Cor. Function $C : [0, +\infty)^2 \to \mathbb{R}$ is the covariance function of a Gaussian process $\{X_t\}$ on some prob. space $(\mathcal{F}, \mathbb{P})$ if and only if $C$ satisfies (A).

Def. A Gaussian process $\{B_t : t \in [0, +\infty)\}$ on $(\mathcal{F}, \mathbb{P})$ is a standard Brownian motion if

1. $B_0 = 0$
2. $t \mapsto B_t$ is continuous, $\mathbb{P}$-a.s.
3. $m(t) = 0$, $C(t, s) = t \wedge s$.

Remark. (3) is equivalent to (2). $\forall 0 \leq s < t$, $B_t - B_s$ has dist. $N(0, t - s)$ and independent increments, and $B_t - B_s$ is independent of $B_r$ $\forall 0 \leq r < s$.

Properties of B.M. A standard B.M. satisfies that:

1. $\{B_t\}$ is a Markov process, i.e., $\forall 0 \leq s \leq t$, $\forall A \in \mathcal{B}(\mathbb{R})$

   \[ \mathbb{P}(B_t \in A \mid \mathcal{F}_s) = \mathbb{P}(B_t \in A \mid B_s) \]

   conditional distribution.

2. $\{B_t\}$ is a martingale, $\forall 0 \leq s \leq t$, $\mathbb{E}[B_t \mid \mathcal{F}_s] = B_s$

   $\{B_t^{\gamma} + \gamma t\}$ and $\{\exp(\lambda B_t - \frac{\lambda^2}{2} t)\}$ ($\lambda \in \mathbb{C}$) are also martingales.

3. Strong Law of Large Number: $\lim_{t \to \infty} \frac{B_t}{t} = 0$ a.s.

   This is to say that $\{B_t\}$ "lives" on space $\Theta_0(\mathbb{R}^+):= \{f \in C(\mathbb{R}^+) : f(w) = 0 \}$

   \[ \lim_{t \to \infty} \frac{f(t)}{t} = 0 \]

4. Law of Iterated Logarithm: $\limsup_{t \to \infty} \frac{B_t}{t \log \log t} = 1 = \liminf_{t \to \infty} \frac{B_t}{t \log \log t}$ a.s.

5. Invariant transformation $\{Y_t = \frac{1}{\sqrt{c}} B_{ct} : t \geq 0\}$, $\{Y_0 = 0, Y_t = \sqrt{t} - B_{\sqrt{t}} : t \geq 0\}$, $\{Z_t = B_t - B_{T-t} : 0 \leq t \leq T\}$ ($T > 0$), $\{T_B : t \geq 0\}$ are all standard B.M.'s.
6. \( A \gamma (0, \frac{1}{2}), t \in [0, T) \mapsto B_t(w) \) is locally \( \beta \)-Hölder-\( k \) continuous for a.e. \( w \in \mathcal{W} \). i.e. \( \forall T > 0. \exists C = C_T, w > 0 \) s.t.
\[
\sup_{0 \leq t, s \leq T} \frac{|B_t - B_s|}{|t - s|^k} \leq C_T, w.
\]

7. For a.e. \( w \in \mathcal{W} \). \( t \in [0, +\infty) \mapsto B_t(w) \) is nowhere differentiable.

Recall that \( \Theta_0 (\mathbb{R}^+)_t = \{ f \in C(\mathbb{R}^+): f(0) = 0, \lim_{t \to +\infty} \frac{f(t)}{t} = 0 \} \)

Equip \( \Theta_0 (\mathbb{R}^+) \) with \( \| \cdot \|_{\Theta_0} \) where \( \| f \|_{\Theta_0} = \sup_{t \geq 0} \frac{|f(t)|}{1 + t} \)

Then \( (\Theta_0 (\mathbb{R}^+), \| \cdot \|_{\Theta_0}) \) becomes a separable Banach space.

Denote \( \mathcal{B}(\Theta_0) \) the Borel \( \sigma \)-algebra on \( \Theta_0 \).

(Convince yourself that \( \mathcal{B}(\Theta_0) \) is the same as
\[
\mathcal{B}(\times_{\mathbb{R}^+} \{ \{0\} \}) \]
)

Def.: Let \( \{ B_t : t \in [0, +\infty) \} \) be a standard B.M. on \( (\mathcal{N}, \mathcal{F}, \mathbb{P}) \). Consider the mapping \( B : w \in \mathcal{W} \mapsto B_t(w) : t \in [0, 0) \mapsto B_t(w) \in \mathbb{R} \).

It is shown that the distribution of \( B \) under \( \mathbb{P} \) is supported on \( \Theta_0 (\mathbb{R}^+) \), i.e. \( \mathbb{P}_B (\Theta_0) = 1 \). Denote \( \mathcal{W}_0 = \mathbb{P}_B \).

Then, \( (\Theta_0 (\mathbb{R}^+), \mathcal{B}(\Theta_0), \mathcal{W}_0) \) is a prob. space, called the classical Wiener space and \( \mathcal{W}_0 \) is the classical Wiener measure.

Remark: Alternatively, consider B.M. on \( [0, 1] \), \( \{ B_t : 0 \leq t \leq 1 \} \).

Take \( \mathcal{C}(\mathbb{R}) = \{ f \in C([0, 1]): f(0) = 0 \} \) equipped with \( \| \cdot \|_{\mathcal{W}} \)

Then, \( (\mathcal{C}(\mathbb{R}), \mathcal{B}(\mathcal{C}(\mathbb{R})), \mathbb{P}_B) \) is the classical Wiener space.

Wiener measure can be obtained through approximations by random walk.

Theorem (Donsker's Invariance Principle): Given a prob. space \( (\mathcal{N}, \mathcal{F}, \mathbb{P}) \), and \( \{ Z_n : n \in \mathbb{N} \} \) a sequence of i.i.d. random variables on \( \mathcal{N} \) s.t. \( \mathbb{E}[Z_n] = 0 \). \( \text{Var}(Z_n) = 1 \). For \( n \geq 1 \), set \( \mathcal{S}_n(w) = \sum_{i=1}^n Z_i(w) \).

$S_n \left( \frac{m}{n} \right) = \frac{1}{n} \sum_{k=1}^{m} Z_k$ for all $m \in \mathbb{N}$ and linearly interpolate $S_n$ on each interval $\left[ \frac{m-1}{n}, \frac{m}{n} \right]$. To get $\{S_n(t), t \geq 0\}$ a piecewise linear stochastic process. Then $S_n$ converges to B.M. in distribution, i.e. if $\mu_n := \mathbb{P} S_n$, then $\mu_n \Rightarrow \mathcal{N}$ (in the sense of weak convergence).

For any $\lambda^* \in \Theta^*_0(\mathbb{R}^+)$, $\lambda^*$ continuous and linear functional on $\Theta_0(\mathbb{R}^+)$

$$\int_{\Theta_0(\mathbb{R}^+)} <\theta, \lambda^*> M_n(d\theta) \xrightarrow{n \to \infty} \int_{\Theta_0(\mathbb{R}^+)} <\theta, \lambda^*> M(d\theta)$$

We still haven't proved the existence of B.M. yet.

There are two commonly used schemes to show existence of B.M.

1. Start with mean function $m(t) = 0$, and covariance function

   $$C(t,s) = t s \quad \forall t, s \text{ finite}, \quad F = [0, \infty) \quad C_F = (C(t,s))_{t,F = s,F}$$

   is symmetric, positive definite. Then there exists a Gaussian process $\{X_t : t \in [0, \infty) \}$

   such that $\{X_t\}$ has the desired finite dimensional distributions.

   But to prove that $\{X_t\} \in \mathcal{C}(\mathbb{R}^+)$, we need to use Kolmogorov's Continuity Theorem.

**Theorem (Kolmogorov's Continuity Theorem)** Let $\{X_t : t \in [0, T]\}$ be a stochastic process satisfying that $\exists \beta \in (1, \infty) \quad C > 0 \text{ and } r > 0$

   such that

   $$\mathbb{E}[|X_t - X_s|^p] \leq C |t-s|^{1+\frac{r}{\beta}} \quad \forall t, s \in [0, T]^2$$

   Then there exists another process $\{\hat{X}_t : t \in [0, T]\}$ such that $\forall \tau \in [0, T]$. $X_t = \hat{X}_t$ a.s.

   and for any $s \in (0, \beta)$. $\{\hat{X}_t\}$ is Hölder-$s$ continuous a.s.
In the case of B.M. w.t.s. $E[|B_t - B_s|^p] = C_n |t-s|^{p/n}$ \( \forall n \geq 1 \)
so $p = 2n$, $r = n - 1$. \( \frac{r}{p} \) can get arbitrarily close to $\frac{1}{2}$
so \( \hat{B}_t \) is locally Hölder-$\frac{r}{p}$ continuous a.s. \( \forall \epsilon \in (0, \frac{1}{2}) \)

2. A more constructive approach to show existence of B.M.
Start with piece-wise linear continuous function on $\mathbb{R}^+$
but using "finer" building blocks.

Let $\{X_m : n \geq 0, m \geq 1\}$ be i.i.d. $N(0,1)$ random variables. We use them
to construct process $B^{(n)}$ to approximate B.M.
If $\{\hat{B}_t : t \geq 0\}$ is a B.M. then $\{B_m \cdot 2^{-m} : m \geq 1\}$ is a Gaussian family with

(*) $E[B_{2m} \cdot 2^{-m}] = 0$ and $E[B_{2m} \cdot 2^{-m} \cdot B_{2m' \cdot 2^{-m'}}] = (\alpha m \cdot 2^{-m}) \forall m, m' \geq 1$

We want $B^{(n)}$ to satisfy (*) for all $n$.

Build $B^{(n)}$ inductively:
1. $B^{(n)}(0) = 0$ and $B^{(n)}(m) = \sum_{j=0}^{m} X_j, 0 \leq m \leq 1$
   Linearly interpolate $B^{(n)}$ on every interval $[m, m+1]$
   Obviously $B^{(n)}$ satisfies (*)
2. Assume that $B^{(n)}$ has been constructed satisfying (*)
   Define $B^{(n+1)}$ by:
   
   $B^{(n+1)}(m) :=$
   
   $\begin{cases} 
   B^{(n)}(2m) & \text{if } m = 2k \\
   B^{(n)}((2k+1)2^{-m}) + \frac{1}{2} \cdot \sum_{j=0}^{m} X_{2^m \cdot 2^{-m} - 1} & \text{if } m = 2k - 1 
   \end{cases}$

   The term $2^{-m} \cdot X_{2^m \cdot 2^{-m} - 1}$ is a corrector, introduced to make
   the variance of $B^{(n+1)}(m \cdot 2^{-m})$ match the desired value $m \cdot 2^{-m}$
   
   \[
   \text{Var}(B^{(n+1)}(2k \cdot 2^{-m})) = \text{Var}(\frac{1}{2} B^{(n)}((2k-1)2^{-m}) + \frac{1}{2} B^{(n)}(2k \cdot 2^{-m}))
   \]
   
   \[
   = \frac{1}{4} (2k-1)2^{-m} + \frac{1}{4} (2k-2)2^{-m} + \frac{1}{4} 2k \cdot 2^{-m}
   \]
   
   \[
   = (2k-\frac{3}{2})2^{-m} \quad \text{(still off by $\frac{1}{2} \cdot 2^{-m} = 2^{-m-1}$)}
   \]

One can verify that $B^{(n+1)}$ also satisfies (*)
Finally, linear interpolate $B^{(n+1)}$ and proceed to $B^{(n+2)}$ similarly.