189-466A, Fall 2000, D. Jakobson

Review of point set topology and metric spaces

A *distance* (p, q) between points p, q in a metric space satisfies

- $\mathrm{dist}(p, q) > 0$ if $p \neq q$; $\mathrm{dist}(p, p) = 0$.
- $\mathrm{dist}(p, q) = \mathrm{dist}(q, p)$.
- $\mathrm{dist}(p, q) + \mathrm{dist}(q, r) \geq \mathrm{dist}(p, r)$.

A sequence p_k converges to p in a metric space X iff for any $\varepsilon > 0$ there exists a natural number N such that for every $k > N$, $\mathrm{dist}(p_k, p) < \varepsilon$. A sequence p_k is *Cauchy* iff for any $\varepsilon > 0$ there exists a natural number N such that for every $k, l > N$, $\mathrm{dist}(p_k, p_l) < \varepsilon$. A sequence in $\mathbf{R}^n$ converges iff all the coordinate sequences converge.

Let X, Y be metric spaces, and let f be a function from X to Y. f is *continuous at* $x \in X$ iff one of the following two equivalent conditions holds:

- for any sequence $x_k \to x$ in X, $f(x_k) \to f(x)$ in Y;
- for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\mathrm{dist}_X(x', x) < \delta$ then $\mathrm{dist}_Y(f(x'), f(x)) < \varepsilon$.

The function f is *continuous* iff one of the following three equivalent conditions holds:

- f is continuous at every point of X;
- for any open set $U \subset Y$, $f^{-1}(U)$ is open in X.
- for any closed set $V \subset Y$, $f^{-1}(V)$ is closed in X.

A function $f : \mathbf{R}^n \to \mathbf{R}^m$ is continuous iff all its coordinate functions are continuous.

$\mathbf{R}^n$ review:

- inner products and norm in $\mathbf{R}^n$; Cauchy-Schwarz inequality;
- different definitions of distance give rise to the same open sets in $\mathbf{R}^n$.

Some useful facts about $\mathbf{R}^n$:

- a subset of $\mathbf{R}^n$ is *compact* iff it is *closed* and *bounded.*
- a ball in $\mathbf{R}^n$ is *convex.*
- an *open* subset of $\mathbf{R}^n$ is connected iff it's path connected (not true for arbitrary subsets of $\mathbf{R}^n$).

Open and closed sets:

- a set A is open iff every point x in A is an *interior point* of A, i.e. if x has a neighborhood (an open ball centered a t x) which is contained in A.
- a set A is closed iff every *limit point* x of A (a limit of a sequence of points in A) is contained in A;
- A is open iff its complement is closed;
- arbitrary union of open sets is open, arbitary intersection of closed sets is closed;
- *finite* intersection of open sets is open, *finite* union of closed sets is closed;
- the empty set and the whole space are both open and closed.

Interior, exterior, and boundary:

- a point x is in the *interior* of A (denoted Int$A$) iff it's an interior point of A;
- a point x is in the *exterior* of A (denoted Ext$A$) iff it's an interior point of the complement A;

- a point x is on the *boundary* of A (denoted $\mathrm{Bd}A$) iff it's neither an interior nor an exterior point of A, i.e. if in every neighborhood of x there are points from A and the complement of A;
- $\mathrm{Int}A, \mathrm{Ext}A$ and $\mathrm{Bd}A$ are disjoint; their union is the whole space;
- A is open iff $A = \mathrm{Int}A$, B is closed iff $\mathrm{Bd}B \subseteq B$.

Let f, g be continuous functions into $\mathbf{R}$, and let a, b be real numbers. Then $af + bg, fg$ are continuous, and f/g is continuous provided $g \neq 0$. Also, if $f : X \to Y$ is continuous, $g : Y \to Z$ is continuous on $f(X)$, then their composition $g(f(x))$ is a continuous function from X to Z. Continuous functions map compact (connected, path connected) sets into compact (connected, path connected) sets.

Compact sets:

- A is *compact* iff every sequence in A has a subsequence converging to a point in A;
- a closed subset of a compact set is compact;
- a continuous function attains a maximum and a minimum on a compact set (extreme value theorem);
- a continuous function on a compact set is uniformly continuous.

Complete sets:

- A is *complete* iff every Cauchy sequence in A converges to a point in A;
- a closed subset of a complete metric space is complete;
- $\mathbf{R}^n$ is complete;
- the spaee of continuous functions on an interval where $\mathrm{dist}(f, g) = \max|f - g|$ is complete (here convergence is equivalent to the uniform convergence).

Convex, path connected and connected sets:

- a subset A of a vector space X is *convex* iff for every $x, y \in A$ the *segment* from x to y (i.e. the set $t \cdot x + (1 - t) \cdot y, 1 \geq t \geq 0$) is contained in A;
- A is *path(wise) connected* iff every two points in A can be joined by a path (a continuous mapping from a closed interval into X) in A;
- A is *connected* iff it is not a union of two nonempty, disjoint, relatively open (in A) sets;
- a metric space X is connected if the only subsets of X which are both open and closed are the empty set and X itself;
- (convex) implies (path connected) implies (connected), but not vice versa!
- a subset or $\mathbf{R}$ is connected iff it's an interval;
- a continuous function into $\mathbf{R}$ maps connected sets into intervals (i.e. intermediate value theorem holds for connected sets);
- a ball in $\mathbf{R}^n$ is convex.

Two more useful facts about such sets:

- an arbitrary intersection of convex sets is convex;
- an arbitrary union of (path) connected sets whose intersection is nonempty is also (path) connected.