Measures on spaces of Riemannian metrics
CMS meeting
December 6, 2015

D. Jakobson (McGill), jakobson@math.mcgill.ca


December 6, 2015
Conjectures in Quantum Chaos: Random Wave conjecture, GOE level spacing conjecture, conjectures about nodal sets etc.

Integration on manifolds of metrics:
D. Ebin defined $L^2$ metric on manifolds of metrics. Restricted to the space of hyperbolic metrics on a compact surface, we get Weil-Petersson metric on moduli spaces of hyperbolic metrics (singular manifolds!). Many people studied differentiation on manifolds of metrics; such notions as Levi-Civita connection, geodesics, parallel transport etc. have been defined.

Want to define integration. Our spaces of metrics are infinite-dimensional, so it is convenient to define Gaussian measures on those spaces.
Fix a compact smooth Riemannian manifold $M^n$. We shall discuss several measures on manifolds of metrics on $M$.

- Measures on conformal classes of metrics: concentrated near a reference metric $g_0$, supported on regular (e.g. Sobolev, real-analytic) metrics a.s. Applications to the study of Gauss curvature.

- Measures on manifolds of metrics with the fixed volume form, applications to the study of $L^2$ (Ebin) distance function, and to integrability of the diameter, eigenvalue and volume entropy functionals.

- **Remark:** All measures are invariant by the action of diffeomorphisms.
Conformal class: \( g_0 \) - reference metric on \( M \). Conformal class of \( g_0 \): \( \{ g_1 = e^f \cdot g_0 \} \); \( f \) is a random (suitably regular) function on \( M \).

\( \Delta_0 \) - Laplacian of \( g_0 \). Spectrum:

\( \Delta_0 \phi_j + \lambda_j \phi_j = 0, \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \). Define \( f \) by

\[
f(x) = - \sum_{j=1}^{\infty} a_j c_j \phi_j(x),
\]

\( a_j \sim \mathcal{N}(0, 1) \) are i.i.d standard Gaussians, \( c_j = F(\lambda_j) \to 0 \) (damping):
The covariance function
\[ r_f(x, y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)\phi_j(y), \text{ for } x, y \in M. \]

For \( x \in M, f(x) \) is mean zero Gaussian of variance
\[ r_f(x, x) = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)^2. \]

Examples:
- Random Sobolev metric: \( c_j = \lambda_j^{-s} \), \( \implies \)
  \[ r_f(x, y) = \sum_j \frac{\phi_j(x)\phi_j(y)}{\lambda_j^{2s}}, \text{ spectral zeta function}. \]
- Random real-analytic metric \( c_j = e^{-\lambda_j t} \), \( \implies \)
  \[ r_f(x, y) = \sum_j \phi_j(x)\phi_j(y)e^{-2\lambda_j t}, \text{ heat kernel}. \]
Sobolev regularity: If

$$
\mathbb{E}\|f\|^2_{H^s} = \sum_j c_j^2(1 + \lambda_j)^s < \infty
$$

then $$f \in H^s(M)$$ a.s. Weyl’s law + Sobolev embedding imply

**Proposition:** If $$c_j = O(\lambda_j^{-s})$$, $$s > n/2$$, then $$f \in C^0$$ a.s; if $$c_j = O(\lambda_j^{-s})$$, $$s > n/2 + 1$$, then $$f \in C^2$$ a.s.
[CJW]: Let $n = 2$, and let $g_0$ have non-vanishing Gauss curvature ($M \neq T^2$). Can estimate the probability that after a random conformal perturbation, the Gauss curvature will change sign somewhere on $M$.

- Techniques: curvature transformation in 2d under conformal changes of metrics, large deviation estimates (Borell, Tsirelson-Ibragimov-Sudakov, Adler-Taylor).

- $n \geq 3$: related results for scalar curvature and $Q$-curvature.
Random (Sobolev) embeddings into $\mathbb{R}^k$: 1-dimensional i.i.d. Gaussians $\rightarrow k$-dimensional i.i.d. Gaussians.

F. Morgan (1979): $M = S^1$, $k = 3$: a.s. results about minimal surfaces spanned by random “knots.”


We shall use similar ideas to define Gaussian measures on manifolds of metrics with fixed volume form; transverse to conformal classes.
Random (Sobolev) embeddings into $\mathbb{R}^k$: 1-dimensional i.i.d. Gaussians $\rightarrow$ $k$-dimensional i.i.d. Gaussians.

F. Morgan (1979): $M = S^1, k = 3$: a.s. results about minimal surfaces spanned by random “knots.”


We shall use similar ideas to define Gaussian measures on manifolds of metrics with fixed volume form; transverse to conformal classes.
Random (Sobolev) embeddings into $\mathbb{R}^k$: 1-dimensional i.i.d. Gaussians $\rightarrow$ $k$-dimensional i.i.d. Gaussians.

F. Morgan (1979): $M = S^1$, $k = 3$: a.s. results about minimal surfaces spanned by random “knots.”


We shall use similar ideas to define Gaussian measures on manifolds of metrics with \textit{fixed volume form}; transverse to conformal classes.
Random (Sobolev) embeddings into $\mathbb{R}^k$: 1-dimensional i.i.d. Gaussians $\rightarrow k$-dimensional i.i.d. Gaussians.

F. Morgan (1979): $M = S^1, k = 3$: a.s. results about minimal surfaces spanned by random “knots.”


We shall use similar ideas to define Gaussian measures on manifolds of metrics with *fixed volume form*; transverse to conformal classes.
Metrics = sections of $\text{Pos}(M) \subset \text{Sym}(M) \subset \text{Hom}(TM, T^*M)$
(positive-definite, symmetric maps); symmetric matrices in local coordinates. $\text{GL}(T_xM)$ acts on $\text{Pos}_x(M)$ with stabilizer isomorphic to $O(n)$.
Fix a volume form $\nu$, consider $\text{Met}_\nu(M)$. $\text{SL}(T_xM)$ acts on the fibre $\text{Pos}_x^\nu(M)$ by

$$h.g_x = h^T \circ g_x \circ h;$$

stabilizer isomorphic to $SO(n)$. We have

$$\text{Pos}_x^\nu(M) \cong \text{SL}_n(\mathbb{R})/SO_n$$

Fix $g^0 \in \text{Met}_\nu$; $d\nu(x) = \sqrt{|\det g^0(x)|}dx_1 \wedge \ldots \wedge dx_n$. Let $G_x = \text{SL}(T_xM)$, $K_x = \text{SO}(g^0_x)$. 
Metrics = sections of $\text{Pos}(M) \subset \text{Sym}(M) \subset \text{Hom}(TM, T^*M)$ (positive-definite, symmetric maps); symmetric matrices in local coordinates. $\text{GL}(T_xM)$ acts on $\text{Pos}_x(M)$ with stabilizer isomorphic to $O(n)$.

Fix a volume form $\nu$, consider $\text{Met}_\nu(M)$. $\text{SL}(T_xM)$ acts on the fibre $\text{Pos}^\nu_x(M)$ by

$$h.g_x = h^T \circ g_x \circ h;$$

stabilizer isomorphic to $\text{SO}(n)$. We have

$$\text{Pos}^\nu_x(M) \cong \text{SL}_n(\mathbb{R})/\text{SO}_n$$

Fix $g^0 \in \text{Met}_\nu$; $d\nu(x) = \sqrt{|\det g^0(x)|} \, dx_1 \wedge \ldots \wedge dx_n$. Let $G_x = \text{SL}(T_xM), K_x = \text{SO}(g^0_x)$. 
- $f_x$ frame in $T_xM$ orthonormal w.r.to $g^0_x$, $A_x \subset G_x$ positive diagonal matrices of determinant 1 (w.r. to $f_x$).
  Every $g^1_x \in \text{Pos}^v_x(M)$ can be represented as
  \[ g^1_x = (k_x a_x) g^0_x, \quad k_x \in K_x, a_x \in A_x; \]
  unique up to $S_n$ acting on $f_x$.
- Assumption: $M$ is parallelizable ($\exists$ global section of the frame bundle). Examples:
  - All 3-manifolds;
  - All Lie groups;
  - The frame bundle of any manifold;
  - The sphere $S^n$ iff $n \in \{1, 3, 7\}$.
  Necessary condition: vanishing of the 2nd Stiefel-Whitney class. For orientable $\iff$ spin.
\( f_x \) frame in \( T_xM \) orthonormal w.r.to \( g^0_x \), \( A_x \subset G_x \) positive diagonal matrices of determinant 1 (w.r. to \( f_x \)).

Every \( g^1_x \in Pos^\forall_x(M) \) can be represented as

\[
g^1_x = (k_x a_x) g^0_x, \quad k_x \in K_x, a_x \in A_x;
\]

unique up to \( S_n \) acting on \( f_x \).

- Assumption: \( M \) is parallelizable (\( \exists \) global section of the frame bundle). Examples:
  - All 3-manifolds;
  - All Lie groups;
  - The frame bundle of any manifold;
  - The sphere \( S^n \) iff \( n \in \{1, 3, 7\} \).

Necessary condition: vanishing of the 2nd Stiefel-Whitney class. For orientable \( \iff \) spin.
$M$ parallelizable. Choose a global section $f^0$ of the frame bundle orthonormal w.r. to $g^0$.

To define $g^1_x$, we apply to $f^0$ a composition of a rotation $k_x \in SO(T_xM)$ and a diagonal unimodular transformation $a_x \in SL(T_xM)$ which will define an orthonormal basis $f^1_x$ for $g^1_x$. By construction, $g^0$ and $g^1$ will have the same volume form.

We let $a_x = \exp(H(x))$, where $H : M \to a \cong \mathbb{R}^{n-1}$ is the Lie algebra of $\text{Diag}_0(n) \subset SL_n$. Similarly, let $k_x = \exp h(x)$, where $h : M \to so_n$, the Lie algebra of $SO_n$. Choice of $g^0 +$ parallelizability makes the above construction well-defined.
We define Gaussian measures on \( \{ H : M \to a \} \) and \( \{ h : M \to \mathcal{S}_n \} \) as in Morgan. In the sequel, only need \( H \); constructions are analogous.
Let

\[ H(x) = \sum_{j=1}^{\infty} \pi_n(A_j) \beta_j \psi_j(x), \]  

(1)

where

• \( \Delta \psi_j + \lambda_j \psi_j = 0; \)
• \( A_j \) are i.i.d standard Gaussians in \( \mathbb{R}^n; \)
• \( \pi_n : \mathbb{R}^n \to \{ x \in \mathbb{R}^n : x \cdot (1, \ldots, 1) = 0 \} \simeq \mathbb{R}^{n-1} \) - projection into the hyperplane \( \sum_{j=1}^{n} x_j = 0; \)
• \( \beta_j = F_2(\lambda_j) > 0, \) where \( F_2(t) \) is (eventually) monotone decreasing function of \( t, F(t) \to 0 \) as \( t \to \infty. \)
Smoothness: Morgan showed

**Proposition 1:** If $\beta_j = O(j^{-r})$ where $r > (q + \alpha)/n + 1/2$, then $H$ converges a.s. in $C^{q,\alpha}(M, \mathbb{R}^{n-1})$.

**Proposition 2:** If $\beta_j = O(\lambda_j^{-s})$ where $s > q/2 + n/4$, then $H$ converges a.s. in $C^q(M, \mathbb{R}^{n-1})$. 
Smoothness: Morgan showed

Proposition 1: If \( \beta_j = O(j^{-r}) \) where \( r > (q + \alpha)/n + 1/2 \), then \( H \) converges a.s. in \( C^{q,\alpha}(M, \mathbb{R}^{n-1}) \).

Proposition 1 + Weyl’s law \( \implies \)

Proposition 2: If \( \beta_j = O(\lambda_j^{-s}) \) where \( s > q/2 + n/4 \), then \( H \) converges a.s. in \( C^q(M, \mathbb{R}^{n-1}) \).
Lipschitz distance $\rho$:

$$\rho(g_0, g_1) = \sup_{x \in M} \sup_{0 \neq \xi \in T_x M} \left| \ln \frac{g_1(\xi, \xi)}{g_0(\xi, \xi)} \right|$$ \hspace{1cm} (2)

A related expression appeared in the paper by Bando-Urakawa. If $g_1(x) = (k(x)d(x))g_0(x)$ then $\rho$ only depends on the diagonal part $d(x)$.

**Tail estimate for $\rho$:**

One can show that for large $R$,

$$\text{Prob}\{\rho(g_0, g_1) > R\} \leq 2^n(n + \epsilon) \cdot \text{Prob}\left\{\sup_{x \in M} d_1(x) > R/2\right\}$$ \hspace{1cm} (3)
**Definition of** $d_1$:

Recall from (1): $H(x) = \sum_{j=1}^{\infty} \pi_n(A_j)\beta_j\psi_j(x)$.

Define $D(x) = (d_1(x), \ldots, d_n(x))$ by

$$D(x) = \sum_{j=1}^{\infty} A_j\beta_j\psi_j(x)$$

(“don’t project $A_j$”).

The covariance function for $d_1(x)$:

$$r_{d_1}(x, y) = \sum_{k=1}^{\infty} \beta_k^2\psi_k(x)\psi_k(y),$$

Define $\sigma^2$ by

$$\sigma^2 := \sigma(d_1)^2 := \sup_{x \in M} r_{d_1}(x, x).$$

(4)
Borell-TIS theorem applied to the random field $d_1$ implies

**Proposition 3.** Let $\sigma$ be as in (4). Then

$$
\lim_{R \to \infty} \frac{\ln \Prob\{\rho(g_0, g_1) > R\}}{R^2} \leq \frac{-1}{8\sigma^2}.
$$

(5)

More precise result:

**Proposition 4.** There exists $\alpha > 0$ such that for a fixed $\epsilon > 0$ and for large enough $R$, we have

$$
\Prob\{\rho(g_1, g_0) > R\} \leq 2^n (n + \epsilon) \exp\left(\frac{\alpha R}{2} - \frac{R^2}{8\sigma^2}\right).
$$
ρ controls diameter and eigenvalues:
Proposition 5.
Assume that \( d\text{vol}(g_0) = d\text{vol}(g_1) \) and \( \rho(g_0, g_1) < R \). Then

\[
e^{-R} \leq \frac{\text{diam}(M, g_1)}{\text{diam}(M, g_0)} \leq e^R
\]

(6)

and

\[
e^{-2R} \leq \frac{\lambda_k(\Delta(g_1))}{\lambda_k(\Delta(g_0))} \leq e^{2R}.
\]

(7)
Propositions 4 and 5 imply

**Theorem 6.**
Let \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) be a monotonically increasing function such that for some \( \delta > 0 \)

\[
h(e^y) = O\left(\exp\left[y^2\left(1/(8\sigma^2) - \delta\right)\right]\right).
\]

Then \( h(\text{diam}(g_1)) \) is integrable with respect to the probability measure \( d\omega(g_1) \) constructed earlier.

and

**Theorem 7.** Let \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) be a monotonically increasing function such that for some \( \delta > 0 \)

\[
h(e^{2y}) = O\left(\exp\left[y^2\left(1/(8\sigma^2) - \delta\right)\right]\right).
\]

Then \( h(\lambda_k(\Delta(g_1))) \) is integrable with respect to the probability measure \( d\omega(g_1) \) constructed earlier.
Similar results can be established for *volume entropy*,

$$h_{vol} = \lim_{s \to \infty} \frac{\ln \text{vol} B(x, s)}{s}$$
- $L^2$ or *Ebin* distance between can be computed as follows [Ebin, Clarke]:

$$
\Omega_2^2(g^0, g^1) := \int_M d_{2,x}(g^0(x), g^1(x))^2 dv(x)
$$

where $d_{2,x}(g^0(x), g^1(x))$ is the distance in $\text{SL}_n/\text{SO}_n$;

$$
= \int_M \langle H(x), H(x) \rangle_{g^0(x)} dv(x).
$$

- For us $\Omega_2^2$ is a random variable whose distribution we shall compute. Note: only depends on $H$, hence it suffices to consider the Gaussian measure on $\{H : M \to a\}$. 

\( L^2 \) or \textit{Ebin} distance between can be computed as follows [Ebin, Clarke]:

\[
\Omega_2^2(g^0, g^1) := \int_{M} d_{2, x}(g^0(x), g^1(x))^2 dv(x)
\]

where \( d_{2, x}(g^0(x), g^1(x)) \) is the distance in \( SL_n/\text{SO}_n \);

\[
= \int_{M} \langle H(x), H(x) \rangle_{g^0(x)} dv(x).
\]

For us \( \Omega_2^2 \) is a \textit{random variable} whose distribution we shall compute. Note: only depends on \( H \), hence it suffices to consider the Gaussian measure on \( \{ H : M \rightarrow \alpha \} \).
In local coordinates, let

$$a_x = \text{diag}(\exp(b_1(x)), \exp(b_2(x)), \ldots, \exp(b_n(x))),$$

where $\sum_{j=1}^{n} b_j(x) = 0, \forall x \in M$. Then

$$d_x(g^0_x, g^1_x)^2 = \sum_{j=1}^{n} b_j(x)^2.$$ 

Accordingly,

$$\Omega_2(g^0, g^1)^2 = \int_M \left( \sum_{j=1}^{n} b_j(x)^2 \right) dv(x).$$
\( \pi_n : \mathbb{R}^n \rightarrow \{ x : \sum x_j = 0 \} \) standard projection. \( P_n \) - matrix of \( \pi_n \) (in the usual basis of \( \mathbb{R}^n \)) with singular values

\[
(1, \ldots, 1, 0) := \mu_{i,n}, 1 \leq i \leq n.
\]

Then in distribution

\[
\Omega_2^2 \overset{D}{=} \sum_j \beta_j^2 \sum_{i=1}^n \mu_{i,n}^2 W_{i,j}
\]

where \( W_{i,j} \sim \chi_1^2 \) are i.i.d. We get \( \Omega_2^2 \overset{D}{=} \sum_j \beta_j^2 V_j \) where \( V_j \sim \chi_{n-1}^2 \) are i.i.d.
Theorem 8.
Moment generating function of $\Omega^2_2$:

$$M_{\Omega^2_2}(t) = E(\exp(t\Omega^2_2)) = \prod_{j} \prod_{i=1}^{n} M_{\chi^2_1}(t\mu^2_i,n\beta^2_j)$$

$$= \prod_{j} \prod_{i=1}^{n} (1 - 2t\mu^2_i,n\beta^2_j)^{-1/2} = \prod_{j} (1 - 2t\beta^2_j)^{-(n-1)/2}$$

Characteristic function of $\Omega^2_2$:

$$\prod_{j} \prod_{i=1}^{n} (1 - 2it\mu^2_i,n\beta^2_j)^{-1/2} = \prod_{j} (1 - 2it\beta^2_j)^{-(n-1)/2}$$
Corollary 9.
Tail estimates for $\Omega_2^2$: applying results of Laurent-Massart, one can show that

$$\text{Prob}\{\Omega_2^2 \geq s^2\} \leq \exp(-s^2/(2\beta_1^2)).$$