 \neq \text{const}$. Then $\exists a_0 > 0$, $T_0 > 0$ (that depend on $g_0, g_1$) such that for any $0 < a < a_0$ and for any $0 < t < T_0$, we have $P(a, T, g_1) > P(a, T, g_0)$.

• **Proof:** By Gauss-Bonnet, $\int_M K_0 dA_0 = \int_M K_1 dA_1$. Since $A(M, g_0) = A(M, g_1)$; and since $K_0 \equiv \text{const}$ and $K_1 \neq \text{const}$, it follows that $b_0 := \min_{x \in M}(K_0(x))^2 > \min_{x \in M}(K_1(x))^2 := b_1$. Accordingly, as $T \to 0^+$, we have

$$\frac{\sigma^2_V(g_1, T)}{\sigma^2_V(g_0, T)} \asymp \frac{b_0}{b_1} > 1.$$  

The result follows easily from Theorem 3.
• **Theorem 4.** $M \neq T^2$. Let $g_0$ and $g_1$ have equal areas, $R_0$ and $R_1$ have constant sign, $K_0 \equiv const$ and $K_1 \not\equiv const$. Then $\exists a_0 > 0$, $T_0 > 0$ (that depend on $g_0$, $g_1$) such that for any $0 < a < a_0$ and for any $0 < t < T_0$, we have $P(a, T, g_1) > P(a, T, g_0)$.

• **Proof:** By Gauss-Bonnet, $\int_M K_0 dA_0 = \int_M K_1 dA_1$. Since $A(M, g_0) = A(M, g_1)$; and since $K_0 \equiv const$ and $K_1 \not\equiv const$, it follows that $b_0 := \min_{x \in M}(K_0(x))^2 > \min_{x \in M}(K_1(x))^2 := b_1$. Accordingly, as $T \to 0^+$, we have

$$\frac{\sigma_V^2(g_1, T)}{\sigma_V^2(g_0, T)} \sim \frac{b_0}{b_1} > 1.$$  

The result follows easily from Theorem 3.
• **Large $T$ asymptotics:**

$\lambda_1$ - the smallest nonzero eigenvalue of $-\Delta_0$. Let $m = m(\lambda_1)$ be the multiplicity of $\lambda_1$, and let

$$F := \sup_{x \in M} \frac{\sum_{j=1}^{m} \phi_j(x)^2}{K_0(x)^2}.$$ 

(3)

• One can show that

$$\lim_{T \to \infty} \frac{\sigma_v^2(T)}{Fe^{-\lambda_1 T}} = 1.$$ 

• **Theorem 5.** Let $g_0$ and $g_1$ be two metrics (of equal area) on a compact surface $M$, such that $K_0$ and $K_1$ have constant sign, and such that $\lambda_1(g_0) > \lambda_1(g_1)$. Then there exist $a_0 > 0$ and $0 < T_0 < \infty$ (that depend on $g_0$, $g_1$), such that for all $a < a_0$ and $T > T_0$ we have $P(a, T; g_0) < P(a, T; g_1)$. 
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**Large $T$ asymptotics:**

$\lambda_1$ - the smallest nonzero eigenvalue of $-\Delta_0$. Let $m = m(\lambda_1)$ be the multiplicity of $\lambda_1$, and let

$$F := \sup_{x \in M} \sum_{j=1}^{m} \frac{\phi_j(x)^2}{K_0(x)^2}.$$  \hfill (3)

One can show that

$$\lim_{T \to \infty} \frac{\sigma_V^2(T)}{Fe^{-\lambda_1 T}} = 1.$$  

**Theorem 5.** Let $g_0$ and $g_1$ be two metrics (of equal area) on a compact surface $M$, such that $K_0$ and $K_1$ have constant sign, and such that $\lambda_1(g_0) > \lambda_1(g_1)$. Then there exist $a_0 > 0$ and $0 < T_0 < \infty$ (that depend on $g_0, g_1$), such that for all $a < a_0$ and $T > T_0$ we have $P(a, T; g_0) < P(a, T; g_1)$. 
• To summarize: Small $T \Rightarrow$ metrics with $K_0 \equiv const$ extremal.
  
  • Large $T \Rightarrow$ metrics with the largest $\lambda_1$ extremal.
  
  • Genus 0: $(S^2, \text{round})$ extremal for both small $T$ and large $T$ (Hersch). **Conjecture:** extremal for all $T$.
  
  • Genus $\gamma \geq 2$: Small $T \Rightarrow$ hyperbolic metrics extremal.
  
  • Large $T$: By a 1985 theorem of R. Bryant, hyperbolic metrics never maximize $\lambda_1$ in their conformal class.
  
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• **Question:** Which metrics are extremal for intermediate $T$?
• We next indicate how to obtain a better estimate for $P(a)$ for $M = S^2$. ∃! conformal class $[g_0]$ on $S^2$; $g_0$ is the round metric, $K_0 \equiv 1$.

• The isometry group acts transitively on $(S^2, g_0)$, so the random fields $f(x), v(x)$ are *isotropic* and in particular have *constant variance*. That allows us to apply results of Adler and Taylor and obtain more precise *asymptotic* estimates for $P(a)$.
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Since $\Delta_0$ on $(S^2, g_0)$ is highly degenerate, we normalize our random Fourier series differently.

- $\mathcal{E}_m$ - space of spherical harmonics of degree $m$, dimension $N_m = 2m + 1$; the corresponding eigenvalue is $E_m = m(m + 1)$. Let $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$ be an orthonormal basis of $\mathcal{E}_m$.

- Let $f(x) = -\sqrt{|S^2|} \sum_{m \geq 1, k} \frac{\sqrt{c_m}}{E_m \sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$, where $a_{m,k}$ are standard Gaussian i.i.d. and $c_m > 0$ are (suitably decaying) constants satisfying $\sum_{m=1}^{\infty} c_m = 1$.

- It follows that $v = \sqrt{|S^2|} \sum_{m \geq 1, k} \frac{\sqrt{c_m}}{\sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$ has unit variance and covariance $r_v(x, y) = \sum_{m=1}^{\infty} c_m P_m(\cos(d(x, y)))$, where $P_m$ is the Legendre polynomial.
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• In the new normalization, if \( c_m = O(M^{-s}) \), \( s > 7 \), then 
\( (\Delta_0 f)(x) \in C^2(S^2) \) a.s.

• Applying results of A-T, we can prove

• **Theorem 6:** Notation as above, let 
\( c_m = O(m^{-s}) \), \( s > 7 \). Let 
\( C = \frac{1}{\sqrt{2\pi}} \sum_{m \geq 1} c_mE_m \). Then there exists \( \alpha > 1 \), s.t. in the limit \( a \to 0 \), \( P(a) \) satisfies

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P(a) = \frac{C}{a} \exp \left( -\frac{1}{2a^2} \right) + \frac{2}{\sqrt{2\pi}} \exp \left( -\frac{1}{2a^2} \right) \\
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• Note that we now have an *asymptotic* expression for \( P(a) \).
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• We next estimate the probability of the event \( \{ ||K_1 - K_0||_\infty < u \} \), \( u > 0 \); we shall do that for \( g_1 = e^{af} g_0 \), in the limit \( a \to 0 \). The result below hold for any compact orientable surface, including \( \mathbb{T}^2 \).

• To state the result, we define a new random field \( w \) on \( M \):

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w = \Delta_0 f + 2K_0 f.
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**Theorem 7:** Assume that the random metric is chosen so that the random fields $f, w$ are a.s. $C^0$. Let $a \to 0$ and $u \to 0$ so that $(u/a) \to \infty$. Then

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\log \text{Prob}(\|K_1 - K_0\|_\infty > u) \sim -\frac{u^2}{2a^2\sigma_w^2}.
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The proof uses Borell-TIS inequality. The condition $(u/a) \to \infty$ ensures that the application of Borell-TIS gives an asymptotic result for $\log \text{Prob}(\|K_1 - K_0\|_\infty > u)$.

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• Improve estimates in [CJW] for the scalar curvature in higher dimensions.

• Consider “rough” metrics that arise in 2D quantum gravity.

• Study the case when \( a \to 0 \).

• Study Ricci and sectional curvatures in high dimensions.

• Consider the space of all metrics, not just those in a conformal class (interesting in dimension \( n \geq 3 \)).

• Study differential geometry of random metrics, e.g. distance between two points, diameter etc.

• Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.

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