A few facts about H_0^1 & Θ_0: $\forall h \in H_0^1$. $t \in [0,+\infty)$.

$$|h(t)| = \left|\int_{[0,t]} \dot{h}(\tau)\, d\tau\right| \le \sqrt{t}\cdot \|h\|_{H_0^1} \Rightarrow \|h\|_{\Theta_0} = \sup_{t\in[0,+\infty)} \frac{|h(t)|}{1+t} \le \frac{1}{2}\|h\|_{H_0^1}$$

$\Rightarrow$ H_0^1 is continuously embedded in Θ_0.

Since $C_c^\infty(\mathbb{R}^+)$ is dense in Θ_0. $C_c^\infty(\mathbb{R}^+) \subseteq H_0^1$ $\Rightarrow$ H_0^1 is dense in Θ_0

- Θ_0^* the dual space of Θ_0, can be identified as

$$\Theta_0^* := \left\{ \lambda \in \mathcal{M}([0,+\infty)) : \lambda(\{0\}) = 0.\ \int_{[0,+\infty)} (1+t)\,|\lambda|(dt) < \infty \right\}$$

space of regular Borel measures — total variation measure

$\forall \theta \in \Theta_0$ $\quad \langle \theta, \lambda\rangle = {}_{\Theta_0}\langle \theta, \lambda\rangle_{\Theta_0^*} = \int_{[0,+\infty)} \theta(t)\,\lambda(dt)$ e.g. $\lambda = \delta_t$. $\langle \theta, \delta_t\rangle = \theta(t)$

$|\langle\theta,\lambda\rangle| \le \|\theta\|_{\Theta_0} \int_{[0,+\infty)} (1+t)|\lambda|(dt) \qquad \|\lambda\|_{\Theta_0^*} = \int_{[0,+\infty)} (1+t)\,|\lambda|(dt)$

- $\forall h \in H_0^1$. $\quad |\langle h, \lambda\rangle| \le \|h\|_{\Theta_0} \|\lambda\|_{\Theta_0^*} \le \frac{1}{2}\|h\|_{H_0^1}\cdot\|\lambda\|_{\Theta_0^*}$

  $\Rightarrow \lambda\restriction_{H_0^1}$ is linear and continuous on H_0^1. $\Rightarrow \exists$ unique $h_\lambda \in H_0^1$. s.t.

  $\langle h, \lambda\rangle = (h, h_\lambda)_{H_0^1}$ $\quad \forall h \in H_0^1$. $\quad \|h_\lambda\|_{H_0^1} \le \frac{1}{2}\|\lambda\|_{\Theta_0^*} \Rightarrow \lambda \mapsto h_\lambda$ is continuous and one-to-one.

- $\{h_\lambda \in H_0^1 : \lambda \in \Theta_0^*\}$ is dense in H_0^1, because if $\exists g \in H_0^1$ s.t. $(h_\lambda, g)_{H_0^1} = 0$. $\forall \lambda \in \Theta_0^*$

  $\Rightarrow \langle g, \lambda\rangle = 0 \quad \forall \lambda \in \Theta_0^* \Rightarrow g = 0$.

- $\forall \theta \in \Theta_0$. $\quad \theta \in H_0^1 \iff \exists C > 0$. s.t. $|\langle\theta,\lambda\rangle| \le C\|h_\lambda\|_{H_0^1}$. $\forall \lambda \in \Theta_0^*$

  and if $\theta \in H_0^1$. $\quad \|\theta\|_{H_0^1} = \sup\{|\langle\theta,\lambda\rangle| : \lambda \in \Theta_0^* \text{ w/ } \|h_\lambda\|_{H_0^1} \le 1\}$

("$\Leftarrow$" The mapping $h_\lambda \mapsto \langle\theta,\lambda\rangle$ can be extended to a unique linear and continuous functional on H_0^1. so $\exists h_\theta \in H_0^1$ s.t

$$\langle h_\theta, \lambda\rangle = (h_\theta, h_\lambda)_{H_0^1} = \langle\theta,\lambda\rangle \quad \forall \lambda \in \Theta_0^* \Rightarrow \theta = h_\theta \in H_0^1)$$

- Θ_0^* is separable under the weak* topology (because the weak* topology on Θ_0^* is second countable) $\Rightarrow \exists \{\lambda_n : n \in \mathbb{N}\} \subseteq \Theta_0^*$ s.t. $\{h_{\lambda_n} : n \in \mathbb{N}\}$ is an o.n.b. of H_0^1