Math 564, Fall 2011 D. Jakobson

TAKE HOME MIDTERM due Friday, October 21, 2011.

Do any 8 of the following 10 problems. Every problem is worth 10 points.

Problem 1. Define the measure on $[0,1]$ by

$$\mu([a,b)) = \log_2 \frac{1+b}{1+a}.$$

a) Prove that μ is preserved by the map $f(x) = \{1/x\}$, where $\{y\}$ denotes the fractional part of y.

Every real number in $x \in [0,1]$ can be expanded into a (finite or infinite) *continued fraction*

$$x = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}},$$

sometimes denoted by $x = [n_1, n_2, n_3, \dots]$.

b) Prove that finite continued fractions correspond to rational numbers, while infinite fractions correspond to irrational numbers.

c) Prove that the function f in part a) can be written as a *shift map*,

$$f([n_1, n_2, n_3, \dots]) = [n_2, n_3, \dots].$$

Hint: You have to show that the measure of the *preimage* $\mu(f^{-1}(A)) = \mu(A)$; it suffices to consider intervals.

Problem 2. We keep the notation from Problem 1.

a) Describe the measure μ in problem 4 in the space of sequences $[n_1, n_2, n_3, \dots]$.

b) Describe all the *periodic* continued fractions, $x = [n_1, \dots, n_k, n_1, \dots, n_k, \dots]$.

Hint:

a) It is enough to specify the measure of a *cylinder* in the space of sequences, e.g. the set of all continued fractions with $n_1 = a_1, n_2 = a_2, ..., n_k = a_k$ for a fixed k-tuple $(a_1, ..., a_k)$ of natural numbers.

b) Example of a periodic continued fraction is the golden ratio, $1/(1 + 1/(1 + 1/...))$; you should show that every periodic (or eventually periodic) continued fraction has a certain property, you don't have to prove a converse.

Problem 3. Let $x, y \in [0,1]$, $x = [0.n_1 n_2 \dots]$ and $y = [0.m_1 m_2 \dots]$. Define $f(x,y) = k$, if $n_k = m_k$, but $n_j \neq m_j$ for $1 \leq j \leq k-1$; and $f(x,y) = \infty$ if no such k exists. Prove that f is Lebesgue measurable on $[0,1] \times [0,1]$, and that it is finite almost everywhere.

Problem 4. Let $f \in L^1(X, \mu)$ and $\mu(X) = 1$. Prove that there exists a monotone function $g \in L^1([0,1])$, such that for all $t \in [0,1]$,

$$\inf_{\mu(A)=t} \int_A f(x)d\mu(x) = \int_0^t g(\tau)d\tau$$

$$\sup_{\mu(A)=t} \int_A f(x)d\mu(x) = \int_{1-t}^1 g(\tau)d\tau$$

Hint: consider the function

$$h(a) = \mu(\{x \in X : f(x) \leq a\}).$$

Problem 5.

a) Compute the area $A(r)$ of the ball of radius r in $\mathbf{R}^2, S^2$, and $\mathbf{H}^2$. Hint: the volume element in polar coordinates (r,θ) is given by $rdrd\theta$ in $\mathbf{R}^2$; $\sin rdrd\theta$ on S^2; and $\sinh rdrd\theta$ in $\mathbf{H}^2$. Where does the volume grow faster? Compute the first 3 terms in the Taylor series expansion of the volume as $r \to 0$; what do you get?

b) Next, compute he length $L(r)$ of the circle of radius r in $\mathbf{R}^2, S^2$, and $\mathbf{H}^2$. Hint: the length element in polar coordinates (r,θ) is given by $dr^2 + r^2 d\theta^2$ in $\mathbf{R}^2$; $dr^2 + \sin^2 rd\theta^2$ on S^2; and $dr^2 + \sinh^2 rd\theta^2$ in $\mathbf{H}^2$.

c) Describe the behavior of the ratio $A(r)/L(r)$ as $r \to 0$.

d) Describe the behavior of the ratio $L(r)/A(r)$ as $r \to \infty$ in $\mathbf{R}^2$ and $\mathbf{H}^2$; and as $r \to \pi$ in S^2.

Problem 6.

a) Compute $L(r)$ and $A(r)$ on an infinite k-regular tree, $k \geq 2$. Describe the behavior of the ratio $L(r)/A(r)$ as $r \to \infty$.

b) Do the same for the graph $\mathbf{Z}^2$.

Problem 7. Let $x = a_0 + a_1 p + a_2 p^2 + \ldots \in \mathbf{Z}_p$. We define $||x|| = p^{-k}$, where k is the smallest integer s.t. $a_k \neq 0 \pmod p$. Recall that we have defined a measure $\mu = \mu_p$ on $\mathbf{Z}_p$ (Assignment 2, Part 2, Problem 2), by requiring that $\mu(p^n \mathbf{Z}_p) = p^{-n}$. Let $s > 0$. Compute $\int_{\mathbf{Z}_p} ||x||^{-s} \mu(x)$.

Hint: Let $W_k = \{x \in \mathbf{Z}_p : ||x|| = k\}$; those are measurable disjoint sets. Then the integral I satisfies $I = \sum_{k=0}^{\infty} p^{-ks} \mu(W_k)$.

Problem 8. Let $f \in L^1(\mu)$. Prove that for each $\epsilon > 0$, there exists $\delta > 0$, such that $\int_E |f| d\mu < \epsilon$ for any measurable E with $\mu(E) < \delta$.

Problem 9. Let $f : X \to \mathbf{R}$ be a function. Describe all those $n \in \mathbf{N}$ for which measurability of $g(x) = (f(x))^n$ implies measurability of $f(x)$.

Problem 10.

a) Let $\mathbf{Z}^2$ act on $\mathbf{R}^2$ by translations. Prove that the number $N(R)$ of points on an orbit of $(0,0)$ (i.e. points with integer coordinates) lying in $B(0,R)$ is asymptotic to πR^2, the area of the ball, as $R \to \infty$.

b) Give an upper bound on the "remainder" $E(R) = N(R) - \pi R^2$ as $R \to \infty$. Getting good upper bounds for $E(R)$ is a very interesting problem!