Math 581, Winter 2018 D. Jakobson

PROBLEM SET 1 due date to be announced.

Do all the problems. Every problem is worth 5 points. Some problems may not be graded because of time constraints.

PDE: Evans, Chapter 6, No. 2, 6, 7, 9, 10.
Geometric Analysis:
GA1. Let (M, g) be a Riemannian manifold, and X_i be an orthonormal frame of vector fields in a neighbourhood U of a given point $p \in M$. Viewing each X_i as a first order differential operator, $X_i(u) = \mathcal{L}_{X_i}(u)$, we define X_i^* by the relation

$$\int_M (X_i u) v d\mathrm{vol}_g = \int_M u(X_i^* v) d\mathrm{vol}_g$$

for any smooth functions u, v, where at least one of them has a compact support contained in U. Show that the Laplace-Beltrami operator satisfies the formula

$$(\Delta_g u)(x) = -(\sum_{i=1}^{n} X_i^* X_i u)(x), \qquad \forall x \in U.$$

GA2. Let (M, g_M) and (N, g_N) be two Riemannian manifolds. Recall that the warped product metric h on $M \times N$ is defined as the direct sum $g_M \psi^2 g_N$, where ψ is a positive function on M. Equivalently, in local coordinates $(x_1, \ldots, x_m)$ on M and $(y_1, \ldots, y_n)$ on N, the metric h has the form

$$(g_M)_{ij} dx^i dx^j + \psi^2(x)(g_N)_{kl} dy^k dy^l.$$

Compute the volume form (and the volume if finite) of $M \times N$ in the warped product metric h, and show that the Laplace operator satisfies the following formula:

$$\Delta_h u = \Delta_M u + n\langle \mathrm{grad}(\log \psi), \mathrm{grad}\ u\rangle_{g_M} + \psi^{-2}\Delta_N u,$$

where n is the dimension of N.