

**Hausdorff Dimension Problem Set**

All problems are extra credit. The deadline for this problem set is later in the semester; please, take your time. Every problem is worth 10 points. This problem set follows the exposition in Falconer's book *Fractal Geometry: Mathematical Foundations and Applications*.

Let  $s \geq 0$ . The Hausdorff  $s$ -dimensional measure of a set  $F \in \mathbf{R}^n$  is defined as follows: Let  $\{U_i\}_{i \in I}$  be a (countable or finite) cover of  $F$  by sets of diameter at most  $\delta > 0$ . Let  $|U| = \text{diam}(U) = \sup_{x, y \in U} |x - y|$ . We let

$$\mathcal{H}_\delta^s(F) := \inf \left\{ \sum_{i \in I} |U_i|^s \right\}, \quad (1)$$

where the infimum is taken over all covers of  $F$  by sets diameter at most  $\delta$ . It is clear that  $\mathcal{H}_\delta^s(F)$  increases as  $\delta \rightarrow 0$ .

We next define the Hausdorff  $s$ -dimensional  $\mathcal{H}^s(F)$  by

$$\mathcal{H}^s(F) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F). \quad (2)$$

The limit exists for any subset  $F \in \mathbf{R}^n$  and is usually equal to 0 or  $\infty$ . It is clear that  $\mathcal{H}^s(\emptyset) = 0$ , and that  $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$  for  $E \subset F$ .

**Problem 1.** Let  $\{F_j\}$  be a countable collection of disjoint Borel sets. Show that

$$\mathcal{H}^s(\cup_j F_j) = \sum_j \mathcal{H}^s(F_j).$$

One can also show that for  $s = n$ ,  $\mathcal{H}^n$  is just a constant multiple of the  $n$ -dimensional volume in  $\mathbf{R}^n$  (can you figure out the value of the constant?) Also,  $\mathcal{H}^0$  is just the counting measure.

**Problem 2.**

i) For  $\lambda > 0$ , let  $\lambda F := \{\lambda x : x \in F\}$ . Show that

$$\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F).$$

ii) Let  $f : F \rightarrow \mathbf{R}^m$  be a mapping satisfying

$$|f(x) - f(y)| \leq c|x - y|^\alpha, \quad x, y \in F$$

for some constant  $c > 0, \alpha > 0$  (Hölder with exponent  $\alpha$ ). Show that

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}^s(F).$$

What happens for  $\alpha = 1$  (Lipschitz maps)?

It is clear that for a fixed  $F \in \mathbf{R}^n$  and  $\delta < 1$ , the function  $\mathcal{H}_\delta^s(F)$  is a non-increasing function of  $s$ .

**Problem 3.** Show that if  $\mathcal{H}^s(F) < \infty$ , then

$$\mathcal{H}^t(F) = 0, \quad \forall t > s.$$

Conclude that there exists at most one value of  $s$  for which  $0 < \mathcal{H}^s(F) < \infty$ .

It follows from Problem 3 that there exists a unique value of  $s_0 \geq 0$  such that  $\mathcal{H}^s(F) = \infty, s < s_0$  and  $\mathcal{H}^s(F) = 0, s > s_0$ . We call  $s_0$  the *Hausdorff dimension* of  $F$ ,

$$\dim_H F = s_0.$$

The measure  $\mathcal{H}^{s_0}(F)$  can be finite or equal to 0 or  $\infty$ .