Math 565, Winter 2017 D. Jakobson

TAKE HOME MIDTERM due Monday, March 6, 2017.

Do any 8 of the following 10 problems. Every problem is worth 10 points. You can use any theorem in Folland or any result in the homework assignments.

Problem 1.

a) **Riesz Lemma.** Let X be a normed linear space, and Y a linear subspace of X. If there exists $0 < r < 1$ such that for any $x \in X; ||x|| = 1$, one has $d(x;Y) < r$, then Y is dense in X. Here $d(x;Y) = \inf_{y \in Y} ||x - y||$.

b) Use Part a) to prove that a unit ball in an infinite-dimensional normed linear space is never compact. Hint: pick x_1 on the unit sphere; by induction, let $x_1, \dots x_{n-1}$ span a finite-dimensional subspace Y_{n-1}. Use Part a) to pick x_n "away" from Y_{n-1}.

Problem 2. Folland, Chapter 6, # 29.
Problem 3. Folland, Chapter 6, # 34.
Problem 4. Folland, Chapter 6, # 35, 36, 37 (one problem).
Problem 5. Folland, Chapter 6, # 41.
Problem 6. Folland, Chapter 7, # 15.
Problem 7. Folland, Chapter 7, # 20.
Problem 8. Define the measure on $[0,1]$ by

$$\mu([a,b)) = \log_2 \frac{1+b}{1+a}.$$

a) Prove that μ is preserved by the map $f(x) = \{1/x\}$, where $\{y\}$ denotes the fractional part of y.

Every real number in $x \in [0,1]$ can be expanded into a (finite or infinite) *continued fraction*

$$x = \cfrac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}},$$

sometimes denoted by $x = [n_1, n_2, n_3, \dots]$.

b) Prove that finite continued fractions correspond to rational numbers, while infinite fractions correspond to irrational numbers.

c) Prove that the function f in part a) can be written as a *shift map*,

$$f([n_1, n_2, n_3, \dots]) = [n_2, n_3, \dots].$$

Hint: You have to show that the measure of the *preimage* $\mu(f^{-1}(A)) = \mu(A)$; it suffices to consider intervals.

Problem 9. We keep the notation from Problem 1.

a) Describe the measure μ in problem 4 in the space of sequences $[n_1, n_2, n_3, \dots]$.

b) Describe all the *periodic* continued fractions, $x = [n_1, \dots, n_k, n_1, \dots, n_k, \dots]$.

Hint:

a) It is enough to specify the measure of a *cylinder* in the space of sequences, e.g. the set of all continued fractions with $n_1 = a_1, n_2 = a_2, ..., n_k = a_k$ for a fixed k-tuple $(a_1, ..., a_k)$ of natural numbers.

b) Example of a periodic continued fraction is the golden ratio, $1/(1+1/(1+1/...))$; you should show that every periodic (or eventually periodic) continued fraction has a certain property, you don't have to prove a converse.

Problem 10 (Hanner's inequality). Let $f, g \in L^p$. Then

i) If $1 \leq p \leq 2$, then

$$||f+g||_p^p + ||f-g||_p^p \geq (||f||_p + ||g||_p)^p + |||f||_p - ||g||_p|^p\,;$$

and

$$(||f+g||_p + ||f-g||_p)^p + |||f+g||_p + ||f-g||_p|^p \leq 2^p(||f||_p^p + ||g||_p^p).$$

ii) If $2 \leq p \leq \infty$, the inequalities are reversed.