Motivation If the density function formulation (Feynman's Representation) of $\mathcal{W}_0$ fails, one can try the characteristic function formulation:

Def: Let E be a separable Banach space. A prob. measure $\mathcal{W}$ on $(E, \mathcal{B}_E)$ is a (non-degenerate) Gaussian measure on E iff $\forall x^* \in E^*$, $x^* \neq 0$,

$x \in E \longmapsto \langle x, x^* \rangle \in \mathbb{R}$ is a (non-degenerate) Gaussian r.v. under $\mathcal{W}$.

Back to $(\Theta_0, \mathcal{B}(\Theta_0), \mathcal{W}_0)$, we want to study $\mathcal{W}_0$ through the actions of $\lambda \in \Theta_0^*$. We start with the action of λ on H_0^1.

Let $h \in H_0^1$. $\langle h, \lambda \rangle = \int_{[0,+\infty)} h(t)\,\lambda(dt) = -\int_{[0,+\infty)} h(t)\, d\big(\lambda((t,+\infty))\big)$

$$= \cancel{-h(t)\lambda((t,+\infty))\Big|_0^{+\infty}}^{\,0} + \int_{[0,+\infty)} \lambda((t,+\infty))\,\dot{h}(t)\,dt = (h, h_\lambda)_{H_0^1}$$

where $h_\lambda(t) = \int_{[0,t]} \lambda((\tau,+\infty))\,d\tau$.

Moreover $\|h_\lambda\|_{H_0^1}^2 = \int_{[0,+\infty)} |\lambda((\tau,+\infty))|^2\,d\tau = \int_{[0,+\infty)} \iint_{(\tau,+\infty)^2} \lambda(dt)\lambda(ds)\,d\tau$

$$= \iint_{[0,+\infty)^2} (s\wedge t)\,\lambda(dt)\lambda(ds).$$

Apply λ to Θ_0: $\langle \theta, \lambda \rangle = -\int_{[0,+\infty)} \theta(t)\, d(\lambda(t,+\infty)) = \int_0^{+\infty} \lambda((t,+\infty))\,d\theta(t)$

$\{\theta(t): t \in [0,+\infty)\}$ is a standard B.M. under $\mathcal{W}_0$ $\Rightarrow \int_0^{+\infty} \lambda((t,+\infty))\,d\theta_t$ is a stochastic integral and has distribution $N\left(0, \int_0^{+\infty} |\lambda((t,+\infty))|^2 dt\right)$

$\Rightarrow$ Under $\mathcal{W}_0$, $\langle \cdot, \lambda \rangle$ has distribution $N(0, \|h_\lambda\|_{H_0^1}^2)$.

Theorem. $\mathcal{W}_0$ is the Wiener measure on $(\Theta_0, \mathcal{B}(\Theta_0))$ iff

$$\forall \lambda \in \Theta_0^*, \quad \mathbb{E}^{\mathcal{W}_0}\left[e^{i\langle \cdot, \lambda \rangle}\right] = e^{-\frac{\|h_\lambda\|_{H_0^1}^2}{2}} \longrightarrow \text{char. fcn. formulation}$$

where $h_\lambda(t) = \int_{[0,t]} \lambda((\tau,+\infty))\,d\tau$ is the unique element in H_0^1 s.t. $\lambda\big|_{H_0^1} = (\cdot, h_\lambda)_{H_0^1}$

Proof: "$\Rightarrow$" Done.

"$\Leftarrow$" If $\mathcal{W}_0$ has the char. fcn. as above, then $\forall 0 < r \le s \le t$, $\forall \xi_1, \xi_2 \in \mathbb{R}$, take $\lambda = \xi_1(\delta_t - \delta_s) + \xi_2 \delta_r$ and compute

$$\|h_\lambda\|_{H_0^1}^2 = \iint_{[0,+\infty)^2} u\wedge v\,\lambda(du)\lambda(dv) = \int_{[0,+\infty)}\int_{[u,+\infty)} u\,\lambda(dv)\,\lambda(du) + \int_{[0,+\infty)}\int_{(v,+\infty)} v\,\lambda(du)\lambda(dv)$$

$$= \xi_1\left(\int_{[t,+\infty)} t\,\lambda(dv) - \int_{[s,+\infty)} s\,\lambda(dv)\right) + \xi_2\left(\int_{[r,+\infty)} r\,\lambda(dv)\right)$$

$$+ \xi_1\left(\int_{(t,+\infty)} t\,\lambda(du) - \int_{(s,+\infty)} s\,\lambda(du)\right) + \xi_2\left(\int_{(r,+\infty)} r\,\lambda(du)\right)$$

$= \xi_1 [t\xi_1 - s(\xi_1 - \xi_1)] + \xi_2 r(\xi_1 - \xi_1 + \xi_2) + \xi_1 (t \cdot 0 - s\xi_1) + \xi_2 r(\xi_2 - \xi_1)$

$= \xi_1^2 (t-s) + \xi_2^2 r$

By the hypothesis. $\mathbb{E}^{W_0}\left[e^{i\xi_1(\theta_t - \theta_s) + i\xi_2 \theta_r} \right] = \exp\left(-\frac{\xi_1^2 (t-s)}{2} - \frac{\xi_2^2 r}{2} \right)$

$\Rightarrow$ Under W_0. $\{\theta(t) : t \in [0,+\infty)\}$ has the same dist. as B.M. #.

Cor. Under W_0, $\{\langle \cdot, \lambda \rangle : \lambda \in \Theta_0^*\}$ is a family of centered Gauss. r.v.'s with covariance given by

$$\mathbb{E}^{W_0}[\langle \cdot, \lambda_1 \rangle \langle \cdot, \lambda_2 \rangle] = \iint_{(0,+\infty)^2} t \wedge s \; \lambda_1(dt) \lambda_2(ds)$$

"$t\wedge s$" is the kernel of Δ on $[0,+\infty)$ with Dirichlet bdry condition

The mapping $I : h_\lambda \in H_0^1 \mapsto \langle \cdot, \lambda \rangle \in L^2(\Theta_0, W_0)$ is an isometry.

Since $\{h_\lambda : \lambda \in \Theta_0^*\}$ is dense in H_0^1. I can be uniquely extended to H_0^1

$I : H_0^1 \to L^2(\Theta_0, W_0)$ as an isometry. I is called the Paley-Wiener map.

$\{I(h) : h \in H_0^1\}$ is a centered Gauss. family with covariance given by

Paley-Wiener integrals.

$$\mathbb{E}^{W_0}[I(h) I(g)] = (h, g)_{H_0^1} \quad \forall h, g \in H_0^1$$

Note: Unless $h = h_\lambda$ for some $\lambda \in \Theta_0^*$. $I(h)$ is a r.v., i.e. $I(h)(\theta)$ is defined for a.e. $\theta \in \Theta_0$. $I(h)(\theta) = \int_0^{+\infty} \dot{h}(t) d\theta(t)$

Remark. More generally, given B.M. $\{B_t\}$, given $f \in L^2_{loc}$. $\{\int_0^t f(s) dB_s : t \geq 0\}$ is also a Gauss. process. similar as the B.M.

$\{\int_0^t f(s) dB_s : t \geq 0\}$ has the same distribution as $\{B_{\int_0^t f^2(s) ds} : t \geq 0\}$.

It's a B.M. running at a "different clock".

If $f \in L^2$, then the "clock" only covers a finite time interval.

Remark. Recall that $\{g_{k,n} : k \geq 1. n \geq 0\}$ (as defined in the construction of B.M) is an o.n.b. of H_0^1. Under W_0 $\{I(g_{k,n}) : k \geq 1, n \geq 0\}$ is i.i.d. $N(0,1)$ r.v.'s.

Therefore. $\sum_{k,n} \underbrace{I(g_{k,n})(\theta)}_{\text{plays the role of } \{X_{k,n} : k\geq 1, n\geq 0\}} g_{k,n} \in \Theta_0$ for $\mathcal{W}_0$-a.e. $\theta \in \Theta_0$

In fact, one can verify that $\sum_{k,n} I(g_{k,n})(\theta)\, g_{k,n} = \theta$ $\mathcal{W}_0$-a.s.

(because $I(g_{k,0})(\theta) = \theta(k) - \theta(k-1)$

$$I(g_{k,n})(\theta) = 2^{\frac{n-1}{2}}\left[2\theta((2k-1)2^{-n}) - \theta((k-1)2^{-n+1}) - \theta(k2^{-n+1})\right]$$

Write $\sum_{k,n} I(g_{k,n})(\theta)\, g_{k,n} = \sum_n \theta^{(n+1)} - \theta^{(n)}$

where $\theta^{(n)}$ is the n-th order dyadically piece-wise linear approximation of θ. Certainly $\lim_n \theta^{(n)} = \theta$ uniformly on compacts)

Remark. Given B.M. $\{B_t : t\in[0,+\infty)\}$ and r.v. $Z \sim N(0,1)$ indep. of $\{B_t\}$.

define $U_t = e^{-\frac{t}{2}} Z + e^{-\frac{t}{2}} \int_0^t e^{\frac{\tau}{2}}\, dB_\tau$ $\quad t \in \mathbb{R}$

$\{U_t : t\in[0,+\infty)\}$ is called the stationary Ornstein-Uhlenbeck process.

$\{U_t\}$ is a continuous Gaussian process. with

$$m(t) = 0. \qquad C(t,s) = e^{-\frac{t+s}{2}} + e^{-\frac{t+s}{2}} \int_0^{t\wedge s} e^{\tau}\, d\tau = \underbrace{e^{-\frac{|t-s|}{2}}}_{(\frac{1}{4}-\Delta)^{-1} \text{ on } \mathbb{R}}$$

Back to Feynman's Representation

$$d\mathcal{W}_0 = \text{“} \frac{1}{Z} \exp\left[-\frac{1}{2}\int_0^{+\infty} |\dot{\theta}(t)|^2 dt\right] d\theta \text{”}$$

Let $f \in \Theta_0$ and denote $T_f : \Theta_0 \to \Theta_0$ by $T_f(\theta) = \theta + f$ (translation)

and denote $\mathcal{W}_{T_f} := (\mathcal{W}_0)_* T_f$ = the law of T_f = “translated $\mathcal{W}_0$”

then

$$d\mathcal{W}_{T_f} = \frac{1}{Z}\exp\left[-\frac{1}{2}\int_0^{+\infty} |\dot{\theta}(t) - \dot{f}(t)|^2 dt\right] d\theta$$

$$= \frac{1}{Z}\exp\left[-\frac{1}{2}\int_0^{+\infty} |\dot{\theta}(t)|^2 dt + (\dot{\theta}, \dot{f})_{L^2} - \frac{1}{2}\int_0^{+\infty} |\dot{f}(t)|^2 dt\right] d\theta$$

$$= \exp\left[\int_0^{+\infty} \dot{f}\, d\theta - \frac{1}{2}\|f\|^2_{H_0^1}\right] d\mathcal{W}_0$$

$$= \exp\left(I(f)(\theta) - \frac{1}{2}\|f\|^2_{H_0^1}\right) d\mathcal{W}_0$$

The Feynman's representation suggests that if $f \in H_0^1$. the translated $\mathcal{W}_0$ is absolutely continuous w.r.t. $\mathcal{W}_0$. and the Radon-Nikodym derivative is given by

$$\frac{d\mathcal{W}_{T_f}}{d\mathcal{W}_0} = e^{I(f) - \frac{1}{2}\|f\|^2_{H_0^1}} \to \text{Cameron-Martin formula}$$

If $f \notin H_0^1$. then $\mathcal{W}_{T_f} \perp \mathcal{W}_0$ i.e. $\mathcal{W}_{T_f}$ is singular w.r.t. $\mathcal{W}_0$

The proof of Cameron-Martin formula will be given in a more general setting in the future.

We now turn to the abstract theory.

<u>Set up</u>: H is a separable Hilbert space. E is a separable Banach space

$H \subseteq E$ continuously embedded as a dense subspace

$\forall x^* \in E^*$. $\exists!\ h_{x^*} \in H$ s.t. $\langle \cdot, x^* \rangle \restriction_H = (\cdot, h_{x^*})_H$.

$\mathcal{W} \in \mathcal{M}^1(E)$ is a prob. measure on $(E, \mathcal{B}_E)$.

<u>Def</u>. $(H, E, \mathcal{W})$ is called an abstract Wiener space (AWS). if $H, E, \mathcal{W}$ are as in "set-up" and $\forall x^* \in E^*$. $\mathbb{E}^{\mathcal{W}}[e^{i\langle \cdot, x^* \rangle}] = e^{-\frac{1}{2}\|h_{x^*}\|_H^2}$

H is the Cameron-Martin space. $\mathcal{W}$ is the Wiener measure (non-degenerate Gaussian

Similarly. we can define the Paley-Wiener map: $\mathcal{I}: H \to L^2(E, \mathcal{W})$

and the Paley-Wiener integrals $\{\mathcal{I}(h) : h \in H\}$.

We first look at an important result about Wiener measure $\mathcal{W}$

<u>Theorem</u> (Fernique's Theorem) Given any separable Banach space E and a Gaussian measure $\mathcal{W}$ on E. $\exists\, \delta > 0$ s.t. $\int_E \exp(\delta \|x\|_E^2)\, d\mathcal{W} < \infty$.

i.e. $\mathbb{E}^{\mathcal{W}}[e^{\delta \|\cdot\|_E^2}] < \infty$

<u>Proof</u>. (This statement is obvious in finite dimensions because of the exponential PDF).

We use a simple fact about Gaussian measure:

if $(x, x') \in E \times E$ is sampled under $\mathcal{W} \times \mathcal{W}$. i.e. x' is an independent copy of x, and $y = \frac{x + x'}{\sqrt{2}}$ and $y' = \frac{x - x'}{\sqrt{2}}$. then the dist. of (y, y') under $\mathcal{W} \times \mathcal{W}$ is the same as (x, x'). i.e. $\underset{\text{law}}{\mathcal{L}}((y, y')) = \mathcal{W} \times \mathcal{W}$

Let $0 < s \le t$ be fixed.

$$\mathcal{W}(\|x\|_E \le s)\,\mathcal{W}(\|x\|_E \ge t) = \mathcal{W}^2(\|x\|_E \le s,\ \|x'\|_E \ge t)$$

$$= \mathcal{W}^2(\|y'\|_E \le s,\ \|y\|_E \ge t) = \mathcal{W}^2(\|x - x'\|_E \le \sqrt{2}s,\ \|x + x'\|_E \ge \sqrt{2}t)$$

$$\le \mathcal{W}^2(|\,\|x\|_E - \|x'\|_E| \le \sqrt{2}s,\ \|x\|_E + \|x'\|_E \ge \sqrt{2}t)$$

$$\le \mathcal{W}^2\left(\|x\|_E \ge \frac{1}{\sqrt{2}}(t-s),\ \|x'\|_E \ge \frac{1}{\sqrt{2}}(t-s)\right)$$

$$= \left(\mathcal{W}\left(\|x\|_E \ge \frac{1}{\sqrt{2}}(t-s)\right)\right)^2$$

Now, choose $R > 0$ sufficiently large s.t. $\mathcal{W}(\|x\|_E \le R) \ge \frac{3}{4}$ and define $\{t_n : n \ge 0\}$ by $t_0 = R$, $t_n = R + \sqrt{2}\, t_{n-1}$ for $n \ge 1$.

Then $\mathcal{W}(\|x\|_E \le R)\,\mathcal{W}(\|x\|_E \ge t_n) \le \left(\mathcal{W}(\|x\|_E \ge t_{n-1})\right)^2$

$$\Rightarrow \underbrace{\frac{\mathcal{W}(\|x\|_E \ge t_n)}{\mathcal{W}(\|x\|_E \le R)}}_{=:\ r_n} \le \left(\frac{\mathcal{W}(\|x\|_E \ge t_{n-1})}{\mathcal{W}(\|x\|_E \le R)}\right)^2 \Rightarrow r_n \le (r_{n-1})^2 \le \cdots \le (r_0)^{2^n}$$

$$r_0 = \frac{\mathcal{W}(\|x\|_E \ge R)}{\mathcal{W}(\|x\|_E \le R)} \le \frac{1}{3}$$

$$\Rightarrow \mathcal{W}(\|x\|_E \ge t_n) \le \left(\frac{1}{3}\right)^{2^n}$$

On the other hand, $t_n = R\,\dfrac{(\sqrt{2})^{n+1} - 1}{\sqrt{2} - 1} \le 4R \cdot 2^{\frac{n}{2}} \Rightarrow \mathcal{W}(\|x\|_E \ge 4R\,2^{\frac{n}{2}}) \le 3^{-2^n}$

Take $\delta > 0$ small (δ TBD), then

$$\int_E e^{\delta \|x\|_E^2}\, d\mathcal{W} = \int_{\|x\|_E \le 4R} \cdots\, d\mathcal{W} + \sum_{n=0}^{+\infty} \int_{4R\,2^{\frac{n}{2}} \le \|x\|_E \le 4R\,2^{\frac{n+1}{2}}} \cdots\, d\mathcal{W}$$

$$\le e^{\delta(4R)^2}\,\mathcal{W}(\|x\|_E \le 4R) + \sum_{n=0}^{+\infty} e^{\delta(4R)^2 2^{n+1}}\,\mathcal{W}(\|x\|_E \ge 4R\,2^{\frac{n}{2}})$$

$$= e^{16R^2\delta} + \sum_{n=0}^{+\infty} e^{(32R^2\delta - \log 3)\,2^{n^2}} < \infty \quad \text{so long as } \delta < \frac{\log 3}{32R^2} \quad \#.$$