Math 455, Winter 2020 D. Jakobson

PROBLEM SET 1 Due January 23; Problems 13,14, 15 due later

Do any 9 of the following problems. Every problem is worth 10 points. If you do more than 9 problems, you will get extra credit points (that can be used towards future assignments, but NOT towards midterm or final). The deadline for problems 13, 14 is later in the semester; please take your time!

Royden/Fitzpatrick, Chapter 7. Problem 10, 15, 19, 21, 26, 28, 32 and 33 (one problem), 35, 39, 44, 50.

Problem 13. Hausdorff distance. We shall discuss Hausdorff distance between subsets of $X = \mathbb{R}$, but the same definition will work in any metric space X. Let $A, C \subset X$. Then the *Hausdorff distance* between A and C is denoted $D_H(A, C)$ and defined by

$$D_H(A,C) := \max\{\sup_{x\in A}\inf_{y\in C} d(x,y),\ \sup_{y\in C}\inf_{x\in A} d(x,y)\}.$$

This distance can be defined equivalently as follows. Let $A_r := \cup_{x\in A} B(x,r)$, i.e. the r-neighbourhood of the set A.

(a) Show that $d_H(A,C) = \inf\{r \geq 0 : A \subseteq C_r \textit{ and } C \subseteq A_r\}$.

(b) Give and example of $A, C \subseteq \mathbb{R}$ such that $D_H(A,C) = \infty$; show that for bounded A, C, $D_H(A,C) < \infty$.

(c) Show that $D_H(A,C) = 0$ if and only if A and C have the same closure in $\mathbb{R}$.

(d) Let $x \in \mathbb{R}$, and let Y, Z be nonempty subsets of $\mathbb{R}$. Show that $d(x,Y) \leq d(x,Z) + D_H(Y,Z)$. Here $d(x,A) = \inf\{d(x,y) : y \in A\}$.

Problem 14. Hausdorff distance, continued.

(a) Show that d_H defines a distance on the set of all non-empty compact subsets of $\mathbb{R}$, i.e. prove that for such sets $D_H(A,B) = 0$ iff $A = B$, and $D_H(A,C) \leq D_H(A,B) + D_H(B,C)$ (the triangle inequality); the symmetry property is obvious.

(b) Show that $|\mathrm{diam}(A) - \mathrm{diam}(B)| \leq 2 \cdot D_H(A,B)$; recall that $\mathrm{diam}(A) = \sup_{x,y\in A} d(x,y)$.

(c) (EXTRA CREDIT; you can get the maximal score for this problem *without* solving (c)). Recall that a metric space Y is called *complete* iff every Cauchy sequence in Y converges to a limit in Y. For example, $\mathbb{R}^n$ is complete. Show that the set M of all compact nonempty subsets of $\mathbb{R}$ is a complete metric space with respect to the Hausdorff distance D_H.

Problem 15 (Hanner's inequality). Let $f, g \in L^p$. Then

i) If $1 \leq p \leq 2$, then

$$||f+g||_p^p + ||f-g||_p^p \geq (||f||_p + ||g||_p)^p + |\,||f||_p - ||g||_p|^p\,;$$

and

$$(||f+g||_p + ||f-g||_p)^p + |\,||f+g||_p - ||f-g||_p|^p \leq 2^p(||f||_p^p + ||g||_p^p).$$

ii) If $2 \leq p \leq \infty$, the inequalities are reversed.