November 25

σ - Compact

X is σ-compact iff it is the union of countably many compact subsets.

Fact If X is a σ-compact LCH space
(Locally Compact Hausdorff)
then there are open sets
$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$$
with each $\overline{U_n}$ compact and
$$X = \bigcup_{1}^{\infty} U_n$$

Sketch of Proof
$X = \bigcup_{1}^{\infty} K_n$ compact
σ-compact
Use the LC condition to get $U_1 \subseteq K_1$. $U_2 \subseteq K_2 \cup \overline{U_1}$, &c.

Proposition If X is a σ-compact LCH space as above
the U_n are as above
then the sets
$$A_{m,n}(f) = \{ g : X \to \mathbb{C} \; ; \; \sup_{x \in \overline{U_n}} |f(x) - g(x)| < \tfrac{1}{m} \}$$
for $m, n \in \mathbb{N}$ and $f : X \to \mathbb{C}$

form a neighborhood base in the topology of uniform convergence on compact sets.

Proof omitted.

§ Partition of Unity

A partition of unity on a set $E \subseteq X$ is a collection of functions $\{ h_\alpha : X \to [0,1] \; ; \; \alpha \in I \}$ such that
$$\sum_{\alpha \in I} h_\alpha(x) = 1 \quad \text{for all } x \in E$$

The partition of unity is subordinate to the open cover $\mathcal{A}$ of E iff for all $\alpha \in I$ there is a $U \in \mathcal{A}$ such that $\text{support}(h_\alpha) \subseteq U$.

Proposition If X is LCH
$K \subseteq X$ is compact
$\{U_j\}_{j=1}^n$ is an open cover of K
(which may as well be finite since K is compact)
then there is a partition of unity on K subordinate to this open cover.

Sketch Each $x \in K$ has a compact neighborhood
$$N_x \subseteq U_j$$
The interiors $\{\text{int } N_x \,;\, x \in K\}$ form an open cover of K. By compactness, there is a finite subcover: $x_1, \dots, x_m \in K$ compact N_{x_j} with $K \subseteq \bigcup_{j=1}^m N_{x_j}$

This trick is to circumvent the problem that the closures of the original U_j covering K might not be compact. Now we have a cover by precompact sets.
Merge things together:
$$F_j = \bigcup \left(\text{all the } N_{x_k}\text{'s such that } N_{x_j} \subseteq U_j \right)$$
F_j is a finite union of compact sets, so it's compact subset of U_j.

By Urysohn's Lemma, there are
$$g_1, \dots, g_n : X \longrightarrow [0,1]$$
such that $g_j \restriction F_j = 1$ and $\operatorname{supp} g_j \subseteq U_j$.

We have $\displaystyle\sum_{j=1}^{n} g_j(x) \geq 1$ for all $x \in K$
because x must be in some F_j and
$g_j(x) = 1$ while the other $g_k(x)$ are ≥ 0.

Rescaling: Use Urysohn's Lemma <u>again</u> to get

$$f \in \mathcal{C}_c(X, [0,1])$$

Such that $f \equiv 1$ on K and

$$\operatorname{supp}(f) \subseteq \left\{ x : \sum_{1}^{n} g_j(x) > 0 \right\}$$

Let $g_{n+1} = 1 - f$.

Then

$$\sum_{j=1}^{n+1} g_j(x) > 0 \quad \text{for all } x \in X.$$

In particular, this sum is non-zero and we can divide by it:

For $j = 1, \dots, n$, let $\cancel{\sum_{j=1}^{n+1} g_j}$

$$h_j(x) = \frac{g_j(x)}{\displaystyle\sum_{k=1}^{n+1} g_k(x)}$$

Then $\operatorname{supp}(h_j) \subseteq \operatorname{supp}(g_j) \subseteq U_j$.

For all $x \in K$,

$$\sum_{j=1}^{n} h_j(x) = \frac{\displaystyle\sum_{j=1}^{n} g_j(x)}{\displaystyle\sum_{j=1}^{n+1} g_j(x)} = \frac{\displaystyle\sum_{1}^{n} g_j(x)}{\displaystyle\sum_{1}^{n} g_j(x)}$$

Since $g_{n+1}(x) = 0$ for $x \in K$ because $f = 1$ on K.

So $\displaystyle\sum_{1}^{n} h_j(x) = 1$ and we have a partition of unity. $\square$

// Tychonov in Russian.

Dima's Theory

Tychonov
↳ Tychonow ← in German
↳ Tychonovv ← misread w as two v's
↳ Tychonoff ← German v is f.

Tychonoff Theorem

If X_α is compact for $\alpha \in I$

then $\prod_{\alpha \in I} X_\alpha$ is also compact (with the product topology)

Preliminaries: Lingo

For $B \subseteq I$, form $\pi_B(x) = \{\pi_\alpha(x)\}_{\alpha \in B}$

If $B_1 \subseteq B_2$, we say that π_{B_2} is an extension of π_{B_1}.

So $\pi_B : X \longrightarrow \prod_{\alpha \in B} X_\alpha$

We can say one point is an extension of another point:

$q \in \prod_{\alpha \in B_2} X_\alpha$ **extends** $p \in \prod_{\alpha \in B_1} X_\alpha$ iff

for all $\alpha \in B_1$ $\quad \pi_\alpha(q) = \pi_\alpha(p)$.

Start of Proof

It suffices to show that for ~~any~~ net in X has a cluster point.

Consider all subsets $B \subseteq I$.

Given a net $\{x^\beta\}_{\beta \in J}$ in X,

we shall look at cluster points of $\{\pi_B(x^\beta)\}_{\beta \in J}$