Math 565, Winter 2017 D. Jakobson

PROBLEM SET 2 Due Tuesday, February 7.

Do all the problems. Every problem is worth 5 points. Some problems will not be graded because of time constraints.

Folland, Ch. 6, No. 27, 30, 31, 33 (extra credit), 38, 39 40 (extra credit), 43, 44, 45.

Problem 11 (extra credit). Let $f : X \to \mathbf{C}$ be measurable, and let $E = \{p : \int_X |f|^p = ||f||_p^p < \infty\}$. It follows from a result in Folland that the set E is connected.

i) Prove that the function $\phi(p) = \ln(||f||_p^p)$ is convex in the the interior of E, and that ϕ is continuous on E.

ii) Can E consist of a single point? Can it be any connected subset of $(0, \infty)$?

iii) Let $||f||_r < \infty$ for some $r < \infty$. Show that $\lim_{r\to\infty} ||f||_r = ||f||_\infty$.

Problem 12 (extra credit). Assume that in addition to assumptions in Problem 11, $\mu(X) = 1$.

i) Prove that if $0 < r < s \leq \infty$, then $||f||_r \leq ||f||_s$. When does equality hold?

ii) Let $||f||_r < \infty$ for some $r > 0$. Prove that $\lim_{r\to 0} ||f||_r = \exp(\int_X log|f|d\mu)$.

Problem 13 (extra credit). Prove that the unit ball in $L^p(\mathbf{R}^n), p \geq 1$ is a countable intersection of half-spaces.

Problem 14 (extra credit).

Let $\mathbf{Z}_p$ denote the set of p-adic integers. The σ-algebra of measurable sets is generated by "cylinder" sets $C_k, k \geq 0$ of the form

$$C_k = \{a_0 + a_1 p + a_2 p^2 + \ldots + a_k p^k + p^{k+1}\mathbf{Z}_p : k \geq 0, 0 \leq a_j \leq p - 1\}.$$

We define a measure μ_p on $\mathbf{Z}_p$ by letting

$$\mu_p(C_k) := p^{-k-1}.$$

a) Show that μ_p is a pre-measure; we shall denote its completion by the same letter.

b) Show that μ_p is invariant under translation by $\mathbf{Z}_p$. What happens under multiplication by $\mathbf{Z}_p$ (compare with the corresponding properties of the Lebesgue measure).

Problem 15 (extra credit). Let $s > 0$. Compute $\int_{\mathbf{Z}_p} ||x||^{-s}\mu(x)$.

Hint: Let $W_k = \{x \in \mathbf{Z}_p : ||x|| = k\}$; those are measurable disjoint sets. Then the integral I satisfies $I = \sum_{k=0}^{\infty} p^{-ks}\mu(W_k)$.