Math 466a, D. Jakobson

Differentiation under the integral sign

Theorem 1. *Let γ be a smooth closed path, and let $\varphi(z)$ be continuous on γ. For $n \geq 1$, denote by $F_n^{\varphi}(z) = F_n(z)$ the function*

$$F_n(z) = \int_\gamma \frac{\varphi(\zeta)d\,\zeta}{(\zeta - z)^n}.$$

Then $F_n(z)$ is analytic for all n in $\mathbf{C}\backslash\gamma$, and

$$F_n'(z) = nF_{n+1}(z)$$

Proof. We begin by proving

Lemma 2. *$F_1(z)$ is continuous.*

Proof of Lemma 2. Let $z_0 \notin \gamma$. Then there is a ball of radius $\delta > 0$ centered at z_0 which doesn't intersect γ. Now, if $z \in B_{\delta/2}(z_0)$, then for any $\zeta \in \gamma$, $|z-\zeta| > \delta/2$. Also,

$$F_1(z) - F_1(z_0) = \int_\gamma \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\right)\varphi(\zeta)d\zeta = (z - z_0)\int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta - z)(\zeta - z_0)}$$

It follows that the last integrand is bounded above by $2M/\delta^2$, where $M = \sup_{z\in\gamma}|\varphi(z)|$. Accordingly,

$$|F_1(z) - F_1(z_0)| < |z - z_0| \cdot \frac{2M\text{length}(\gamma)}{\delta^2}. \tag{1}$$

Thus, $F_1(z) \to F_1(z_0)$ as $z \to z_0$. □

Introduce a new function

$$\psi(\zeta) := \frac{\varphi(\zeta)}{\zeta - z_0}$$

It is continuous on γ, so we can apply Lemma 2 to it. Now,

$$\frac{F_1^{\varphi}(z) - F_1^{\varphi}(z_0)}{z - z_0} = \int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta - z)(\zeta - z_0)} \tag{2}$$

and the last expression is equal to $F_1^{\psi}(z)$ by definition. By Lemma 2 then, the fraction in (2) converges to $F_1^{\psi}(z_0) = F_2^{\varphi}(z_0)$ as $z \to z_0$. This proves Theorem 1 for $n = 1$.

The case of general n is proved by induction. Suppose that we have shown that $F_{n-1}'(z) = (n-1)F_n(z)$. Consider the difference

$$F_n(z) - F_n(z_0) = \int_\gamma \varphi(\zeta)d\zeta \left(\frac{1}{(\zeta - z)^n} - \frac{1}{(\zeta - z_0)^n}\right)$$

Add and subtract the integral

$$\int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta - z)^{n-1}(\zeta - z_0)}$$

The expression becomes

$$\left[\int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta - z)^{n-1}(\zeta - z_0)} - \int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta - z_0)^n}\right] + (z - z_0)\int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta - z)^n(\zeta - z_0)} \tag{3}$$

Applying the induction hypothesis for the function $\psi(\zeta) = \varphi(\zeta)/(\zeta - z_0)$ to the expression in the first bracket of (3), we conclude that it tends to 0 as $z \to z_0$. In the second term in (3), the factor of $(z - z_0)$ is bounded by

$$\frac{2^n M \text{length}(\gamma)}{\delta^{n+1}}$$

if $z \in B_{\delta/2}(z_0)$. Accordingly, we conclude that the expression in (3) tends to zero as $z \to z_0$. Since it is equal to $F_n(z) - F_n(z_0)$, we conclude that $F_n(z)$ is continuous.

To prove that $F_n(z)$ is differentiable, we divide (3) by $(z - z_0)$ and let $z \to z_0$. The quotient in the first bracket tends to

$$(F^{\psi}_{n-1})'(z_0)$$

By induction hypothesis, it is equal to

$$(n-1)F^{\psi}_n(z_0) = (n-1)F_{n+1}(z_0)$$

When we divide the second term in (3) by $(z - z_0)$, we get a function $F^{\psi}_n(z)$, which we proved is continuous at z_0, and tends to $F_{n+1}(z_0)$ as $z \to z_0$. Putting the two results together, we conclude that $F_n(z)$ is differentiable at z_0 and its derivative is equal to $nF_{n+1}(z_0)$. This proves Theorem 1. $\square$