McGill University
Math 564: Advanced Real Analysis

Practice problems — not for credit

Problem 1. Determine whether the family of $\mathcal{F} = \{f_n\}$ functions $f_n(x) = x^n$ is uniformly equicontinuous.
1st Solution: The family $\mathcal{F}$ is clearly uniformly bounded. If it were uniformly equicontinuous, we could apply Arzela-Ascoli's theorem to conclude that a sequence f_n has a subsequence that converges uniformly in $C[0,1]$; the limit would have to be a continuous function $g(x)$. However, it is easy to see that $f_n(x) \to h(x)$ as $n \to \infty$, where

$$h(x) = \begin{cases} 0, & x \in [0,1), \\ 1, & x = 1. \end{cases}$$

This function has a jump discontinuity at $x = 1$ and so the convergence cannot be uniform, hence $\mathcal{F}$ is not uniformly equicontinuous.
2nd Solution: Alternatively, we can show that for any sequence n_k, the sequence of functions f_{n_k} cannot be a Cauchy sequence in $C[0,1]$ (which would be necessary for uniform convergence). Indeed, fix some $m = n_k$, and consider the d_∞ distance between x^m and $x^n, n = n_{k+1}, n_{k+2}, \ldots$ We claim that $\limsup_{n\to\infty} d_\infty(x^m, x^n) \geq 1/2$.

Indeed, choose x_0 s.t. $x_0^m > 3/4$, say. Then let N be such that $x_0^n < 1/4$ for $n > N$. Then for any $n_k > N$, we have $x_0^m - x_0^{n_k} \geq 3/4 - 1/4 = 1/2$, hence the same inequality holds for d_∞, QED.
3rd Solution: Finally, take $x = 1$ in the definition of the uniform equicontinuity. Then for any *fixed* $\delta > 0$, we clearly have $\lim_{n\to\infty}(1-\delta)^n = 0 \neq 1 = f_n(1)$, which shows that for large enough n, $|f_n(1-\delta) - f_n(1)| \geq 1/2$, and so the family $\mathcal{F}$ cannot be uniformly equicontinuous at $x = 1$.

Problem 2. Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is differentiable for each $n \in \mathbb{N}$. Suppose also that $\{f_n'\}$ converges uniformly on $\mathbb{R}$ and that $\{f_n(0)\}$ converges. Then $\{f_n\}$ converges pointwise on $\mathbb{R}$.
Solution. Fix $x \in \mathbb{R}, x \neq 0$, and let $\epsilon > 0$ be given. Since $\{f_n(0)\}$ converges it is Cauchy, so we may choose $N_1 \in \mathbb{N}$ so that

$$m, n \geq N_1 \text{ implies } |f_n(0) - f_m(0)| < \frac{\epsilon}{2}$$

Furthermore, since $\{f_n'\}$ is uniformly convergent it is uniformly Cauchy. Therefore we may choose $N_2 \in \mathbb{N}$ so that

$$m, n \geq N_2 \text{ implies } |f_n'(y) - f_m'(y)| < \frac{\epsilon}{2|x|} \text{ for all } y \in \mathbb{R}$$

Now set $N = \max(N_1, N_2)$ and fix any $m, n \geq N$. Then use the mean value theorem to choose $c \in (0, x)$ so that

$$(f_n - f_m)'(c)(x - 0) = (f_n - f_m)(x) - (f_n - f_m)(0)$$

Then using the above inequalities we find that

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(0) - f_m(0)| + |f_n(x) - f_n(0) + f_m(0) - f_m(x)| \\ &= |f_n(0) - f_m(0)| + |(f_n - f_m)(x) - (f_n - f_m)(0)| \\ &= |f_n(0) - f_m(0)| + |(f_n - f_m)'(c)(x - 0)| \\ &= |f_n(0) - f_m(0)| + |f_n'(c) - f_m'(c)||x| \\ &< \epsilon \end{aligned}$$

We conclude that $\{f_n(x)\}$ is Cauchy and hence convergent. Thus $\{f_n\}$ converges pointwise.

Problem 3. Let the Borel set $A \subset [0,1]$ satisfy the following property: there exists $0 \le \tau < 1$ such that for any interval $I \subset [0,1]$, $m(A \cap I) \le \tau \cdot m(I)$. Prove that $m(A) = 0$ (here m denotes the Lebesgue measure).

Problem 4.

Prove that for $p > 0$,

$$\int_0^1 \frac{x^p}{1-x} \log\left(\frac{1}{x}\right) dx = \sum_{k=1}^{\infty} \frac{1}{(p+k)^2}.$$

Justify all your steps. Hint: expand $1/(1-x)$ in Taylor series and integrate by parts.

Problem 5. Determine whether the following sequences of functions converge uniformly or pointwise (or neither) in the regions indicated; explain why.

a)

$$f_n(x) = \begin{cases} \frac{\sin nx}{nx}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

for $x \in [-\pi, \pi]$.

b) $f_n(x) = x^2/(3 + 2nx^2)$ for $x \in [0,1]$.

c) Find $\lim_{n\to\infty} f_n(x)$ in a); is it continuous?

Solution. The limiting function $g = \lim_{n\to\infty} f_n(x)$ in a) is equal to 1 at $x = 0$ and to 0 for $x \neq 0$ (since $|\sin nx| \le 1$ while $nx \to \infty$ for $0 < |x| \le \pi$). Since f_n is continuous for every n ($\lim_{x\to 0}(\sin(nx)/(nx)) = 1$), the convergence cannot be uniform, since a uniform limit of continuous functions is continuous. So, the functions in a) converge only pointwise. The functions in b) converge uniformly to the zero function on $[0,1]$. Since $f_n(0) = 0$, there is nothing to prove there. For $0 < x \le 1$, we can estimate f_n as follows:

$$0 < \frac{x^2}{3 + 2nx^2} = \frac{1}{2n + 3/x^2} < \frac{1}{2n}$$

Accordingly, as $n \to \infty$, $0 \le f_n(x) \le 1/(2n)$ and thus converges to zero uniformly by the "squeezing principle".

Problem 6. Determine whether the following sets are open, closed (or neither open nor closed) and explain why.

a) The set of all $(x,y,z) \in \mathbf{R}^3$ such that $|\cos(2x + 3y + 5z)| < 1/2$ and $x^2 + y^2 + z^2 < 180$.

b) The set of all continuous functions $f \in C([0,1])$ (with the uniform distance) such that $|f(1/n)| \le 1/n^2$ for every natural $n \ge 1$.

Solution. The functions $f_1(x,y,z) = \cos(2x+3y+5z)$ and $f_2(x,y,z) = x^2+y^2+z^2$ are continuous everywhere (by results about the continuity of sum and composition of continuous functions, and since linear and quadratic functions and $\cos x$ are continuous everywhere). Accordingly, the sets $U_1 = f_1^{-1}((-1/2, 1/2))$ and $U_2 = f_2^{-1}((-\infty, 180))$ are open, since they are inverse images of open sets by continuous functions. Accordingly, the set U in a) is open, since it is an intersection of two

open sets U_1 and U_2. It is easy to see that U is nonempty and that $U \neq \mathbf{R}^3$. Since $\mathbf{R}^3$ is connected, U cannot be both closed and open, so it is not closed.
For b), let B_k be the set of all continuous functions f on $[0,1]$ such that $|f(1/k)| \leq 1/k^2$ for a *fixed* $k \geq 1$. Then the set V in b) is equal to $\cap_{k=1}^{\infty} B_k$. If we show that B_k is closed for all k, then we can conclude that B is also closed as an intersection of closed sets. The set V is nonempty (the zero function lies in V), and its complement is also nonempty (the function $f(x) \equiv 2000$ is not in V). Since the space of continuous functions on $[0,1]$ is connected (being convex), the set V cannot be both open and closed. It remains to be shown that B_k is closed. Suppose a sequence of functions $f_j \in B_k$ converges to a function f (which is continuous by a theorem about uniform convergence, but we are only considering continuous functions anyway, so we may as well assume that it's continous!). Since $\operatorname{dist}(f_j, f) = \max|f_j(x) - f(x)|$ goes to 0 as $j \to \infty$ by the definition of convergence, we see that $|f_j(1/k) - f(1/k)| \to 0$ as $j \to \infty$. Since the interval $[-1/k^2, 1/k^2]$ is closed, we conclude that $f(1/k) \in [-1/k^2, 1/k^2]$, and so $f \in B_k$ and the set B_k is closed, QED.

Problem 7. Prove that for the double integral

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dxdy$$

both repeated integrals exist, but that they are not equal. Why is there no contradiction with Fubini's theorem?

Problem 8. Verify Lusin's theorem for the function $f(x) = \arcsin(1/x^2), x \in (0,1]$.

Problem 9. For which values of α and β, the function $f(x) = x^\alpha (\sin x)^\beta$ is Lebesgue integrable on $(0,1]$?

Problem 10. For $n \geq 0$, let

$$f(x,y) = \begin{cases} 2^{2n}, & 2^{-n} \leq x \leq 2^{-n+1}, 2^{-n} \leq y < 2^{-n+1}; \\ -2^{2n+1}, & 2^{-n-1} \leq x \leq 2^{-n}, 2^{-n} \leq y < 2^{-n+1}; \\ 0, & \text{otherwise.} \end{cases}$$

Show that iterated integrals exist but are not equal to each other.

Problem 11. Prove that the set of points at which a sequence of measurable real-valued functions converges to a finite limit is measurable.

Problem 12. Suppose ν_j is a sequence of positive measures. If $\nu_j \perp \mu$ for all j, then $\sum_j \nu_j \perp \mu$. If $\nu_j \ll \mu$ for all j, then $\sum_j \nu_j \ll \mu$.

Problem 13. If E is a Borel set in $\mathbf{R}^n$, the *density* $D_E(x)$ is defined as

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))}$$

whenever the limit exists.

a) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in \mathbf{R}^n \setminus E$.

b) Find examples of E and x s.t. $D_E(x)$ is a given number $\alpha \in (0, 1)$, or such that $D_E(x)$ does not exist.

Problem 14. Verify the assertions in Example 3.25 in Folland.

Problem 15. Folland, Problem 3.28.

Problem 16. Folland, Problem 4.44.

Problem 17. Folland, Problem 4.54.

Problem 18. Folland, Problem 4.61.

Problem 18. Folland, Problem 4.68.