Math 599 Fall 2016. Sep. 27th. 2016

Review of probability: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space

sample space (Ω), σ-algebra ($\mathcal{F}$), prob. measure ($\mathbb{P}$)

$X: \Omega \to \mathbb{R}^d$ a random variable i.e. $\forall B \in \mathcal{B}(\mathbb{R}^d)$. $X^{-1}(B) \in \mathcal{F}$.

Borel σ-algebra ($\mathcal{B}(\mathbb{R}^d)$)

$X = (X_1, \cdots, X_d)$

Assume that $X \in L^2(\Omega; \mathbb{P})$ i.e. $\mathbb{E}[|X|^2] = \int_\Omega X \cdot X^T \, d\mathbb{P} < \infty$

Denote $\mathbb{R}^d \ni m := \mathbb{E}[X]$ (mean). $C = (C_{ij})_{d\times d} = (\operatorname{Cov}(X_i, X_j))_{d\times d}$

covariance matrix. symmetric. positive definite

Def: $X = (X_1, \cdots, X_d): \Omega \to \mathbb{R}^d$ is a (multivariate) normal/Gaussian random variable, if the probability density funtion of X is

$$f_{m,C} : x = (x_1, \cdots, x_d) \in \mathbb{R}^d \longmapsto f_{m,C}(x) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \frac{1}{\sqrt{\det C}} \exp\left(-\frac{1}{2}(x-m)C^{-1}(x-m)^T\right)$$

i.e. $\forall B \in \mathcal{B}(\mathbb{R}^d)$.

$$\mathbb{P}(X \in B) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \frac{1}{\sqrt{\det C}} \int_B e^{-\frac{1}{2}(x-m)C^{-1}(x-m)^T} \, dx$$

Lebesgue measure (dx)

If $m = 0$. then X is a centered Gaussian random variable

Prop. $X: \Omega \to \mathbb{R}^d$ is a centered Gaussian random variable iff the characteristic function of X, denoted by $ch(X)$, is (Fourier transform)

$$ch(X): \xi \in \mathbb{R}^d \longmapsto ch(X)(\xi) := \mathbb{E}\left[e^{i(X,\xi)_{\mathbb{R}^d}}\right] = e^{-\frac{\xi C \xi^T}{2}}$$

where C is the covariance matrix of X.

We call the distribution of X under $\mathbb{P}$, denoted by $\mathbb{P}_* X$, a Gaussian distribution or a Gaussian measure on $\mathbb{R}^d$, denoted by $N(m, C)$.

<u>Remark</u>: X is called a standard normal / Gaussian random variable if $m=0$, $C=I$

X is standard normal $\Longleftrightarrow$ density $f(x) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} e^{-\frac{|x|^2}{2}}$

$$\Longleftrightarrow \quad ch(X)(\xi) = e^{-\frac{|\xi|^2}{2}}$$

<u>Remark</u>: $X_1 : \Omega \to \mathbb{R}$, $X_2 : \Omega \to \mathbb{R}$ two random variables. (X_1, X_2) is Gaussian then X_1 and X_2 are independent $\Longleftrightarrow$ $\mathrm{Cov}(X_1, X_2) = 0$

$$\Updownarrow$$

$$\forall \xi_1, \xi_2 \in \mathbb{R} \quad \mathbb{E}\left[e^{iX_1\xi_1 + iX_2\xi_2}\right] = \mathbb{E}\left[e^{iX_1\xi_1}\right] \cdot \mathbb{E}\left[e^{iX_2\xi_2}\right]$$

$$\Updownarrow$$

$$\text{density } f_{(X_1, X_2)}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2).$$

Rewrite $\mathbb{R}^d = H$ with $(x, y)_H := (x, C^{-1}y)_{\mathbb{R}^d}$ $\quad (H, \|\cdot\|_H)$ Hilbert sp.

Write $\lambda_H(dx) := (\det C)^{-\frac{1}{2}} dx$ $\quad \lambda_H(dx)$ is the translation invariant measure on H that assigns 1 to unit box under $\|\cdot\|_H$.

Rewrite the density and the charateristic function of X as follows: if X is a centered Gaussian random variable with covariance matrix C.

then $\quad f(x) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} e^{-\frac{\|x\|_H^2}{2}} \lambda_H(dx)$

and $\quad ch(X)(\xi) = \mathbb{E}\left[e^{i(X, \xi)_H}\right] = e^{-\frac{\|\xi\|_H^2}{2}}$ $\quad \Big\} \; (*)$

<u>Remark</u>. Any (centered, non-degenerate) Gaussian measure (i.e. the distribution of a centered Gaussian random variable) in finite dimensions can be written in the form of a standard Gaussian distribution.

The "natural" hosting place of a Gaussian measure is a Hilbert space!

<u>Remark</u>: The configuration $(*)$ implies that if $\{h_1, \cdots, h_d\}$ is an o.n.b. of H then the family of random variables $\{(X, h_j) : 1 \le j \le d\}$ is a family of independent identically distributed (i.i.d.) standard normal random variables in $\mathbb{R}$.

Question: What if $d=+\infty$? Assume H is a separable Hilbert space, $\dim(H)=+\infty$

i.e. $\exists$ countable dense subset

"Assume" $X:\Omega\to H$ is a centered Gauss. random variable similar as above

then $\|X(\omega)\|_H^2<\infty$ for a.e. $\omega\in\Omega$.

However, if $\{h_n: n\in\mathbb{N}\}$ is an o.n.b. of H, then $\{(X,h_n)_H: n\in\mathbb{N}\}$ is i.i.d. standard normal random variables.

$$\Rightarrow \quad \|X(\omega)\|_H^2=\sum_{n=1}^{+\infty}(X(\omega),h_n)_H^2=+\infty \quad \text{a.s.}$$

Contradiction! This means that a Gaussian measure with (*) configuration in infinite dimensions will not "fit" in a Hilbert space.

The early study of infinite dimensional Gaussian measures was conducted in the context of Brownian motion. (Robert Brown (Botanist 1827)

Thiele 1880. Einstein 1905, ... Wiener. Lévy. Cameron-Martin 1940-50, ...

abstract theory is studied by Segal 1950s. and completed by Gross 1966)

We will first give a crash course on Brownian motion.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a generic probability space and $\{X_t: t\in[0,+\infty)\}$ a $\mathbb{R}$-valued stochastic process, i.e.

- $\forall t\geq 0$, $X_t:\Omega\to\mathbb{R}$ is a random variable.
- $\forall \omega\in\Omega$, $t\in[0,+\infty)\mapsto X_t(\omega)\in\mathbb{R}$ is a sample path of $\{X_t\}$
- $\forall\ 0\leq t_1<t_2<\cdots<t_d<\infty$, the distribution of $(X_{t_1}, X_{t_2}, \cdots X_{t_d})$ on $\mathbb{R}^d$ is, denoted by $\mu_{\{t_1,t_2,\cdots,t_d\}}$, is a finite dimensional distribution of $\{X_t\}$

- Consider the mapping $X: \Omega \to \mathbb{R}^{[0,+\infty)} := \{$space of all the paths on $[0,\infty)\}$ given by $\omega \in \Omega \mapsto X_\cdot(\omega) = \{X_t(\omega) : t \geq 0\} \in \mathbb{R}^{[0,+\infty)}$.

Then the distribution of X is a prob. measure on $\mathbb{R}^{[0,+\infty)}$

denoted by $\mathbb{P}_* X$

infinite dimensional

(We equip $\mathbb{R}^{[0,+\infty)}$ with the σ-algebra generated by all pointwise projections)
$\sigma\left(\{\pi_t : f \in \mathbb{R}^{[0,+\infty)} \mapsto f(t) \in \mathbb{R} : t \geq 0\}\right)$

For any $F = \{t_1, t_2, \cdots, t_N\} \subseteq [0,+\infty)$. let μ_F be a prob. meas. on $\mathbb{R}^N$.

Def: Let $\{\mu_F : F \text{ finite}, F \subseteq [0,+\infty)\}$ be a family of finite dimensional prob. measures. Then, $\{\mu_F\}$ is consistent if $\forall$. F, G finite. $\subseteq [0,+\infty)$ and $F \subseteq G$. e.g. $F = \{t_1, \cdots t_N\}$ $G = \{t_1, \cdots, t_N, t_{N+1}, \cdots t_M\}$

then $\mu_G|_{\mathbb{R}^N} = \mu_F$

→ restriction onto the first N coordinates.

Theorem (Kolmogorov's Consistency Theorem) A family of finite dimensional prob. measures $\{\mu_F : F \text{ finite}. F \subseteq [0,+\infty)\}$ is the finite dim. dist. of a stochastic process $\{X_t : t \in [0,+\infty)\}$ on some prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ iff $\{\mu_F\}$ is consistent.

Proof is omitted here. Refer to §12.1 of "Real Analysis and Probability" by R. Dudley.

Def: A stochastic process $\{X_t : t \geq 0\}$ is a Gaussian process if any finite dim. dist. of $\{X_t\}$ is a Gauss. dist.

$\forall\ F = \{t_1, t_2, \cdots, t_N\} \subseteq [0,+\infty)$. $\mu_F = N(m_F, C_F)$

for some $m_F \in \mathbb{R}^N$ and C_F symm. positive def. $n \times n$ matrix.

Remark. All finite dim. dist. of a Gaussian process $\{X_t\}$ are determined by

① mean function: $t \in [0,\infty) \mapsto m(t) := \mathbb{E}[X_t]$

② covariance function. $(t,s) \in [0,\infty)^2 \mapsto C(t,s) = \mathrm{Cov}(X_t, X_s)$

(Δ) $C(t,s) = C(s,t)$. $\forall\ F \subseteq [0,\infty)$. F finite $\left(C(t,s)\right)_{s \in F,\, t \in F}$ is symm. positive definite

Cor: Function $C: [0,+\infty)^2 \to \mathbb{R}$ is the covariance function of a Gauss. process $\{X_t\}$ on some prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ $\iff$ C satisfies (Δ).

Def: A Gaussian process $\{B_t : t\in[0,+\infty)\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard Brownian motion (B.M.) if ① $B_0 \equiv 0$ ② $t\in[0,+\infty) \mapsto B_t(\omega)$ is continuous. $\forall \omega\in\Omega$

③ $m(t) = 0$ $C(t,s) = t\wedge s$.

Remark: ③ is equivalent to ③' $\forall\, 0\le s<t$. $B_t - B_s$ has dist. $N(0, t-s)$

independent increment $\longrightarrow$ and $B_t - B_s$ is indep. of B_r $\forall\, 0\le r\le s$.

Properties of B.M. A standard B.M. satisfies that

① $\{B_t\}$ is a Markov process. i.e. $\forall\ 0\le s\le t$. $\forall A \in \mathcal{B}(\mathbb{R})$

$\mathcal{F}_s := \sigma(B_r : 0\le r\le s)$. $\mathbb{P}(B_t\in A \mid \mathcal{F}_s) = \mathbb{P}(B_t \in A \mid B_s)$

$\searrow$ conditional distribution.

② $\{B_t\}$ is a martingale. $\forall\, 0\le s\le t$. $\mathbb{E}[B_t \mid \mathcal{F}_s] = B_s$

$\{B_t^2 - t\}$ and $\{\exp(\lambda B_t - \frac{\lambda^2}{2}t)\}$ $(\lambda\in\mathbb{C})$ are also martingales

③ Strong Law of Large Number: $\lim\limits_{t\to\infty} \frac{B_t}{t} = 0$ a.s.

this is to say that $\{B_t\}$ "lives" on space $\Theta_0(\mathbb{R}^+) := \left\{ f\in C(\mathbb{R}^+) : f(0)=0,\ \lim\limits_{t\to\infty}\frac{f(t)}{t} = 0 \right\}$

④ Law of Iterated Logarithm: $\limsup\limits_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1 = -\liminf\limits_{t\to\infty}\frac{B_t}{\sqrt{2t\log\log t}}$ a.s.

⑤ Invariant transformation $\{X_t = \frac{1}{\sqrt{c}} B_{ct} : t\ge 0\}$ $(c>0)$. $\{Y_0 = 0,\ Y_t = tB_{\frac{1}{t}} : t>0\}$

$\{Z_t = B_T - B_{T-t} : 0\le t\le T\}$ $(T>0)$. $\{-B_t : t\ge 0\}$

are all standard B.M.s