Cor. Function $C: [0,+\infty)^2 \to \mathbb{R}$ is the covariance function of a Gauss. process $\{X_t\}$ on some prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ $\iff$ C satisfies (Δ).

Def. A Gaussian process $\{B_t : t \in [0,+\infty)\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard Brownian motion (B.M.) if ① $B_0 \equiv 0$ ② $t \in [0,+\infty) \mapsto B_t(\omega)$ is continuous. $\forall \omega \in \Omega$

③ $m(t) = 0$ $\quad C(t,s) = t \wedge s$.

Remark. ③ is equivalent to ③' $\forall\, 0 \le s < t$. $B_t - B_s$ has dist. $N(0, t-s)$

independent increment $\longrightarrow$ and $B_t - B_s$ is indep. of B_r $\forall\, 0 \le r \le s$.

Properties of B.M. A standard B.M. satisfies that

① $\{B_t\}$ is a Markov process. i.e. $\forall\, 0 \le s \le t$. $\forall A \in \mathcal{B}(\mathbb{R})$

$\mathcal{F}_s := \sigma(B_r : 0 \le r \le s)$. $\mathbb{P}(B_t \in A \mid \mathcal{F}_s) = \mathbb{P}(B_t \in A \mid B_s)$

conditional distribution.

② $\{B_t\}$ is a martingale. $\forall\, 0 \le s \le t$. $\mathbb{E}[B_t \mid \mathcal{F}_s] = B_s$

$\{B_t^2 - t\}$ and $\{\exp(\lambda B_t - \frac{\lambda^2}{2} t)\}$ $(\lambda \in \mathbb{C})$ are also martingales

③ Strong Law of Large Number: $\lim\limits_{t \to \infty} \dfrac{B_t}{t} = 0$ a.s.

this is to say that $\{B_t\}$ "lives" on space $\Theta_0(\mathbb{R}^+) := \left\{ f \in C(\mathbb{R}^+) : f(0) = 0 \quad \lim\limits_{t\to\infty} \dfrac{f(t)}{t} = 0 \right\}$

④ Law of Iterated Logarithm: $\limsup\limits_{t \to \infty} \dfrac{B_t}{\sqrt{2t \log\log t}} = 1 = -\liminf\limits_{t \to \infty} \dfrac{B_t}{\sqrt{2t \log\log t}}$ a.s.

⑤ Invariant transformation $\quad \{X_t = \frac{1}{\sqrt{c}} B_{ct} : t \ge 0\}$ $(c > 0)$. $\{Y_0 = 0,\ Y_t = t B_{\frac{1}{t}} : t > 0\}$.

$\{Z_t = B_T - B_{T-t} : 0 \le t \le T\}$ $(T > 0)$. $\{-B_t : t \ge 0\}$

are all standard B.M.s

⑥ $\forall \gamma \in (0, \frac{1}{2})$, $t \in [0, +\infty) \mapsto B_t(\omega)$ is locally Hölder-γ continuous for a.e. $\omega \in \Omega$.

i.e. $\forall T > 0$, $\exists C = C_{T,\omega} > 0$ s.t.

$$\sup_{0 \le t, s \le T} \frac{|B_t - B_s|}{|t-s|^\gamma} \le C_{T,\omega}.$$

⑦ For a.e. $\omega \in \Omega$, $t \in [0, +\infty) \mapsto B_t(\omega)$ is nowhere differentiable.

Recall that $\Theta_0(\mathbb{R}^+) := \{ f \in C(\mathbb{R}^+) : f(0) = 0 \quad \lim_{t \to \infty} \frac{f(t)}{t} = 0 \}$

Equip $\Theta_0(\mathbb{R}^+)$ with $\|\cdot\|_{\Theta_0}$ where $\|f\|_{\Theta_0} := \sup_{t \in [0, +\infty)} \frac{|f(t)|}{1+t}$

Then $(\Theta_0(\mathbb{R}^+), \|\cdot\|_{\Theta_0})$ becomes a separable Banach space.
Denote $\mathcal{B}(\Theta_0)$ the Borel σ-algebra on Θ_0.
(Convince yourself that $\mathcal{B}(\Theta_0)$ is the same as
$\sigma(\{ \pi_t : f \in \Theta_0(\mathbb{R}^+) \mapsto f(t) \in \mathbb{R} : t \ge 0 \})$)

Def. Let $\{B_t : t \in [0, \infty)\}$ be a standard B.M. on $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the mapping $B : \omega \in \Omega \mapsto B_\cdot(\omega) : t \in [0, \infty) \mapsto B_t(\omega) \in \mathbb{R}$.
It is shown that the dist. of B under $\mathbb{P}$ is supported on $\Theta_0(\mathbb{R}^+)$
i.e. $\mathbb{P}_* B(\Theta_0) = 1$. Denote $\mathcal{W}_0 = \mathbb{P}_* B$
Then, $(\Theta_0(\mathbb{R}^+), \mathcal{B}(\Theta_0), \mathcal{W}_0)$ is a prob. space, called the classical Wiener space and $\mathcal{W}_0$ is the classical Wiener measure

Remark Alternatively, consider B.M. on $[0,1]$ $\{B_t : 0 \le t \le 1\}$.
Take $C_0([0,1]) := \{ f \in C([0,1]) : f(0) = 0 \}$ equipped with $\|\cdot\|_u$
Then $(C_0([0,1]), \mathcal{B}(C_0([0,1])), \mathbb{P}_* B)$ is the classical Wiener space.

Wiener measure can be obtained through approximations by random walk.

Theorem (Donsker's Invariance Principle) Given a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{Z_n : n \in \mathbb{N}\}$ a sequence of i.i.d. random variables on Ω s.t. $\mathbb{E}[Z_n] = 0$, $\mathrm{Var}(Z_n) = 1$. For $n \ge 1$, set $S_n(\omega) \equiv 0$.

$S_n(\frac{m}{n}) = \frac{1}{\sqrt{n}} \sum_{k=1}^{m} Z_k$ for all $m \in \mathbb{N}$ and linearly interpolate S_n on each interval $[\frac{m-1}{n}, \frac{m}{n}]$. to get $\{S_n(t): t \geq 0\}$ a piecewise linear stochastic process. Then S_n converges to B.M. in distribution, i.e. if $\mu_n := \mathbb{P}_x S_n$ then $\mu_n \Rightarrow \mathcal{W}_0$ (in the sense of weak convergence)

$\forall \lambda^* \in \Theta_0^*(\mathbb{R}^+)$. λ^* continuous and linear functional on $\Theta_0(\mathbb{R}^+)$

$$\int_{\Theta_0(\mathbb{R}^+)} \langle \theta, \lambda^* \rangle \mu_n(d\theta) \xrightarrow{n \to +\infty} \int_{\Theta_0(\mathbb{R}^+)} \langle \theta, \lambda^* \rangle \mathcal{W}_0(d\theta)$$

We still haven't proved the existence of B.M. yet.

There are two commonly used schemes to show existence of B.M.

1. Start with mean function $m(t) = 0$. and covariance function $C(t,s) = t \wedge s$. $\forall F$ finite. $F \subseteq [0, +\infty)$. $C_F = (C(t,s))_{t \in F, s \in F}$ is symm. positive definite. Then $\exists$ a Gaussian process $\{X_t : t \in [0, \infty)\}$ s.t. $\{X_t\}$ has the desired finite dim. distributions.

But to prove that $\{X_t\} \in C(\mathbb{R}^+)$. we need to use Kolmogorov's Continuity Theorem

Theorem (Kolmogorov's Continuity Theorem) Let $\{X_t : t \in [0, T]\}$ be a stochastic process satisfying that $\exists p \in (1, +\infty)$. $C > 0$ and $r > 0$ s.t.

$$\mathbb{E}[|X_t - X_s|^p] \leq C |t-s|^{1+r} \quad \forall t, s \in [0, T]^2.$$

Then $\exists$ another process $\{\tilde{X}_t : t \in [0, T]\}$ s.t. $\forall t \in [0, T]$. $X_t = \tilde{X}_t$ a.s. and for any $\gamma \in (0, r/p)$. $\{\tilde{X}_t\}$ is Hölder-γ continuous a.s.

In the case of B.M. $\forall t, s$ $\mathbb{E}[|B_t - B_s|^{2n}] = C_n |t-s|^n$ $\forall n \geq 1$
so $p = 2n$. $r = n-1$. r/p can get arbitrarily close to $\frac{1}{2}$
So $\{B_t\}$ is locally Hölder-γ continuous a.s. $\forall \gamma \in (0, \frac{1}{2})$

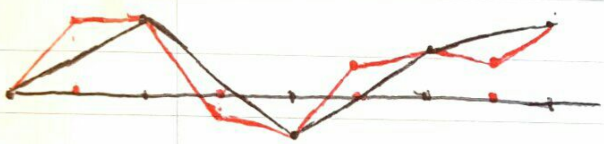
2. A more constructive approach to show existence of B.M.
Start with piece-wise linear continuous function on $\mathbb{R}^+$,
but using "finer" building blocks.

Let $\{X_{m,n} : n \geq 0, m \geq 1\}$ be iid. $N(0,1)$ random variables. We use them to construct process $B^{(n)}$ to approximate B.M.

__Observations__. If $\{B_t : t \geq 0\}$ is a B.M. then $\{B_{m \cdot 2^{-n}} : m \geq 1\}$ is a Gaussian family with
$(*)$ $\mathbb{E}[B_{m 2^{-n}}] = 0$ and $\mathbb{E}[B_{m 2^{-n}} \cdot B_{m' 2^{-n}}] = (m \wedge m') 2^{-n}$ $\forall m, m' \geq 1$.

We want $B^{(n)}$ to satisfy $(*)$ for all n.

Build $B^{(n)}$ inductively: ① $B^{(0)}(0) \equiv 0$. and $B^{(0)}(m) = \sum_{j=1}^{m} X_{j,0}$ $\forall m \geq 1$.
Linearly interpolate $B^{(0)}$ on every interval $[m-1, m]$
Obviously $B^{(0)}$ satisfies $(*)$



② Assume that $B^{(n)}$ has been constructed, satisfying $(*)$.
Define $B^{(n+1)}$ by: $B^{(n+1)}(0) \equiv 0$.

$$B^{(n+1)}(m 2^{-n-1}) = \begin{cases} B^{(n)}(k 2^{-n}) & \text{if } m = 2k \\ B^{(n)}((2k-1) 2^{-n-1}) + 2^{-\frac{n}{2}-1} X_{k, n+1} & \text{if } m = 2k-1 \end{cases}$$

The term "$2^{-\frac{n}{2}-1} X_{k,n+1}$" is a corrector, introduced to make the variance of $B^{(n+1)}(m 2^{-n-1})$ match the desired value $m 2^{-n-1}$

$$\Big(Var(B^{(n)}((2k-1) 2^{-n-1})) = Var\left(\tfrac{1}{2} B^{(n)}((2k-2) 2^{-n-1}) + \tfrac{1}{2} B^{(n)}(2k \cdot 2^{-n-1})\right)$$

$$= \tfrac{1}{4}(2k-2) 2^{-n-1} + \tfrac{1}{2}(2k-2) 2^{-n-1} + \tfrac{1}{4} 2k \cdot 2^{-n-1}$$

$$= (2k - \tfrac{3}{2}) 2^{-n-1} \longleftarrow \text{still off by } \tfrac{1}{2} \cdot 2^{-n-1} = 2^{-n-2} \Big)$$

One can verify that $B^{(n+1)}$ also satisfies $(*)$.
Finally, linear interpolate $B^{(n+1)}$ and proceed to $B^{(n+2)}$ similarly.