D. Jakobson

PROBLEM SET 1, EXTRA CREDIT D

Due date to be announced

Every problem is worth 10 points. This problem set follows the exposition in Falconer's book *Fractal Geometry: Mathematical Foundations and Applications*.

Let $s \ge 0$. The Hausdorff s-dimensional measure of a set $F \in \mathbf{R}^n$ is defined as follows: Let $\{U_i\}_{i\in I}$ be a (countable or finite) cover of F by sets of diameter at most $\delta > 0$. Let $|U| = \operatorname{diam}(U) = \sup_{x,y\in U} |x-y|$. We let

(1)
$$\mathcal{H}^s_{\delta}(F) := \inf\{\sum_{i \in I} |U_i|^s\},\$$

where the infimum is taken over all covers of F by sets diameter at most δ . It is clear that $\mathcal{H}^s_{\delta}(F)$ increases as $\delta \to 0$.

We next define (for suitable sets!) the Hausdorff s-dimensional measure $\mathcal{H}^{s}(F)$ by

(2)
$$\mathcal{H}^{s}(F) := \lim_{\delta \to 0} \mathcal{H}^{s}_{\delta}(F).$$

The limit exists for any subset $F \in \mathbf{R}^n$ and is usually equal to 0 or ∞ . It is clear that $\mathcal{H}^s(\emptyset) = 0$, and that $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$ for $E \subset F$.

Problem 13 (extra credit). Let $\{F_j\}$ be a countable collection of disjoint Borel sets. Show that

$$\mathcal{H}^s(\cup_j F_j) = \sum_j \mathcal{H}^s(F_j).$$

One can also show that for s = n, \mathcal{H}^n is just a constant multiple of the *n*-dimensional volume in \mathbb{R}^n (can you figure out the value of the constant?) Also, \mathcal{H}^0 is just the counting measure.

Problem 14 (extra credit).

i) For $\lambda > 0$, let $\lambda F := \{\lambda x : x \in F\}$. Show that

$$\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F).$$

ii) Let $f: F \to \mathbf{R}^m$ be a mapping satisfying

$$|f(x) - f(y)| \le c|x - y|^{\alpha}, \qquad x, y \in F$$

for some constant $c > 0, \alpha > 0$ (Hölder with exponent α). Show that

$$\mathcal{H}^{s/\alpha}(f(F)) \le c^{s/\alpha} \mathcal{H}^s(F).$$

What happens for $\alpha = 1$ (Lipschitz maps)?

It is clear that for a fixed $F \in \mathbf{R}^n$ and $\delta < 1$, the function $\mathcal{H}^s_{\delta}(F)$ is a non-increasing function of s.

Problem 15 (extra credit). Show that if $\mathcal{H}^{s}(F) < \infty$, then

$$\mathcal{H}^t(F) = 0, \qquad \forall t > s.$$

Conclude that there exists at most one value of s for which $0 < \mathcal{H}^s(F) < \infty$.

It follows from Problem 3 that there exists a unique value of $s_0 \ge 0$ such that $\mathcal{H}^s(F) = \infty, s < s_0$ and $\mathcal{H}^s(F) = 0, s > s_0$. We call s_0 the Hausdorff dimension of F,

$$\dim_H F = s_0$$

The measure $\mathcal{H}^{s_0}(F)$ can be finite or equal to 0 or ∞ .