Math 565, Winter 2016 D. Jakobson

TAKE HOME FINAL Due Monday, April 24, 2017.

Do any 8 of the following 10 problems. Every problem is worth 10 points. You can use any result in Folland or any result in the homework assignments.

Problem 1. For $1 \leq p \leq \infty$, let $W^{1,p}(\mathbf{T}^1)$ be the set of all A.C. functions on $\mathbf{T}^1$ s.t. $f' \in L^p(\mathbf{T}^1)$. Show that if $f \in W^{1,p}(\mathbf{T}^1)$, then $\hat{f} \in l^1(\mathbf{Z})$. Hint: it suffices to assume $p \leq 2$; use the relation between $\hat{f}(k)$ and $\hat{f}'(k)$, together with the inequalities of Hölder and Hausdorff-Young.

Problem 2. Let $1/p + 1/q = 1$. Show that $W^{1,p}(\mathbf{T}^1) \subset \mathrm{Lip}_{1/q}(\mathbf{T}^1)$. If $\alpha > 1/q$, then $W^{1,p}(\mathbf{T}^1) \not\subset \mathrm{Lip}_\alpha(\mathbf{T}^1)$. Hint: use fundamental theorem of calculus.

Problem 3. For $f \in C(\mathbf{T})$ we define the *modulus of continuity* $\omega(f,h)$ by

$$\omega(f,h) = \sup_{|y| \leq h, t} |f(t+y) - f(t)|.$$

For $f \in L^1(\mathbf{T})$ we define the *integral modulus of continuity* $\Omega(f,h)$ by

$$\Omega(f,h) = ||f(t+h) - f(t)||_{L^1}.$$

Clearly, $\Omega(f,h) \leq \omega(f,h)$.

a) For $n \neq 0$, show that $|\hat{f}(n)| \leq \frac{1}{2}\Omega(f, \pi/|n|)$.

b) Conclude that for $f \in \mathrm{Lip}_\alpha(\mathbf{T})$, we have $\hat{f}(|n|) = O(|n|^{-\alpha})$

Problem 4. Folland, Chapter 8, Problem 17.

Problem 5. Let $G(\mathbf{R}^2) \subset C^\infty(\mathbf{R}^2)$ denote the space of all smooth functions $g(x,y)$ satisfying

$$g(x+m, y+n) = g(x,y)e^{2\pi i m y}, \qquad \forall (m,n) \in \mathbf{Z}^2.$$

Show that

a) For every $f \in \mathcal{S}(\mathbf{R})$, the function $\sum_{k \in \mathbf{Z}} e^{-2\pi i k y} f(x+k)$ belongs to G.

b) Any function $g \in G$ can be obtained from some $f \in \mathcal{S}(\mathbf{R})$ using the formula above. Hint: let $f(x) = \int_0^1 g(x,y)dy$.

Problem 6. Folland, Chapter 8, Problem 42.

Problem 7. Please, do *both* problems: Folland, Chapter 8, Problem 49 *and* Problem 50.

Problem 8. Let Γ be a simple C^1 closed curve in $\mathbf{R}^2$ with length L and enclosing area A. Prove the *Isoperimetric ineqaulity* $A \leq L^2/(4\pi)$.

Hint: parametrize Γ by arclength s, so that $(x(s), y(s))$ satisfy $x'(s)^2 + y'(s)^2 = 1$. Suppose the length of Γ is 2π. Expand $x(s), y(s), x'(s), y'(s)$ in Fourier series. Use Parseval's identity and Green's formula.

Problem 9. Please, do *both* problems: Folland, Chapter 8, Problem 27 *and* Problem 29.

Problem 10. Folland, Chapter 8, Problem 32.