Math 599 Nov. 1st.

Recall: Let H be a separable Hilbert space. E be a separable Banach space and H is continuously embedded in E as a dense subspace.

$\forall x^* \in E^*$. $\exists !\ h_{x^*} \in H$ s.t. ${}_E\langle \cdot, x^* \rangle_{E^*}\big|_H = (\cdot, h_{x^*})_H$

$\mathcal{W} \in \mathcal{M}_1(E)$ is a centered Gaussian measure

space of prob. measures on $(E, \mathcal{B}_E)$

$(H, E, \mathcal{W})$ is an abstract Wiener space (AWS) if $\forall x^* \in E^*$

$$\mathbb{E}^{\mathcal{W}}[e^{i\langle \cdot, x^* \rangle}] = \exp(-\tfrac{1}{2}\|h_{x^*}\|_H^2).$$

H is the Cameron-Martin space. $\mathcal{W}$ is the Wiener measure.

The isometry $\mathcal{I}: H \to L^2(E;\mathcal{W})$ s.t. $\mathcal{I}(h_{x^*}) = \langle \cdot, x^* \rangle$ $\forall x^* \in E^*$ is the Paley-Wiener map. Its images $\{\mathcal{I}(h) : h \in H\}$ are Paley-Wiener integrals. (centered Gaussian family. covariance given by

$$\mathrm{Cov}(\mathcal{I}(h), \mathcal{I}(g)) = (h, g)_H.$$

Fernique's Theorem: Given any separable Banach space E and a Gaussian measure $\mathcal{W}$ on E.

$\exists\, \delta > 0$ s.t. $\mathbb{E}^{\mathcal{W}}[e^{\delta\|\cdot\|_E^2}] < \infty$.

To day, we see more results on the structure of infinite dimensional Gaussian measure

Question 1. Given separable Banach space E and a centered Gauss. meas. $\mathcal{W}$ does there always exist a Cameron-Martin space H?

Theorem 1 Given E and $\mathcal{W}$ (same as above). $\exists!$ separable Hilbert space H s.t. $(H, E, \mathcal{W})$ is an AWS.

Proof. We first prove the uniqueness. Suppose H is such a Hilbert space. then

$\forall x^*, y^* \in E^*$ $\quad \langle h_{x^*}, y^* \rangle = (h_{x^*}, h_{y^*})_H = \langle h_{y^*}, x^* \rangle$

In particular. $\langle h_{x^*}, x^* \rangle = \|h_{x^*}\|_H^2 = \int_E \langle x, x^* \rangle^2 d\mathcal{W}$

So. by the Polarization identity.

$$\langle h_{x^*}, y^* \rangle = \tfrac{1}{4}\|h_{x^*} + h_{y^*}\|_H^2 - \tfrac{1}{4}\|h_{x^*} - h_{y^*}\|_H^2$$

$$= \int_E \langle x, x^* \rangle \langle x, y^* \rangle\, d\mathcal{W}$$

$$= \left\langle \int_E \langle x, x^* \rangle x\, d\mathcal{W},\ y^* \right\rangle$$

On the other hand. $\left\| \int_E \langle x, x^* \rangle x\, d\mathcal{W} \right\|_E \le \int_E \|x\|_E\, |\langle x, x^* \rangle|_E\, d\mathcal{W} < \infty$

so. $\int_E \langle x, x^* \rangle x\, d\mathcal{W} \in E$. Since y^* is arbitrary. $h_{x^*} = \int_E \langle x, x^* \rangle x\, d\mathcal{W}$

Next, for general $h \in H$, assume that $h_{x_n^*} \to h$ in H (since $\{h_{x^*} : x^* \in E^*\}$ is dense in H) where $x_n^* \in E^*$. Then $\{\langle \cdot, x_n^* \rangle : n \geq 1\}$ forms a Cauchy sequence in $L^2(E; \mathcal{W})$

Denote $\psi_h := \lim_n \langle \cdot, x_n^* \rangle$ in $L^2(E; \mathcal{W})$. Then, $\forall y^* \in E^*$

$$\langle h, y^* \rangle = \lim_n \langle h_{x_n^*}, y^* \rangle = \lim_n \int_E \langle x, x_n^* \rangle \langle x, y^* \rangle \, d\mathcal{W} = \int_E \psi_h(x) \langle x, y^* \rangle \, d\mathcal{W}$$

if $h_{x_n^*} \to h$ in H, then $h_{x_n^*} \to h$ in E

$$= \left\langle \int_E \psi_h(x) x \, d\mathcal{W}, y^* \right\rangle \Rightarrow h = \int_E \psi_h(x) x \, d\mathcal{W}$$

Therefore, $\forall h \in H$, $\exists \psi_h \in \overline{\{\langle \cdot, x^* \rangle : x^* \in E^*\}}^{L^2} := \Phi$, s.t. $h = \int_E \psi_h(x) x \, d\mathcal{W}$

and such a ψ_h is unique (for if $\exists \tilde{\psi}_h = \lim_n \langle \cdot, \tilde{x}_n^* \rangle$ in L^2 with $h_{\tilde{x}_n^*} \to h$ in H,

then $\|\psi_h - \tilde{\psi}_h\|_{L^2}^2 = \lim_n \|\langle \cdot, x_n^* - \tilde{x}_n^* \rangle\|_{L^2}^2 = \lim_n \|h_{x_n^*} - h_{\tilde{x}_n^*}\|_H^2 = 0$).

Conversely, $\forall \phi \in \Phi$, assume that $\{y_n^*\} \subseteq E^*$ s.t. $\langle \cdot, y_n^* \rangle \to \phi$ in $L^2(E; \mathcal{W})$

Then $\{h_{y_n^*} : n \geq 1\}$ forms a Cauchy seq. in H. $\Rightarrow \exists h \in H$ s.t. $h_{y_n^*} \to h$ in H.

Similarly, $h = \int_E \phi(x) x \, d\mathcal{W}$. Therefore $H = \left\{ \int_E \phi(x) x \, d\mathcal{W} : \phi \in \Phi \right\}$ (*)

To show the existence, all we need to do is to verify that the set on the RHS above is the Cameron-Martin space. For $h \in$ RHS of (*), i.e. $h = \int_E \phi(x) x \, d\mathcal{W}$ with $\phi \in \overline{\{\langle \cdot, x^* \rangle : x^* \in E^*\}}^{L^2}$, define $\|h\|_H := \|\phi\|_{L^2}$. $\|h\|_E \leq \int_E |\phi(x)| \, \|x\|_E \, d\mathcal{W} \leq C \|\phi\|_{L^2} = C \|h\|_H$

$h \in E$ and RHS of (*) is continuously embedded in E.

To show it is also dense, take $x^* \in E^*$, assume that $\langle h, x^* \rangle = 0$ $\forall h \in$ RHS of (*)

$\Rightarrow \forall \phi \in \Phi$, $\int_E \phi(x) \langle x, x^* \rangle \, d\mathcal{W} = 0$. In particular, $\int_E \langle x, x^* \rangle^2 \, d\mathcal{W} = 0 \Rightarrow x^* = 0$

Finally, $\forall x^* \in E^*$, $\|h_{x^*}\|_H^2 = \left\| \int_E \langle x, x^* \rangle x \, d\mathcal{W} \right\|_H^2 = \|\langle \cdot, x^* \rangle\|_{L^2}^2$

$\Rightarrow \langle \cdot, x^* \rangle$ under $\mathcal{W}$ has distribution $N(0, \|h_{x^*}\|_H^2)$

So, the RHS of (*) is indeed the Cameron-Martin space for $(E, \mathcal{W})$ $\square$

<u>Remark</u>: Any infinite dim. Gauss. meas. (on a separable Banach space) exists in the form of AWS.

We now prove the Cameron–Martin Theorem in the general AWS setting

Theorem 2 (Cameron–Martin) Let $(H, E, \mathcal{W})$ be an AWS for $g \in E$, define $\tau_g : E \to E$ by $\tau_g(x) = x + g$. Denote $\mathcal{W}_{\tau_g}$ the distribution of τ_g under $\mathcal{W}$, or i.e. $\mathcal{W}_{\tau_g} = (\tau_g)_* \mathcal{W}$. Then,

① if $g \in H$, then $\mathcal{W}_{\tau_g} \ll \mathcal{W}$ and

$$d\mathcal{W}_{\tau_g}/d\mathcal{W} = \exp\left(\mathcal{I}(g) - \tfrac{1}{2}\|g\|_H^2\right) \rightarrow \text{Cameron–Martin formula}$$

② if $g \notin H$, then $\mathcal{W}_{\tau_g} \perp \mathcal{W}$

Proof: ① Assume $g \in H$, denote $E_g := \exp(\mathcal{I}(g) - \frac{1}{2}\|g\|_H^2)$.

$\forall x^* \in E^*$ $\quad \mathbb{E}^{\mathcal{W}_{\tau_g}}[e^{i\langle \cdot, x^*\rangle}] = \int_E e^{i\langle x+g, x^*\rangle} d\mathcal{W}$

$$= e^{i\langle g, x^*\rangle} \cdot \mathbb{E}^{\mathcal{W}}[e^{i\langle \cdot, x^*\rangle}] = e^{i\langle g, x^*\rangle} \cdot e^{-\frac{1}{2}\|h_{x^*}\|_H^2}$$

On the other hand,

$$\mathbb{E}^{\mathcal{W}}[e^{i\langle \cdot, x^*\rangle} E_g] = \int_E e^{i\langle x, x^*\rangle + \mathcal{I}(g)(x)} d\mathcal{W} \cdot e^{-\frac{1}{2}\|g\|_H^2}$$

$$= \int_E e^{(i\mathcal{I}(h_{x^*}) + \mathcal{I}(g))(x)} d\mathcal{W} \cdot e^{-\frac{1}{2}\|g\|_H^2}$$

$$= e^{-\frac{1}{2}\|h_{x^*}\|_H^2} \cdot e^{i(h_{x^*}, g)_H} \cdot e^{\frac{1}{2}\|g\|_H^2} \cdot e^{-\frac{1}{2}\|g\|_H^2} = e^{-\frac{1}{2}\|h_{x^*}\|_H^2} \cdot e^{i\langle g, x^*\rangle}$$

$\Rightarrow \forall x^* \in E^*$. $\mathbb{E}^{\mathcal{W}_{\tau_g}}[e^{i\langle \cdot, x^*\rangle}] = \mathbb{E}^{\mathcal{W}}[e^{i\langle \cdot, x^*\rangle} E_g]$

$\Rightarrow \mathcal{W}_{\tau_g} = E_g \cdot \mathcal{W}$ (i.e. $E_g = d\mathcal{W}_{\tau_g}/d\mathcal{W}$)

② Recall that for $g \in E$, $g \in H \Leftrightarrow \underbrace{\sup\{|\langle g, x^*\rangle| : x^* \in E^*\ \|h_{x^*}\|_H = 1\}}_{<\infty}$

Now assume that $\mathcal{W}_{\tau_g} \not\perp \mathcal{W}$, and let R be the Radon–Nikodym derivative of its absolutely continuous part. Given $x^* \in E^*$ with $\|h_{x^*}\|_H = 1$, let $\mathcal{F}_{x^*}$ be the σ-field generated by $\langle \cdot, x^*\rangle$. Then $\forall A \in \mathcal{F}_{x^*}$ $\exists B \in \mathcal{B}(\mathbb{R})$ s.t. $A = \{\langle \cdot, x^*\rangle \in B\}$. Therefore,

$$\mathcal{W}_{\tau_g}(A) = \mathcal{W}(\{x \in E : \langle x+g, x^*\rangle \in B\}) = N(\langle g, x^*\rangle, \|h_{x^*}\|_H^2)(B)$$

$$= \frac{1}{\sqrt{2\pi}\|h_{x^*}\|} \int_B e^{-\frac{y^2}{2\|h_{x^*}\|_H^2}} \cdot e^{\frac{\langle g, x^*\rangle y}{\|h_{x^*}\|_H^2}} dy \cdot e^{-\frac{\langle g, x^*\rangle^2}{2\|h_{x^*}\|_H^2}}$$

$$= \mathbb{E}^{\mathcal{W}}[\exp(\langle g, x^*\rangle\langle \cdot, x^*\rangle) \mathbb{1}_B] \exp(-\tfrac{1}{2}\langle g, x^*\rangle^2)$$

$\Rightarrow \mathcal{W}_{\tau_g}|_{\mathcal{F}_{x^*}} \ll \mathcal{W}|_{\mathcal{F}_{x^*}}$ and the R-N derivative is given by

$$G(x) = \exp\left(\langle g, x^*\rangle\langle x, x^*\rangle - \tfrac{1}{2}\langle g, x^*\rangle^2\right)$$

Hence, $G \geq \mathbb{E}^{\mathcal{W}}[R \mid \mathcal{F}_{x^*}]$ for $\forall A \in \mathcal{F}_{x^*}$ $\mathcal{W}_{\tau_g}(A) \geq \int_A R\, d\mathcal{W}$

so $G \geq \mathbb{E}^{\mathcal{W}}[\sqrt{R} \mid \mathcal{F}_{x^*}]^2 \Rightarrow \sqrt{G} \geq \mathbb{E}^{\mathcal{W}}[\sqrt{R} \mid \mathcal{F}_{x^*}]$