Analysis IV

Math 355

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Winter 2011

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Chapter 1

Introduction

Section 1.1

Banach-Tarski Paradox

Consider B_1, the unit ball in 3-space. Chop it into pieces, then using rotations and translations (rigid motions), we can produce two balls equal in radius. Thus, we want

$$\lim_{n\to\infty}\int_x f_n(x)dx = \int_x \lim_{n\to\infty} f_n(x)dx$$

This, however, is not necessarily true for Riemann integration.

Section 1.2

Existence & Uniqueness Of Lebesgue's Integral

We have the set

$$C_c(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : f \text{ continuous}, supp(f) \text{ compact}\}$$

where

$$supp(f) = \{x \in \mathbb{R}^n : f(x) \neq 0\}$$

1. $f \in C_c(\mathbb{R}^n)$
   Norm is

   $$\int_{\mathbb{R}^n} |f(x)|dx = \int_{supp(f)} |f(x)|dx = \|f\|_1$$

   $$\hat{C}_c(\mathbb{R}) = C_c(\mathbb{R}^n)/\sim$$

   where the slash is modulo which is an equivalence relation. We have

   $$f \sim g \Longleftrightarrow \int_x |f(x) - g(x)|dx = 0$$

   Also, $\hat{C}_c(\mathbb{R}^n), \|\cdot\|$ is a normed space. Now, if we let $L_1(\mathbb{R}^n)$ be the Banach space of all Lebesge measurable functions, then we have that $C_c(\mathbb{R}^n)$ is dense in $L_1(\mathbb{R}^n)$.

2. Define a bounded linear functional $\phi : \hat{C}_c(\mathbb{R}^n) \to \mathbb{R}$ where

   $$\phi(f) = \int_{\mathbb{R}^n} f dx$$

Theorem 1. *The functional ϕ from above extends uniquely to a bounded linear functional $\Phi : L_1(\mathbb{R}^n) \to \mathbb{R}$ such that $\Phi(f) = \phi(f)$ if $f \in C_c(\mathbb{R}^n)$, and we call $\Phi(f)$ Lebesgue's integral.*

Example 1. *Let $X = [0,1] \times [0,1]$ and let*

$$\Delta_{X \backslash \mathbb{Q}^2}(x) = \begin{cases} 1 & x \in X \backslash \mathbb{Q}^2 \\ 0 & x \in \mathbb{Q}^2 \end{cases}$$

so the Lebesgue integral

$$\int_X \Delta_{X \backslash \mathbb{Q}^2}(x) dx = 1$$

The approach is to define

1. Measurable sets and measure.
2. Measurable functions and integrals.

So, consider $\mathbb{R}^n$ for $n = 2$ and take a large rectangle $X = [M_1, M_2] \times [N_1, N_2] \neq \emptyset$ and we want to measure this set.

Definition 1. *A measure is a function $\mu : \mathcal{F} \to \mathbb{R}$ such that*

(i) Extend area.

(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$.

(iii) $A \subseteq B \Longrightarrow \mu(A) \leq \mu(B)$.

We now go over the Lebesgue measurable sets in X.

1. Rectangles:

$$I_i = [a_i, b_i], [a_i, b_i), (a_i, b_i], (a_i, b_i)$$

   so $R = I_1 \times I_2$ then

$$\mathcal{R}(x) = \{R \subset X : R \text{ rectangle}\}$$

   and $\phi, X \in \mathcal{R}$. Here, the measure is $m : \mathcal{R}(X) \to \mathbb{R}$ and $m(R) = |b_i - a_i| \cdot |b_2 - a_2| \geq 0$ and $m(\emptyset) = 0$. Also, if R is the disjoint union of the R_i's where $1 \leq i \leq n$, then

$$m(R) = \sum_{i=1}^{n} m(R_i)$$

2. Elementary sets:
   Let

$$E = \bigcup_{i=1}^{n} R_i$$

   where $R \in \mathcal{R}(X)$, then let

$$\mathcal{E}(X) = \left\{ E = \bigcup_{i=1}^{n} \mathbb{R}_i : R_i \in \mathcal{R}(X) \right\}$$

Theorem 2. *Let $E, F \in \mathcal{E}(X)$, then*

$$E \cup F, E \cap F, E \backslash F \in \mathcal{E}(X)$$

Now, $\hat{m} : \mathcal{E}(X) \to \mathbb{R}$ and if the R_i's are disjoint, and

$$E = \bigcup_{i=1}^{n} R_i = \bigcup_{j=1}^{k} R'_j$$

then

$$\hat{m}(E) = \sum_{i=1}^{n} m(R_i) = \sum_{j=1}^{k} m(R'_j)$$

Theorem 3 (Additivity). *If E and F are disjoint, then*

$$\hat{m}(E \cup F) = \hat{m}(E) + \hat{m}(F)$$

Definition 2. *We say that the rectangle R is an **HV Rectangle** if its edges are horizontal and vertical.*

Definition 3. *We say that $A \sqcup B$ is a **Disjoint Union** and it is equivalent to $A \cup B$ where $A \cap B = \emptyset$.*

We let $X = [M_1, M_2] \times [N_1, N_2]$.

1. Rectangle (hv):

   $$\mathcal{R}(X) = \{R \subseteq X : R \textbf{ hv Rectangle}\}$$

   and $\emptyset, X \in \mathcal{R}(X)$, so $m : \mathcal{R}(X) \to \mathbb{R}$ where

   $$m(R) = area(R)$$

   and

   $$m(R_1 \sqcup R_2) = m(R_1) + m(R_2)$$

2. Elementary Sets (Technical Tool to Approximate Measure Sets):

   $$\mathcal{E}(X) = \{E = \sqcup_{i=1}^{n} R_i : n \in \mathbb{N}, R_i \text{ hv Rectangle}\}$$

   and

   $$\hat{m} : \mathcal{E}(X) \to \mathbb{R}$$

   so if

   $$E = \sqcup_{j=1}^{k} R'_j = \sqcup_{i=1}^{n} R_i$$

   then

   $$\hat{m}(E) = \sum_{j=1}^{k} m(R'_j) = \sum_{i=1}^{n} m(R_i)$$

3. Lebesgue's Measurable Sets (LM Sets):
   First, we define

   $$Exp(X) = \{\text{All subsets of X}\}$$

   Second, we define **Outer Measure** to be

   $$\mu^* : Exp(X) \to \mathbb{R}$$

so that if $A \in X$, then

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} m(R_n) : A \subseteq \bigcup_{n=1}^{\infty} R_n, R_n \text{ hv Rectangle} \right\}$$

Third, we define **Inner Measure** to be

$$\mu_* : Exp(X) \to \mathbb{R}$$

so that if $A \subseteq X$, then

$$\mu_*(A) = m(X) - \mu^*(X \backslash A)$$

In general, we always have that

$$\mu_*(A) \leq \mu^*(A)$$

Fourth, we define **Lebesgue Measurable Sets** to be the sets A such that

$$\mu_*(A) = \mu^*(A)$$

and we say that

$$\mathcal{M}(X) = \{A \subseteq X : \mu_X(A) = \mu^*(A)\}$$

is the **Space of Lebesgue Measurable Sets**.

This is the right thing to do because of the following theorem

THEOREM 4. *The following properties are true:*

(i) Extends Area

$$E \in \mathcal{E}(X) \Longrightarrow E \in \mathcal{M}(X)$$

and $\mu(E) = \hat{m}(E)$.

(ii) Classification of LM Sets

$$A \in \mathcal{M}(X) \Longleftrightarrow \forall\, \epsilon > 0,\ \exists\, E \in \mathcal{E}(X)\ s.t.$$
$$\mu^*(A \triangle E) < \epsilon$$

where $A \triangle E = (A \cup E) \backslash (A \cap E)$.

(iii) Algebra
Let $A, B \in \mathcal{M}(X)$, *then* $\emptyset, X, A \cap B, A \cup B$ *and* $A \backslash B \in \mathcal{M}(X)$.

(iv) Additivity
Let $A, B \in \mathcal{M}(X)$ *be disjoint, then*

$$\mu(A \sqcup B) = \mu(A) + \mu(B)$$

and

$$\mu\left(\sqcup_{n=1}^{k} A_k\right) = \sum_{n=1}^{k} \mu(A_n)$$

(v) σ-Algebra

$$\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{M}(X) \Longrightarrow \bigcup_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} A_n \in \mathcal{M}(X)$$

(vi) *σ-Additivity*
If $\{A_n\}_{n=1}^k \in \mathcal{M}(X)$ are disjoint, then

$$\mu\left(\sqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

(vii) *Monotonicity*
Let $A \subseteq B$, then

$$\mu(A) \leq \mu(B)$$

(viii) *Continuity*

a. If $\cdots \subseteq A_2 \subseteq A_1 \in \mathcal{M}(X)$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

b. If $A_1 \subseteq A_2 \subseteq \cdots \in \mathcal{M}(X)$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

(ix) *Let $A \in X$ and let $\mu^*(A) = 0$, then $A \in \mathcal{M}(X)$ and $\mu(A) = 0$.*

(x) *Translation Invariance*
Let $A \in \mathcal{M}(X)$, then

$$A + x_0 = \{x + x_0 : x \in A\} \in \mathcal{M}(X)$$

and

$$\mu(A + x_0) = \mu(A)$$

Proposition 5. *Every open set is a countable union of hv rectangles.*

To make a final note, we have that

$$\mathcal{R}(X) \subseteq \mathcal{E}(X) \subseteq \{\text{open/closed sets}\} \subseteq \mathcal{M}(X)$$

Definition 4. *We say that A is a* ***Lebesgue Measurable Set On All Of*** *$\mathbb{R}^n$ or just $A \in \mathcal{M}(\mathbb{R}^n)$ whenever*

$$A \cap X_{m,n} \in \mathcal{M}(X_{m,n})$$

where

$$X_{m,n} = [m, m+1] \times [n, n+1]$$

with measure

$$\mu(A) = \sum_{m,n=-\infty}^{\infty} \mu(A \cap X_{m,n})$$

Section 1.3

Non-Measurable Sets

Suppose that $x, y \in X$ and let α be an irrational number, so $\alpha \notin \mathbb{Q}$, and $x \sim y \iff x - y = n\alpha \mod 1$ where $n \in \mathbb{Z}$.

Theorem 6 (Axiom Of Choice). *We can choose a single representative from each equivalence class. We note that this is NOT a theorem, it is simply a widely accepted mathematical axiom.*

Let us call the set of all these representatives Φ.

Proposition 7. *Φ is not Lebesgue measurable.*

Proof.

Let

$$\Phi_n = \{x + n\alpha \mod 1 : x \in \Phi\}$$

We now claim that all the Φ_n's are pairwise disjoint. Suppose then for a contradiction that $x \in \Phi_n \cap \Phi_m$ where $n \neq m$. This implies that there exists $x_1 \in \Phi$ and $y_1 \in \Phi$ such that

$$x_1 + n\alpha y_1 + m\alpha \mod 1$$

then

$$x_1 - y_1 = (m - n)\alpha \mod 1$$

and so $x_1 \sim y_1$, and thus $y_1 \in \{x_1\}$ - equivalence class. But Φ contains only a single element from each equivalence class. This contradiction yields our first claim.

We now claim that

$$\bigcup_n \Phi_n = S^1$$

Now, suppose for another contradiction that Φ is in fact measurable. Then, there exists $0 \leq a = \mu(\Phi)$. Now, suppose that $a > 0$. Then

$$\mu(\Phi_n) = \mu(\Phi_m) = a$$

Then

$$(*) \qquad \mu([0,1]) = 1 = \sum_{n=-\infty}^{\infty} a = \infty$$

Thus, $a = 0$. But then $(*)$ gives that

$$0 = \mu([0,1]) = 1 \qquad \square$$

Chapter 2

Collections Of Sets

Definition 5. *We say that the collection $\mathcal{R} \neq \emptyset$ of sets is a **Ring** if for any $A, B \in \mathcal{R}$, we have that*

$$A \cap B, A \triangle B \in \mathcal{R}$$

Corollary 8. *$\mathcal{R}$ is closed under operations of taking unions, intersections, differences, and symmetric differences.*

Remark. $A \cup B = (A \cap B) \cup (A \triangle B)$.
We also note that every ring contains $\emptyset$ since $\emptyset = A \backslash A$ so that the smallest possible ring is $\{\emptyset\}$.

Definition 6. *We say that the set E is **Unit** in $\mathcal{R}$ if for any set $A \in \mathcal{R}$, we have that $A \cap E = A$.*

Remark. It is important to note that E is unique.

Definition 7. *A ring $\mathcal{R}$ together with a unit E is called an **Algebra**.*

Example 2. *We will list some rings and algebras. First, let A be any set.*

1. *Let $\mathcal{R}$ be all the subsets of A. Then $E = A$.*
2. *Let $\mathcal{R} = \{\emptyset, A\}$.*
3. *Let $\mathcal{R}$ b e all the finite subsets of A. $\mathcal{R}$ has a unit (and it would be A) if and only if A is a finite set.*
4. *Let $\mathcal{R}$ be all bounded subsets of $\mathbb{R}$. Then $\mathcal{R}$ has no unit.*

Theorem 9. *If $\mathcal{R}_\alpha$ is a ring for all $\alpha \in I$, then $\cap_{\alpha \in I} \mathcal{R}_\alpha$ is also a ring.*
Proof.
We know that $A, B \in \cap_{\alpha \in I} \mathcal{R}_\alpha$ if and only if $A, B \in \mathcal{R}_\alpha$ for each $\alpha \in I$. Then $A \cap B, A \triangle B \in \mathcal{R}_\alpha$ for each $\alpha \in I$. Thus,

$$A \cap B, A \triangle B \in \bigcup_{\alpha \in I} \mathcal{R}_\alpha \qquad \square$$

Definition 8. *$\mathcal{A}$ is called a **Semi-Ring** if*

(i) $\emptyset \in \mathcal{A}$

(ii) $A, B \in \mathcal{A} \Longrightarrow A \cap B \in \mathcal{A}$

(iii) If $A \in \mathcal{A}$ and $A_1 \subseteq A$, where $A_1 \in \mathcal{A}$ then there exists $A_2, \ldots, A_k$ such that

$$A = A_1 \cup A_2 \cup \cdots \cup A_k$$

and $A_i \cap A_j = \emptyset$ for $i \neq j$ where $A_i \in \mathcal{A}$ for all $i \leq k$.

Remark. Every ring is a semi-ring.

DEFINITION 9. *From the definition above, we say that the sets $A_1, \ldots, A_k$ form a* ***Finite Partition*** *of $\mathcal{A}$.*

LEMMA 10. *Let $A_1, \ldots, A_n \in \mathcal{A}$ where $\mathcal{A}$ is a semi-ring and $A_i \cap A_j = \emptyset$, if $i \neq j$ and $A_i \subseteq A$ for all $i \leq n$. Then there exists sets $A_{n+1}, \ldots, A_s$ such that*

$$(*) \quad A = A_1 \cup \cdots \cup A_n \cup A_{n+1} \cup \cdots \cup A_s$$

and such that $(*)$ is a finite partition of A (i.e. the A_i's are pairwise disjoint).
Proof.
We proceed by induction. If $n = 1$, then this holds by the definition of a semi-ring. For the rest of this induction, see Kolmogorov & Fomin.

LEMMA 11. *If $A_1, \ldots, A_n \in \mathcal{A}$ where $\mathcal{A}$ is a semi-ring, then there exists finitely many disjoint sets $B_1, \ldots, B_t$ such that every for each A_k there exists I_k such that*

$$A_k = \bigcup_{s \in I_k} B_s$$

Proof.
We proceed by induction. Trivially, it holds for $n = 1$ (just take $B_1 = A_1$). Now, suppose that it's true for $n = m$. We wish to show that it holds for $n = m + 1$. Let $A_1, \ldots, A_m, A_{m+1} \in \mathcal{A}$ and let $B_1, \ldots, B_t$ be the elements of $\mathcal{A}$ that satisfy the conclusion of the lemma for $A_1, \ldots, A_m$. Now, let $B_{s,1} = A_{m+1} \cap B_s$. We know that $B_{1,1}, \ldots, B_{t,1}$ are disjoint subsets of A_{m+1} so we can apply Lemma 10.
Lemma 10 implies that

$$A_{m+1} = \cup_{s=1}^{t} B_{s,1} = \bigcup_{p=1}^{q} B'_p$$

so that the B'_p form a finite partition. Also, by the definition of a semi-ring, there exists a finite partition

$$B_s = B_{s,1} \cup B_{s,2} \cup \cdots$$

It is clear that $\{B_{1,1}, \ldots, B_{1,f_1}; B_{2,1}, B_{2,2}, \ldots, B_{2,f_2}; B_{t,1}, \ldots, B_{t,f_t}; B'_1, \ldots, B'_q\}$ would satisfy the conclusion of this lemma fo $A_1, \ldots, A_{m+1}$ $\square$

LEMMA 12. *Let $\mathcal{A}$ be a semi-ring. Then, the ring generated by $\mathcal{A}$, $\mathcal{R}(\mathcal{A})$ is the collection $\mathcal{B}$*

$$A = \bigcup_{k=1}^{n} A_k$$

where $A_k \in \mathcal{A}$.
Proof.
This proof makes use of Lemmas 10 and 11 and can be found in Kolmogorov & Fomin.

THEOREM 13. *If $\mathcal{A}$ is an arbitrary collection of sets, then there exists a unique ring $\mathcal{R}(\mathcal{A})$*

(i) Containing all sets of $\mathcal{A}$.

(ii) $\mathcal{R}(\mathcal{A})$ is contained in every ring satisfying (i).

*It is called a **Minimal Ring** over $\mathcal{A}$, or the **Ring Generated By** $\mathcal{A}$.*
Proof.
We shall prove this in a special case with semi-rings.

Section 2.1

Sigma Algebras

Definition 10. *We say that $\mathcal{R}$ is a σ**-ring** if and only if $A_1, \ldots, A_n, \ldots$ where $A_i \in \mathcal{R}$ implies that*

$$\bigcup_{j=1}^{\infty} A_i \in \mathcal{R}$$

Now, if $\mathcal{R}$ is an algebra of subsets of a set X, then

$$\left(\bigcup_n A_n\right)^C = \bigcap_n (A_n)^C$$

If $A_n \in \mathcal{R}$, then $(A_n)^C \in \mathcal{R}$ which implies that

$$\bigcap_{n=1}^{\infty} A_n^C \in \mathcal{R}$$

Definition 11. *We say that $\mathcal{R}$ is a δ**-ring** if and only if $A_1, \ldots, A_n, \ldots$ where $A_i \in \mathcal{R}$ implies that*

$$\bigcap_{j=1}^{\infty} A_i \in \mathcal{R}$$

Definition 12. *If a σ-ring is an algebra, then it is also a δ-ring. Such algebras are called **Borel Algebras**.*

Theorem 14. *If Σ is a collection of sets, then there exists "the smallest" σ-algebra containing all the sets in Σ and contained in every σ-algebra that contains Σ. It is called the **Minimal** σ**-Algebra** over Σ, or just the* **Minimal**$\mathcal{B}$ **− Algebra** *generated by Σ. In $\mathbb{R}$, the Borel σ-algebra is the σ-algebra generated by all (closed) intervals.*

We now wish to know what functions do to rings. Let $f : X \to Y$ where $y = f(x)$, and suppose that $\mathcal{M}$ is a collection of subsets of X and that $\mathcal{N}$ is a collection of subsets of Y.

Definition 13. *For notational purposes, we say*

$$f(\mathcal{M}) = \{f(B), B \in \mathcal{M}\}$$

and we say

$$f^{-1}(\mathcal{N}) = \{f^{-1}(C) : C \in \mathcal{N}\}$$

Proposition 15. *We list some properties of the above objects:*

(i) *If $\mathcal{R}$ is a ring of subsets of Y, then $f^{-1}(\mathcal{R})$ is also a ring of subsets of X.*
Proof.
If $A, B \in \mathcal{R}$ then

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

and

$$f^{-1}(A \backslash B) = f^{-1}(A) \backslash f^{-1}(B)$$

so that

$$\begin{aligned} f^{-1}(A \triangle B) &= f^{-1}((A \backslash B) \cup (B \backslash A)) \\ &= f^{-1}(A \backslash B) \cup f^{-1}(B \backslash A) \\ &= (f^{-1}(A) \backslash f^{-1}(B)) \cup (f^{-1}(B) \backslash f^{-1}(A)) \\ &= f^{-1}(A) \triangle f^{-1}(B) \end{aligned}$$

(ii) *If $\mathcal{A}$ is an algebra, then $f^{-1}(\mathcal{A})$ is an algebra. That is, there exists $E \in \mathcal{A}$ such that $f^{-1}(E) =$ unit in $f^{-1}(\mathcal{A})$.*

(iii) *If $\mathcal{A}$ is a σ-algebra, then $f^{-1}(\mathcal{A})$ is also a σ-algebra.*

(iv) *The ring generated by $f^{-1}(\mathcal{N})$ is equal to*

$$f^{-1}(\textit{ring generated by } \mathcal{N})$$

(v) *The σ-algebra generated by $f^{-1}(\mathcal{N})$ is equal to*

$$f^{-1}(\sigma\textit{-algebra generated by } \mathcal{N})$$

Chapter 3

Measures

Suppose that $\mathcal{R}$ is a semi-ring, then we define

$$\mu : \mathcal{R} \to \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$$

If $A = \cup_{k=1}^{\infty} A_k$ is a finite partition of A where $A_k \in \mathcal{R}$ for each k. Then

$$\mu(A) = \sum_{k=1}^{n} \mu(A_k)$$

Proposition 16. *Let $\mathcal{R}$ be a semi-ring and let $\mu : \mathcal{R} \to \mathbb{R}_+$ be a measure. Let $A_1, \ldots, A_k$ be pairwise dijoint subsets of the set A such that $A_j, A \in \mathcal{R}$ for each j. Then*

$$\sum_{j=1}^{k} \mu(A_j) \leq \mu(A)$$

Proof.
Let $B_1, \ldots, B_l$ be such that

$$A = A_1 \cup \cdots \cup A_k \cup B_1 \cup \cdots \cup B_l$$

then

$$\mu(A) = \mu(A_1) + \cdots + \mu(A_k) + \mu(B_1) + \cdots + \mu(B_l)$$

where each of the measures of the B_i's are non-negative. $\square$

Theorem 17. *Let $\mathcal{S}$ be a semi-ring and let μ be a measure. Let $A_1, \ldots, A_n; A \in \mathcal{S}$ (not necessarily disjoint) be such that*

$$A \subseteq \bigcup_{k=1}^{n} A_k$$

then

$$\mu(A) \leq \sum_{k=1}^{n} \mu(A_k)$$

Proof.
By a property of semi-rings, there exists sets $B_1, \ldots, B_t$ such that the B_i's are pariwise disjoint and

$$A = \bigcup_{s \in I_0} B_s$$

and

$$A_k = \bigcup_{s \in I_k} B_s$$

for each k, so that

$$A = \bigcup_{s \in I_0} B_s \subseteq \bigcup_{k=1}^{n} \bigcup_{s \in I_k} B_s$$

and so

$$\mu(A) = \sum_{s \in I_0} \mu(B_s) \leq \sum_{k=1}^{n} \sum_{s \in I_k} \mu(B_s) \qquad \square$$

Corollary 18. *If $A_1 \subseteq A_2$, then $\mu(A_1) \leq \mu(A_2)$.*

Definition 14. *An **Extension** μ of a measure m satisfies*

(i) Ring $\mathcal{S}_m \subseteq \mathcal{S}_\mu$.

(ii) For any $A \in \mathcal{S}_m$, we have that $\mu(A) = m(A)$.

Theorem 19. *Any measure $m(A)$ on a semi-ring $\mathcal{S}_m$ has a unique extension $\mu(A)$ such that*

$$\mathcal{S}_\mu = \{\textit{ring generated by } \mathcal{S}_m\} = \mathcal{R}(\mathcal{S}_m)$$

Proof (Sketch).
Recall that any set in $\mathcal{R}(\mathcal{S}_m)$ has a partition

$$A = \bigcup_{k=1}^{n} B_k$$

where $B_k \in \mathcal{S}_m$. Now, let

$$(*) \qquad \mu(A) = \sum_{k=1}^{n} m(B_k)$$

We now claim that $(*)$ doesn't depend on the sets B_k's that we used. To show this, we note that any two ways to assemble A have a common refinement.
This gives an extension of m to $\mathcal{R}(\mathcal{S}_m)$.

For uniqueness, suppose that μ_1 and μ_2 both extend m. Let $A \in \mathcal{R}(\mathcal{S}_m)$, then

$$A = \bigcup_{k=1}^{n} B_k$$

where the B_k's form a partition of A, and so

$$\mu_1(A) = \sum_{k=1}^{n} \mu_1(B_k) = \sum_{k=1}^{n} m(B_k) \qquad \square$$

We note that for the complete proof, see Kolmogorov & Fomin.

Definition 15. *We say that μ is **Countably/Completely/σ-Additive** if we have a countable family of pairwise disjoint subsets in $\mathcal{S}_\mu$, $A_1, \ldots, A_n, \ldots$ that we can measure, and if we have*

$$A = \bigcup_{n=1}^{\infty} A_n$$

then

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$$

Example 3. *Here, we list some countably additive measures.*

1. *Lebesgue Measure is σ-additive.*
2. *Let $p_1, \ldots, p_n, \ldots$ be non-negative numbers such that*

$$\sum_{n=1}^{\infty} p_n = 1$$

and let

$$\mathcal{S}_\mu = \{\textit{all subsets of } \mathbb{N}\}$$

where

$$\mu(\{n\}) = p_n$$

If $I \subseteq \mathbb{N}$, then

$$\mu(I) = \sum_{n \in I} p_n \Longrightarrow \mu(\mathbb{N}) = \sum_{n=1}^{\infty} = 1$$

This measure that we just described is σ-additive.

We note that there do exist measures that are additive but are not σ-additive.

Theorem 20. *If m defined on $\mathcal{S}_m$ is σ-additive, then its extension $\mu = r(m)$ to the ring $\mathcal{R}(\mathcal{S}_m)$ is also σ-additive.*

Proof (Sketch).

Suppose that $A \in \mathcal{R}(\mathcal{S}_m)$ and that for each k, we have $B_k \in \mathcal{R}(\mathcal{S}_m)$. Now, suppose that the B_k's are disjoint and

$$A = \bigcup_{k=1}^{\infty} B_k$$

We want that

$$\mu(A) = \sum_{k=1}^{\infty} \mu(B_k)$$

We know that for a finite combination of $A_j \in \mathcal{S}_m$ and $B_{n,i}$, we can write

$$A = \bigcup_j A_j$$

where the A_j's are disjoint, and we can write

$$B_n = \bigcup_i B_{n,i}$$

where the B_i's are disjoint. Now, define

$$C_{n,i,j} = A_j \cap B_{n,i}$$

then we have that

$$A = \bigcup_n \left(\bigcup_i C_{n,i,j} \right) \qquad B_{n,i} = \bigcup_j C_{n,i,j}$$

so that

$$m(A_j) = \sum_n \sum_i m(C_{n,i,j}) \qquad m(B_{n,i}) = \sum_j m(C_{n,i,j})$$

and so

$$\mu(A) = \sum_j m(A_j) = \sum_j \sum_n \sum_i m(C_{n,i,j})$$

$$\mu(B_n) = \sum_i m(B_{n,i}) = \sum_i \sum_j m(C_{n,i,j})$$

and finally, since n *lives in* $\mathbb{N}$*, our sum over all* n *is an infinite series, and so*

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) \qquad \square$$

Section 3.1

The Borel-Cantelli Lemma

Definition 16. *Consider a sequence* $A_1, \ldots, A_n, \ldots \subseteq X$*, then we define*

$$B = \limsup_{n\to\infty} A_n = \bigcap_{n\geq 1} \left(\bigcup_{k\geq n} A_k \right) = \{x : x \in A_k \textit{ for infinitely many } k\}$$

Lemma 21 (Borel-Cantelli). *Suppose that*

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty$$

Then

$$\mu(B) = 0$$

Theorem 22. *If a measure* μ *is* σ*-additive, and* $A_1, \ldots, A_n, \ldots \in \mathcal{S}_\mu$*, and*

$$A \subseteq \bigcup_{n=1}^{\infty} A_n$$

then

$$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Proof (Idea).
First, let

$$B_n = (A \cap A_n) \backslash \left(\bigcup_{k=1}^{n-1} A_k \right)$$

and

$$A = \bigcup_{n=1}^{\infty} B_n$$

and so our B_n*'s are disjoint. Now we apply* ***Theorem 20*** *to finish the proof.*

Section 3.2

Lebesgue Extension of Measure on a Semi-Ring with Unit

Consider a σ-additive measure m defined on $\mathcal{S}_m$ with the unit E.

Definition 17. *Let $A \subseteq E$, then we define the **Outer Measure** μ^* to be*

$$\mu^*(A) = \inf \left\{ \sum_n m(B_n) : A \subseteq \bigcup_n B_n \right\}$$

for $B_n \in \mathcal{S}_m$.

Definition 18. *Let $A \subseteq E$, then we define the **Inner Measure** μ_* to be*

$$\mu_*(A) = m(E) - \mu^*(E \backslash A)$$

where μ^* is the outer measure.

Proposition 23. *We have that*

$$\mu_*(A) \leq \mu^*(A)$$

Proof.
We know that

$$\mu_*(A) = m(E) - \mu^*(E \backslash A)$$

and we want to show that

$$m(E) \leq \mu^*(A) + \mu^*(E \backslash A)$$

so

$$\mu^*(A) = \inf_{A \subseteq \cup_n B_n} \sum_n m(B_n)$$

$$\mu^*(E \backslash A) = \inf_{E \backslash A \subseteq \cup_m C_m} \sum_m m(C_m)$$

Now, we have

$$E \subseteq \left(\bigcup_n B_n \right) \cup \left(\bigcup_m C_m \right)$$

and this implies that

$$m(E) \leq \inf \sum_{n=1}^{\infty} m(B_n) + \inf \sum_{m=1}^{\infty} m(C_m)$$

and this finally implies that

$$\mu_*(A) = m(E) - \mu^*(E \backslash A) \leq \mu^*(A) - \mu^*(E \backslash A) + \mu^*(E \backslash A) = \mu^*(A) \qquad \square$$

Definition 19. *We say that $A \subseteq E$ is **Lebesgue Measurable** if and only if*

$$\mu_*(A) = \mu^*(A)$$

in which case, we define

$$\mu(A) = \mu_*(A) = \mu^*(A)$$

Remark. Also, if A is measurable, then so is its compliment $E\backslash A$.

PROPOSITION 24. *If μ is a σ-additive extension of m, then*

$$\mu_*(A) \leq \mu(A) \leq \mu^*(A)$$

COROLLARY 25. *If A is Lebesgue measurable, then for any σ-finite extension μ of m, we have that*

$$\mu(A) = \mu_*(A) = \mu^*(A)$$

DEFINITION 20. *A is **Lebesgue Measurable** if and only if*

$$\mu^*(A) + \mu^*(E\backslash A) = m(E)$$

THEOREM 26. *Suppose that*

$$A \subseteq \bigcup_{n=1}^{\infty} A_n$$

then we have that

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

Proof.
For any $\epsilon > 0$, there exists a cover $\{P_{n,k}\}$ each measurable with respect to m such that

$$\sum_{k=1}^{\infty} m(P_{n,k}) \leq \mu^*(A_n) + \frac{\epsilon}{2^n}$$

Now,

$$A \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} P_{n,k}$$

and so

$$\mu^*(A) \leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} m(P_{n,k}) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon$$

but ϵ is arbitrarily small, so this yields our claim. □

THEOREM 27. *If $A \in \mathcal{R}$ where $\mathcal{R}$ is the collection of sets measurable with respect to the old measure, then*

$$\mu_*(A) = m'(A) = \mu^*(A)$$

where m is the old measure.
Proof.
For a detailed proof, see Kolmogorov & Fomin, §33.

LEMMA 28. *For any sets $A, B \subseteq E$, we have that*

$$|\mu^*(A) - \mu^*(B)| \leq \mu^*(A \triangle B)$$

Proof.
WLOG, we can assume that

$$\mu^*(A) \geq \mu^*(B)$$

so, we have

$$A \subseteq B \cup (A \triangle B) \Longrightarrow \mu^*(A) \leq \mu^*(B) + \mu^*(A \triangle B)$$

which completes the proof. $\square$

THEOREM 29. *The set A is measurable if and only if for any $\epsilon > 0$, there exists a set $B \in \mathcal{R}(\mathcal{S}_{\Updownarrow})$ (contained in the minimal ring generated by $\mathcal{S}_m$) such that*

$$\mu * (A \triangle B) < \epsilon$$

Proof (Sketch).

(i) Sufficiency.
*Suppose that for any $\epsilon > 0$, there exists a set B such that $\mu^*(A \triangle B) \leq \epsilon$. Now, by **Lemma 28**, we get that*

$$|\mu^*(A) - m'(B)| \leq \epsilon$$

Now,

$$(E \backslash A) \triangle (E \backslash B) - A \triangle B$$

Now, we have

$$|\mu^*(E \backslash A) - m'(E \backslash B)| < \epsilon$$

and B is elementary, so

$$m'(B) + m'(E \backslash B) = m(E)$$

Now, we finally get that

$$|\mu^*(A) + \mu^*(E \backslash A) - m(E)| < 2\epsilon$$

but this gives that

$$\mu^*(A) + \mu^*(E \backslash A) - m(E) = 0$$

which implies that A is measurable, so we have sufficiency.

(ii) Necessity.
Suppose that

$$\mu^*(A) + \mu^*(E \backslash A) = m(E) = 1$$

Then for all $\epsilon > 0$, there exists elementary sets $\{B_n\}$ and $\{C_n\}$ such that

$$A \subseteq \bigcup_n B_n \qquad E \backslash A \subseteq \bigcup_n C_n$$

and

$$\sum_n m(B_n) \leq \mu^*(A) + \frac{\epsilon}{3} \qquad \sum_n m(C_n) \leq \mu^*(E \backslash A) + \frac{\epsilon}{3}$$

We know that

$$\sum_n m(B_n) < \infty$$

so choose N such that

$$\sum_{n > N} m(B_n) < \frac{\epsilon}{3}$$

and let

$$B = \bigcup_{n=1}^{N} B_n$$

so

$$P = \bigcup_{n>N} B_n$$

forms an open cover of $A \backslash B$, *and let*

$$Q = \bigcup_{n} (B \cap C_n)$$

which is a cover of $B \backslash A$. *Now,*

$$A \triangle B \subseteq P \cup Q$$

so that

$$\mu^*(P) \leq \sum_{n \geq N} m(B_n) < \frac{\epsilon}{3}$$

We now claim that

$$\mu^*(Q) < \frac{2\epsilon}{3}$$

the proof for which can be found in Kolmogorov & Fomin. Using this, we arrive at

$$\mu^*(A \triangle B) \leq \mu^*(P) + \mu^*(Q) < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

which gives necessity. □

Remark. $B = B(\epsilon)$.

Theorem 30. *The collection* $\mathcal{M}$ *of all Lebesgue Measurable sets is a ring.*

Proof (Sketch).
We know that

$$A_1 \cap A_2 = A_1 \backslash (A_1 \backslash A_2)$$

and that

$$A_1 \cup A_2 = E \backslash [(E \backslash A_1) \cap (E \backslash A_2)]$$

It suffices to show that if A_1, A_2 *are measurable, then* $A_1 \backslash A_2$ *is also measurable. Now there exists* B_1, B_2 *such that*

$$\mu^*(A_j \triangle B_j) < \frac{\epsilon}{2}$$

for $j = 1, 2$. *Let* $B = B_1 \backslash B_2$. *Then we claim that*

$$(A_1 \backslash A_2) \triangle (B_1 \triangle B_2) \subseteq (A_1 \backslash B_1) \triangle (A_2 \backslash B_2)$$

We will leave the proof of this claim as an excercise. Assuming the above, we have that

$$\mu^*((A_1 \backslash A_2) \triangle (B_1 \triangle B_2)) \leq \mu^*((A_1 \backslash B_1) \triangle (A_2 \backslash B_2)) < 2\frac{\epsilon}{2} = \epsilon$$

and since ϵ *is arbitrary, we have that* $A_1 \backslash A_2$ *are measurable as needed.* □

THEOREM 31. *Let $E \in \mathcal{M}$ where $\mathcal{M}$ is an algebra with unit E. Then μ is additive on $\mathcal{M}$. That is, if $A_1, \ldots, A_n \in \mathcal{M}$ are disjoint, then*

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k)$$

Proof (Sketch).
It is enough to show this for $n = 2$. So, there exists B_1, B_2 elementary sets such that

$$\mu^*(A_1 \triangle B_1) < \epsilon$$

and

$$\mu^*(A_2 \triangle B_2) < \epsilon$$

and we claim that

$$B_1 \cap B_2 \subseteq (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$$

the proof of which is left as an excercise. Assuming this, we arrive at

$$\mu^*(B_1 \cap B_2) < 2\epsilon$$

and we have that

$$|\mu^*(B_1) - \mu^*(A_1)| = |m'(B_1) - \mu^*(A_1)| < \mu^*(A_1 \triangle B_1) < \epsilon$$

and similarly,

$$|m'(B_2) - \mu^*(A_2)| < \epsilon$$

Now, let $B = B_1 \cup B_2$, then we get

$$\begin{aligned} m'(B) &= m'(B_1) + m'(B_2) - m'(B_1 \cap B_2) \\ &\geq \mu^*(A_1) + \mu^*(A_2) - \epsilon - \epsilon - 2\epsilon \\ \mu^*(A_1) + \mu^*(A_2) - 4\epsilon & \end{aligned}$$

Now we claim that

$$(A \triangle B) \subseteq (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$$

and the proof to this is left as another excercise. Assuming this, we get

$$\begin{aligned} \mu^*(A) &\geq m'(B) - \mu^*(A \triangle B) \\ &\geq m'(B) - 2\epsilon \\ &\geq \mu^*(A_1) + \mu^*(A_2) - 6\epsilon \end{aligned}$$

and since ϵ is arbitrarily small, we finally get that

$$\mu^*(A) \geq \mu^*(A_1) + \mu^*(A_2)$$

and since we get the other inequality for free, we have that

$$\mu^*(A) = \mu^*(A_1) + \mu^*(A_2)$$

and then the fact that $\mu = \mu^*$ completes the proof. $\square$

Theorem 32. *The measure μ is σ-additive on $\mathcal{M}$.*

Proof (Sketch).
Let

$$A = \bigcup_{n=1}^{\infty} A_n$$

and assume that the A_n's are disjoint. Then,

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

***Theorem 31** tells us that*

$$\mu^*(A) \geq \mu^*\left(\bigcup_{k=1}^{N} A_k\right) = \sum_{k=1}^{N} \mu^*(A_k)$$

Now, let $N \to \infty$, which gives

$$\mu^*(A) \geq \sum_{n=1}^{\infty} \mu^*(A_n)$$

and we have equality. $\square$

We say that μ on $\mathcal{M}$ is the **Lebesgue Extension** of m.

Theorem 33. *$\mathcal{M}$ is a Borel σ-algebra with unit E.*

Proof.
It is enough to show that if $A_1, \ldots, A_n, \ldots \in \mathcal{M}$, then $\cup_{n=1}^{\infty} A_n$ is measurable. To show this, let

$$A'_n = A_n \backslash \left(\bigcup_{k=1}^{n-1} A_k\right)$$

then the A'_n are disjoint and

$$\bigcup_{n=1}^{\infty} A'_n = \bigcup_{n=1}^{\infty} A_n$$

and $A'_n \in \mathcal{M}$ then

$$\sum_{k=1}^{n} \mu(A'_k) = \mu\left(\bigcup_{k=1}^{n} A'_k\right) \leq \mu^*(A)$$

Hence,

$$\sum_{n=1}^{\infty} \mu(A'_n) < \infty$$

This means that for any $\epsilon > 0$, there exits $N > 0$ such that

$$\sum_{n>N} \mu(A'_n) < \frac{\epsilon}{2}$$

Now, look at

$$C = \bigcup_{n=1}^{N} A'_n$$

which is measurable. Hence, there exists an elementary set B such that

$$\mu(C \triangle B) < \frac{\epsilon}{2}$$

We have that

$$A \triangle B \subseteq (C \triangle B) \cup \left(\bigcup_{n>N} A'_n \right)$$

and this gives us that

$$\mu^*(A \triangle B) \leq \mu(C \triangle B) + \sum_{n>N} \mu^*(A'_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \square$$

Proposition 34. *The proposition has two parts.*

(i) If $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ where $A_j \in \mathcal{M}$ for each j. Then

$$A = \bigcap_n A_n$$

is measurable, and

$$\mu(A) = \lim_{n \to \infty} \mu(A_n)$$

(ii) The same conclusion for $A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ with a slight difference. We get

$$\mu\left(\bigcup_{n=1}^{\infty} A_n \right) = \lim_{n \to \infty} \mu(A_n)$$

Chapter 4

Measurable Functions

We begin with a definition.

Definition 21. *Consider $(X, \Sigma_1), (Y, \Sigma_2)$ and we have $f : X \to Y$. Then we say that f is* ***Measurable With Respect To*** *Σ_1, Σ_2 if and only if for every $A \in \Sigma_2$, we have $f^{-1}(A) \in \Sigma_1$.*

Suppose that $Y = \mathbb{R}$ and that Σ_2 is the Lebesgue measurable sets. Assume that X has a measure μ that is σ-additive and say that Σ_1 is the algebra of μ measurable sets with $\mu(X) < \infty$ with X as the unit of Σ_1. We then say that in this special case, $f : X \to \mathbb{R}$ is μ-measurable if and only if for any $A \subseteq \mathbb{R}$, we have $f^{-1}(A) \in \Sigma_1$.

Remark. It is true that the Borel sets in $\mathbb{R}$ can be generated (by unions, complements, etc) by $(-\infty, C)$ so that

$$f^{-1}((-\infty, C)) = \{x : f(x) < C\}$$

Section 4.1

The Measurability Of f

Theorem 35. *The function f is μ measurable if and only if for any $C \in \mathbb{R}$, we have*

$$\{x \in X : f(x) < C\} \in \Sigma_1$$

Theorem 36. *Let $f_n(x) \to f(x)$ for every $x \in X$. Now, suppose that f_n is μ measurable for every n. Then f is also μ measurable.*

Proof.
We know that $\{x \in X : f(x) < C\}$ is μ measurable for any $C \in \mathbb{R}$. Now, we claim that

$$(*) \qquad \{x \in X : f(x) < C\} = \bigcup_k \bigcup_n \left(\bigcap_{m>n} \left\{ x \in X : f_m(x) < C - \frac{1}{k} \right\} \right)$$

It is not hard to see that this claim implies the theorem. To prove the claim, we note that if $f(x) < C$, then there exists k such that

$$f(x) < C - \frac{2}{k}$$

Now,

$$f(x) = \lim_{m\to\infty} f_m(x)$$

and so

$$f_m(x) < C - \frac{1}{k}$$

where m is large enough, say $m > n$. This would now mean that x is contained in the RHS set of $(*)$. Thus, $LHS \subseteq RHS$.
Now, suppose that $x \in RHS$, so there exists k and n such that

$$f_m(x) < C - \frac{1}{k}$$

for all $m > n$. Then we have

$$f(x) = \lim_{m\to\infty} f_m(x) \leq C - \frac{1}{k} < C$$

and so $RHS \subseteq LHS$ and so we have equality. $\square$

Definition 22. *We say that $f : X \to \mathbb{R}$ is* ***Simple*** *if f is μ measurable and the image of X under f ($\{f(x) : x \in X\}$) is finite or countable.*

Let $c_1, \ldots, c_n, \ldots$ be values taken by f and let $A_n = \{x : f(x) = c_n\}$. First, we have that $A_n \subseteq X$ is measurable. Now, all A_j are disjoint, so

$$f = \sum_{n=1}^{\infty} c_n \chi_{A_n}$$

Also, the A_n's form a partition of X. These are essentially step functions. A basic example could be attained by letting only finitely many c_j's be non-zero, so that $c_1, \ldots, c_{n-1} \neq 0, c_n = 0$ and $A_n = X \backslash (A_1 \cup \cdots \cup A_{n-1})$.

Aside. We have that

$$\int f(x) d\mu(x) = \sum_{n=1}^{\infty} c_n \mu(A_n)$$

only if the sum converges.

Theorem 37. *The function $f(x)$ is μ measurable if and only if f is a limit of a uniformly convergent sequence of measurable simple functions.*

Proof.
($\Longleftarrow$): This direction is given by Theorem 36.

($\Longrightarrow$): Given $n \in \mathbb{N}$, there exists $m \in \mathbb{Z}$ such that

$$\frac{m}{n} \leq f(x) < \frac{m+1}{n}$$

Now, let

$$f_n(x) = \frac{m}{n}$$

We now claim that f_n is measurable. Let $A_{m,n} = \left\{x : f_n(x) = \frac{m}{n}\right\}$ and A is measurable and

$$f_n = \sum_{m \in \mathbb{Z}} \frac{m}{n} \chi_{A_{m,n}}$$

Now, for any x, we have

$$|f(x) - f_n(x)| < \frac{1}{n}$$

so we get that $f_n \to f(x)$ as needed. $\square$

THEOREM 38. *If f, g is measurable, then so is $f+g$.*

Proof. We use **Theorem 37**. Let $f_n \to f$ and $g_n \to g$ be simple (measurable) functions.
Now we claim that $f_n + g_n$ is simple and measurable. This claim implies the theorem. To prove this, we write

$$f_n = \sum_k c_k \chi_{A_k} \qquad g_n = \sum_l d_l \chi_{B_l}$$

Now, $f_n + g_n$ takes values in $\{c_k + d_l : k, l \in \mathbb{Z}\} = V$. Now, let $z \in V$, then

$$\{x : f_n(x) - g_n(x) = z\} = \bigcup_{(k,l): c_k + d_l = z} (A_k \cap B_l)$$

is measurable for each $k, l \in \mathbb{Z}$ which yields our claim and proves the theorem. □

THEOREM 39 (Compositions). *A Borel measurable function of a μ- measurable function is μ-measurable.*

Proof. Let $\psi : X \to \mathbb{R}$ be μ-measurable and let $\phi : \mathbb{R} \to Y$ be Borel measurable. Then if $A \in \Sigma_Y$, then $\mathbb{R} \supseteq \phi^{-1}(A)$ is Borel measurable. Now, we also have that $\psi^{-1}(\phi^{-1}(A)) \in \Sigma_X$ where Σ_X is the set of μ-measurable functions. Thus, $[\phi \circ \psi]^{-1}(A) \in \Sigma_X$ as needed. □

DEFINITION 23. *We say that $f \sim g$ if and only if*

$$\mu\{x : f(x) \neq g(x)\} = 0$$

and it is easy to see that this is an equivalence relation.

DEFINITION 24. *A property P is satisfied* ***a.e. - (Almost Everwhere)*** *if and only if P holds except on some set of measure zero.*

PROPOSITION 40. *If f, g are continuous on $[a, b]$ such that $f = g$ a.e., then $f(x) = g(x)$ for any $x \in [a, b]$.*

Proof. Suppose that $f(x_0) \neq g(x_0)$ for some point x_0, then there exists an interval I containing x_0 such that $f(y) \neq g(y)$ for each $y \in I$. But then we have

$$\mu\{y : f(y) \neq g(y)\} > 0$$

which contradicts the assumption of equality a.e. and yields our claim. □

PROPOSITION 41. *If $f \sim g$ and g is measurable, then f must also be measurable.*

Proof. We have that

$$\mu\left(f^{-1}((-\infty, c)) \triangle g^{-1}((-\infty, c))\right) = 0$$

but then we have that $f^{-1}((-\infty, c))$ is also μ measurable as needed. □

THEOREM 42 (Lusin). *The function f is measurable on $[a,b]$ if and only if for every $\epsilon > 0$, there exists a continuous function ϕ defined on $[a,b]$ such that*

$$\mu\{x : f(x) \neq \phi(x)\} < \epsilon$$

Proof. To be proven later. ☐

Section 4.2

Sequences Of Measurable Functions

DEFINITION 25. *We say that $f_n(x) \to f(x)$ a.e. if and only if*

$$\mu\{x : f_n(x) \nrightarrow f(x)\} = 0$$

THEOREM 43. *If $f_n(x) \to f(x)$ a.e., and f_n is measurable, then $f(x)$ is measurable.*

THEOREM 44 (Egorov). *Let $f_n(x)$ be a sequence of measurable functions, and let $f_n(x) \to f(x)$ a.e. on X. Then for every $\delta > 0$, there exists X_δ such that*

(i) $\mu(x_\delta) > \mu(X) - \delta$.

(ii) $f_n(x) \to f(x)$ *uniformly on* X_δ.

Proof. Suppose that $f(x)$ is measurable and let

$$\begin{aligned} X_n^m &= \bigcap_{k \geq n} \left\{ x : |f_k(x) - f(x)| < \frac{1}{m} \right\} \\ &= \left\{ x : |f_k(x) - f(x)| < \frac{1}{m}, \forall\, k \geq n \right\} \end{aligned}$$

Now, let

$$X^m = \bigcup_n X_n^m$$

and remark that

$$X_1^m \subseteq X_2^m \subseteq \cdots \subseteq X_n^m \cdots$$

and m is fixed. Now, μ is σ-additive, so we get

$$\mu(X^m) = \lim_{n \to \infty} \mu(X_n^m)$$

and then for any $\delta > 0$, there exists an index $n(m)$ such that

$$\mu(X^m \backslash X_{n(m)}^m) < \frac{\delta}{2^m}$$

Now, we claim that if we let

$$X_\delta = \bigcap_m X_{n(m)}^m$$

then this set will have the required property.
To prove this, we note that we get uniform convergence on X_δ since for all $i \geq n(m)$, we have

$$(*) \qquad |f_i(x) - f(x)| < \frac{1}{m}$$

and since

$$x \in \bigcap_m X^m_{n(m)}$$

the inequality $(*)$ eventually holds for all m.
As for estimating $\mu(X \backslash X_\delta)$, we first note that

$$\mu(X \backslash X^m) = 0 \quad \forall\, m$$

since if $x_0 \in X \backslash X^m$ then for infinitely many indices $\{i_0\}$ we have that $f_{i_0}(x_0) \nrightarrow f(x_0)$ and thus, by assumption, we have that

$$\mu\{x : f_i(x) \nrightarrow f(x)\} = 0$$

Now, we have that

$$\mu(X \backslash X^m_{n(m)}) = \mu(X^m \backslash X^m_{n(m)}) < \frac{\delta}{2^m}$$

and so

$$\begin{aligned}\mu(X \backslash X_\delta) &= \mu\left(X \backslash \left(\bigcap_m X^m_{n(m)}\right)\right) \\ &\leq \mu\left(\bigcup_m (X \backslash X^m_{n(m)})\right)\end{aligned}$$

and thus

$$\sum_m \mu(X \backslash X^m_{n(m)}) < \sum_{m=1}^{\infty} \frac{\delta}{2^m} = \delta$$

which yields our claim. □

Definition 26. *We say that a sequence $f_n(x)$ converges to $f(x)$ **In Measure** if and only if for every $\delta > 0$, we have*

$$\lim_{n \to \infty} \mu\{x : |f_n(x) - f(x)| \geq \delta\} = 0$$

Theorem 45. *Suppose that $f_n(x) \to f(x)$ a.e., then $f_n(x) \to f(x)$ in measure.*

Proof. Let

$$A = \{x : f_n(x) \nrightarrow f(x)\}$$

then $\mu(A) = 0$ by assumption. Now, let

$$X_k(\delta) = \{x : |f_k(x) - f(x)| \geq \delta\}$$

and let

$$R_n(\delta) = \bigcup_{k=n}^{\infty} X_k(\delta)$$

and let

$$M = \bigcup_{n=1}^{\infty} R_n(\delta) = \{x : |f_k(x) - f(x)| \geq \delta \text{ for infinitely many } k\}$$

So, $R_1(\delta) \supseteq R_2(\delta) \supseteq \cdots \supseteq R_n(\delta) \supseteq \cdots$, and thus

$$\mu(R_n(\delta)) \to \mu(M)$$

as $n \to \infty$. We now claim that $M \subseteq A$ which holds from the definition of M. This implies that $\mu(M) = 0$ and thus

$$\mu(R_n(\delta)) \to 0$$

as $n \to \infty$ and thus

$$\mu(X \backslash R_n(\delta) \to \mu(X)$$

as needed. $\square$

Remark 1. We note that the converse of the theorem is false. Convergence a.e. is more powerful than convergence in measure. An example of this can by constructed. Let

$$f_i^k(x) = \chi_{[\frac{i}{k}, \frac{i+1}{k}]}(x)$$

We claim that $f_i^k \to 0$ in measure as $k \to \infty$, but

$$\{x : f_i^k(x) \to 0\} = \emptyset$$

which gives us convergence in measure but not convergence a.e.

Remark 2. If $f_n \to f$ in measure and $f \sim g$, thne $f_n \to g$ in measure as $n \to \infty$.

Theorem 46. *Suppose that $f_n(x) \to f(x)$ in measure. Then, there exists a natural subsequence $\{n_k\}$ such that*

$$\{f_{n_k}(x)\} \to f(x) \quad a.e.$$

Proof. Let $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > \cdots \geq 0$ so that $\epsilon_n \to 0$ as $n \to \infty$. Now, let $\eta_j > 0$ such that

$$\sum_{j=1}^{\infty} \eta_j < \infty$$

and now let n_1 be such that

$$\mu\{x : |f_{n_1}(x) - f(x)| \geq \epsilon_1\} < \eta_1$$

and let n_2 be such that

$$\mu\{x : |f_{n_2}(x) - f(x)| \geq \epsilon_1\} < \eta_2$$

and so on. Now, we claim that $f_{n_k}(x) \to f(x)$ a.e.
Let

$$R_i = \bigcup_{k=i}^{\infty} \{x : f_{n_k}(x) - f(x)| \geq \epsilon_k\}$$

and let

$$Q = \bigcap_{i=1}^{\infty} R_i$$

and we have that $R_1 \supseteq \cdots \supseteq R_n \supseteq \cdots$ and then

$$\mu(R_k) \to \mu(Q)$$

as $k \to \infty$ so that

$$\begin{aligned}\mu(R_i) &\leq \sum_{k=i}^{\infty} \mu\{x : f_{n_k}(x) - f(x)| \geq \epsilon_k\} \\ &\leq \sum_{k=i}^{\infty} \eta_k \to 0\end{aligned}$$

as $i \to \infty$. Thus, $\mu(Q) = 0$ and we claim that $f_{n_k}(x) \to f(x)$ for $x \notin Q$. To show this, let $x_0 \in X \backslash Q$. Then there exists i_0 such that $x_0 \notin R_{i_0}$. It follows that for any $k \geq i_0$, we have that

$$x_0 \notin \{y : f_{n_k}(y) - f(y)| \geq \epsilon_k\}$$

which implies that

$$|f_{n_k}(x_0) - f(x_0)| < \epsilon_k$$

for every $k \geq i_0$ and since $\epsilon_k \to 0$, we are done. $\square$

Chapter 5

The Lebesgue Integral

Section 5.1

The Lebesgue Integral For Simple Functions

Suppose that $f(x)$ is simple with values y_n, $n \geq 1$ such that $y_i \neq y_j$ for $i \neq j$. We then define the **Integral** of f over the set A as

$$(*) \qquad \int_A f(x)d\mu = \sum_n y_n \mu(\{x : x \in A, f(x) - y_n\})$$

Definition 27. *We say that a simple function $f(x)$ is μ-**integrable** over A if the series $(*)$ is absolutely convergent and we call the series $(*)$ the **Integral** of f over A.*

Lemma 47. *Suppose that*

$$A = \bigcup_k B_k, \quad B_i \cap B_j = \emptyset, \;\; i \neq j$$

and that $f(x)$ assumes only one value on each set B_k. Then

$$\int_A f(x)d\mu = \sum_k c_k \mu(B_k)$$

where f is integrable over A if and only if the series above is absolutely convergent.

Proof. It's clear that

$$A_n = \{x : x \in A, f(x) = y_n\}$$

is the union of all sets B_k for which $c_k = y_n$. Thus,

$$\sum_n y_n \mu(A_n) = \sum_n y_n \sum_{c_k = y_n} \mu(B_k) = \sum_k c_k \mu(B_k)$$

Now, since measure is non-negative, we have that

$$\sum_n |y_n| \mu(A_n) = \sum_n |y_n| \sum_{c_k = y_n} \mu(B_k) = \sum_k |c_k| \mu(B_k)$$

so that the series

$$\sum_n y_n \mu(A_n), \quad \sum_k c_k \mu(B_k)$$

are either both absolutely convergent or both divergent. □

We will now list some properties of the Lebesgue integral of f for f simple.

(i) We have

$$\int_A f(x)d\mu + \int_A g(x)d\mu = \int_A (f(x)+g(x))d\mu$$

where the existence of the integrals on the LHS implies the existence of the integral on the RHS.

(ii) We also have for every constant k that

$$k\int_A f(x)d\mu = \int_A (kf(x))d\mu$$

where the existence of the integral on the LHS implies the existence of the integral on the RHS.

(iii) A simple function $f(x)$ bounded on a set A is integrable over A and

$$\left|\int_A f(x)d\mu\right| \leq M\mu(A)$$

where $|f(x)| \leq M$ on A.

Section 5.2
General Definition and Properties of Lebesgue Integral

Definition 28. *We say that a general function $f(x)$ is* ***Integrable*** *over a set A if there exists a sequence of simple functions $f_n(x)$ all integrable over A and uniformly convergent to f. We then say that the limit of the integrals of the f_n*

$$\lim_{n\to\infty}\int_A f_n(x)d\mu = \int_A f(x)d\mu$$

is the ***Integral*** *of f.*

Theorem 48. *The above definition holds if:*

(i) The limit for an arbitrary uniformly convergent sequence of simple functions integrable over A exists.

(ii) The limit, for fixed $f(x)$ is independent of the choice of sequence $f_n(x)$.

(iii) For simple functions, this definition of integrability is equivalent to that of the previous section.

Proof. (i) Because of the three properties of integrals of simple functions discussed in the previous section, we have that

$$\left|\int_A f_n(x)d\mu - \int_A f_m(x)d\mu\right| \leq \mu(A)\sup\{|f_n(x)-f_m(x) : x \in A\}$$

which yields the first result.

(ii) We consider two sequences $\{f_n(x)\}$ and $\{f_n^*(x)\}$ and use the fact that

$$\begin{aligned}\left|\int_A f_n(x)d\mu - \int_A f_n^*(x)d\mu\right| &\leq \mu(A)\sup\{|f_n(x)-f(x) : x \in A\}\\ &\quad + \sup\{|f_n^*(x)-f(x)| : x \in A\}\end{aligned}$$

which yields the second result.

(iii) Now, it's sufficient to consider the sequence $f_n(x) = f(x)$ for all n, and we're done.

$\square$

We now generalize the properties listed in the previous section into full-blown theorems about general functions.

THEOREM 49. *We have*

$$\int_A 1 \cdot d\mu = \mu(A)$$

THEOREM 50. *For every constant k, we have*

$$k \int_A f(x) d\mu = \int_A (kf(x)) d\mu$$

where the existence of the integral on the LHS implies the existence of the integral on the RHS.

THEOREM 51. *We have that*

$$\int_A f(x) d\mu + \int_A g(x) d\mu = \int_A (f(x) + g(x)) d\mu$$

where the existence of the integrals on the LHS implies the existence of the integral on the RHS.

THEOREM 52. *A function $f(x)$ bounded on a set A is integrable over A.*

Proof. We prove this by simply passing to the limit of property (*iii*) of Theorem 48. $\square$

THEOREM 53. *If $f(x) \geq 0$, then*

$$\int_A f(x) d\mu \geq 0$$

assuming that the integral exists.

Proof. For simple functions, the theorem follows immediately from the definition of the integral. In general though, we base the proof on the fact that we can approximate a nonnegative function by simple functions arbitrarily well. $\square$

COROLLARY 54. *If $f(x) \geq g(x)$, then*

$$\int_A f(x) d\mu \geq \int_A g(x) d\mu$$

Corollary 55. *If $m \leq f(x) \leq M$ on A, then*

$$m\mu(A) \leq \int_A f(x)d\mu \leq M\mu(A)$$

Theorem 56. *Suppose that*

$$A = \bigcup_n A_n, \quad A_i \cap A_j = \emptyset, \;\; i \neq j$$

then we have that

$$\int_A f(x)d\mu = \sum_n \int_{A_n} f(x)d\mu$$

where the existence of the integral on the LHS implies the existence of the integrals and absolute convergence of the series on the RHS.

Corollary 57. *If $f(x)$ is integrable over A, then $f(x)$ is integrable over an arbitrary $A' \subseteq A$.*

Theorem 58. *If a function $\phi(x)$ is integrable over A and $|f(x)| \leq \phi(x)$, then $f(x)$ is also integrable over A.*

Proof. If both f and ϕ are simple functions, then A can be written as the union of a countable number of sets on each of which f and ϕ are constant so that

$$f(x) = a_n \quad \phi(x) = \alpha_n$$

where $|a_n| \leq \alpha_n$. The integrability of ϕ implies that

$$\sum_n |a_n|\mu(A_n) \leq \sum_n \alpha_n \mu(A_n) = \int_A \phi(x)d\mu$$

Thus, we have that f is also integrable, and

$$\begin{aligned} \left|\int_A f(x)d\mu\right| &= \left|\sum_n a_n\mu(A_n)\right| \\ &\leq \sum_n |a_n|\mu(A_n) \\ &= \int_A |f(x)|d\mu \\ \leq \int_A \phi(x)d\mu \end{aligned}$$

and the theorem is proved by passing to the limit. ∎

Theorem 59. *The integrals*

$$I_1 = \int_A f(x)d\mu, \quad I_2 \int_A |f(x)|d\mu$$

either both exist or both do not.

Theorem 60 (Chebychev Inequality). *If $\phi(x) \geq 0$ on A, then*

$$\mu(\{x : x \in A, \phi(x) \geq c\}) \leq \frac{1}{c} \int_A \phi(x) d\mu$$

Proof. First, set

$$A' = \{x : x \in A, \phi(x) \geq c\}$$

and we have

$$\int_A \phi(x) d\mu = \int_{A'} \phi(x) d\mu + \int_{A \setminus A'} \phi(x) d\mu \geq \int_{A'} \phi(x) d\mu \geq c\mu(A')$$

as needed. □

Corollary 61. *If*

$$\int_A |f(x)| d\mu = 0$$

then $f(x) = 0$ a.e.

Proof. By the Chebychev Inequality, we get

$$\mu\left\{x : x \in A, |f(x)| \geq \frac{1}{n}\right\} \leq n \int_A |f(x)| d\mu = 0$$

for all n. Therefore,

$$\mu\{x : x \in A, f(x) \neq 0\} \leq \sum_{n=1}^{\infty} \mu\left\{x : x \in A, |f(x)| \geq \frac{1}{n}\right\} = 0$$

as needed. □

Theorem 62. *Suppose that $f_n \to f$ on A and that*

$$|f_n(x)| \leq \phi(x)$$

for every n where $\phi(x)$ is integrable over A. Then f is integrable over A and

$$\int_A f_n(x) d\mu \to \int_A f(x) d\mu$$

Proof. It is easy to see that $|f(x)| \leq \phi(x)$. Now, let $A = \{x : k - 1 \leq \phi(x) < k\}$ and let

$$B_m = \bigcup_{k \geq m+1} A_k = \{x : \phi(x) \geq m\}$$

By Theorem 56, we know that

$$\int_A \phi(x) d\mu = \sum_k \int_{A_k} \phi(x) d\mu$$

and that the series above converges absolutely. Thus, we get that

$$\int_{B_m} \phi(x) d\mu = \sum_{k \geq m+1} \int_{A_k} \phi(x) d\mu$$

and the convergence of the first sum gives that there exists an m such that

$$\int_{B_m} \phi(x) d\mu < \frac{\epsilon}{5}$$

and $\phi(x) < m$ is true on $A \backslash B_m$. By Egorov's Theorem, we get that $A \backslash B_m$ can be written as $C \cup D$ where $\mu(D) < \frac{\epsilon}{5m}$ and the sequence f_n converges uniformly to f on C. Now, pick N so large that

$$|f_n(x) - f(x)| < \frac{\epsilon}{5\mu(C)}$$

for each $n > N$ and $x \in C$. This gives that

$$\begin{aligned}\int_A (f_n(x) - f(x)) d\mu = \int_{B_m} f_n(x) d\mu - \int_{B_m} f(x) d\mu + \int_D f_n(x) d\mu - \int_D f(x) d\mu \\ + \int_C (f_n(x) - f(x)) d\mu < 5\frac{\epsilon}{5} = \epsilon\end{aligned}$$

which yields our claim. □

Remark. The values of f on a set of measure zero don't effect the integral but we'll still want to assume that f_n converges a.e. in the above theorem.

Corollary 63. *Suppose that $|f_n(x)| \le M$ and that $f_n \to f$. Then*

$$\int_A f_n(x) d\mu \to \int_A f(x) d\mu$$

Theorem 64. *Suppose that*

$$f_1(x) \le f_2(x) \le \cdots \le f_n(x) \le \cdots$$

on a set A where the functions $f_n(x)$ are integrable and their integrals are bounded from above. That is,

$$\int_A f_n(x) d\mu \le K$$

Then we have that the limit

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists a.e. on A. Also, we have that f is integrable on A and

$$\int_A f_n(x) d\mu \to \int_A f(x) d\mu$$

This theorem also holds for a decreasing sequence of functions all of whom are bounded from below.

Proof. WLOG, assume that $f(x) \ge 0$, since we can write $g_n(x) = f_n(x) - f_1(x)$. Let

$$\Omega = \{x : x \in A, f_n(x) \to \infty\}$$

Now, if we let

$$\Omega_n^{(r)} = \{x : x \in A, f_n(x) > r\}$$

then it is clear to see that

$$\Omega = \bigcap_r \bigcup_n \Omega_n^{(r)}$$

By the Chebychev Inequality, we get that

$$\mu(\Omega_n^{(r)}) \leq \frac{K}{r}$$

Now, since $\Omega_1^{(1)} \subseteq \cdots \subseteq \Omega_n^{(r)} \subseteq \cdots$, it follows that

$$\mu\left(\bigcup_n \Omega_n^{(r)}\right) \leq \frac{K}{r}$$

for every r, and so

$$\mu(\Omega) \leq \frac{K}{r}$$

Now, since r is arbitrary, we get that $\mu(\Omega) = 0$. This shows that the monotone sequence f_n has a finite limit f a.e. on A. Now, let $\phi(x) = r$ for each x so that

$$r - 1 \leq f(x) < r$$

It remains to show that $\phi(x)$ is integrable on A, since the rest follows from Theorem 62. Let A_r be the set of all points $x \in A$ such that $\phi(x) = r$ and define

$$B_s = \bigcup_{r=1}^{s} A_r$$

Now, since f_n and f are bounded on B_s, and $\phi(x) \leq f(x) + 1$, we get that

$$\begin{aligned}\int_{B_s} \phi(x) d\mu &\leq \int_{B_s} f(x) d\mu + \mu(A) \\ &= \lim_{n\to\infty} \int_{B_s} f_n(x) d\mu + \mu(A) \leq K + \mu(A)\end{aligned}$$

but

$$\int_{B_s} \phi(x) d\mu = \sum_{r=1}^{s} r\mu(A_r)$$

and since the partial sums above are bounded, the series

$$\sum_{r=1}^{\infty} r\mu(A_r) = \int_A \phi(x) d\mu$$

must then converge. Thus, $\phi(x)$ is integrable over A. $\square$

Corollary 65. *Suppose that $\psi_n(x) \geq 0$ and that*

$$\sum_{n=1}^{\infty} \int_A \psi_n(x) d\mu = M < \infty$$

then the series $\sum_{n=1}^{\infty} \psi_n(x)$ converges a.e. on A and

$$\int_A \left(\sum_{n=1}^{\infty} \psi_n(x)\right) d\mu = \sum_{n=1}^{\infty} \int_A \psi_n(x) d\mu$$

Proof. Let

$$F_n = \sum_{k=1}^{n} \psi_k(x)$$

Then $F_n(x)$ is an increasing sequence. Thus

$$\int_X F_n(x) d\mu = \sum_{k=1}^{n} \int_X \psi_k d\mu \leq M$$

Then Theorem 64 implies that $F_n(x) \to \phi(x) < \infty$ a.e. where

$$\phi(x) = \sum_{n=1}^{\infty} \psi_n(x)$$

and so

$$\int \phi d\mu = \lim_{n\to\infty} \int_X F_n d\mu$$

as needed. □

EXAMPLE 4. *Let $f : X \to \mathbb{R}$ and let μ be a σ-additive measure on X. We claim that f is measurable if and only if for each $r \in \mathbb{Q}$, $f^{-1}((-\infty, r))$ is measurable. To this end, we need to show that for each $t \in \mathbb{R}$, we have that the set*

$$f^{-1}((-\infty, -t))$$

is measurable. Now $f(x) \geq t$ if and only if $f(x) \geq r$ for each $r \in \mathbb{Q}, r \leq t$. Then

$$X \backslash [f^{-1}((-\infty, -t))] = \bigcap_{\substack{r \leq t \\ r \in \mathbb{Q}}} X \backslash [f^{-1}((-\infty, r))]$$

where each set in the intersection is measurable. We now have a countable intersection of measurable sets which is also measurable. Thus, the preimage is measurable.

THEOREM 66 (Fatou). *Suppose that $f_n : X \to \mathbb{R}$ is measurable and that $f_n(x) \to f$ a.e. on X and*

$$\int_X f_n(x) d\mu \leq K$$

Then $f(x)$ is integrable on X and

$$\int_X f(x) d\mu \leq K$$

We note that an alternate way to state this is that

$$\int_X (\liminf_{n\to\infty} f_n(x)) d\mu \leq \liminf_{n\to\infty} \int_X f_n(x) d\mu$$

Proof. Let $\phi_n(x) = \inf_{k \geq n}\{f_k(x)\}$ and note that

$$\{x : \phi_n(x) < C\} = \bigcup_{k \geq n} \{f_k(x) < C\}$$

where the LHS set is measurable, and each set in the RHS intersection is also measurable. Thus, ϕ_n is measurable. Now, $0 \leq \phi_n \leq f_n$. Thus, we have

$$\int_X \phi_n d\mu \leq \int_X f_n d\mu \leq K$$

and also $\phi_1(x) \leq \cdots \leq \phi_n(x) \leq \cdots$. We remark that if we don't assume that

$$f_n(x) \to f(x)$$

Then

$$\lim_{n\to\infty} \phi_n(x) = \liminf_{n\to\infty} f_n(x)$$

and so

$$\lim_{n\to\infty} \phi_n(x) = f(x)$$

and now we apply Theorem 64 to get the desired result. Now, to prove the alternate statement of the theorem, we remark that the smallest number K that works is exactly

$$\liminf_{n\to\infty} \int_X f_n(x) d\mu$$

as needed. $\square$

THEOREM 67. *Let*

$$X = \bigcup_{n\geq 1} A_n, \quad (A_i \cap A_j = \emptyset, i \neq j)$$

and suppose that

$$\sum_n \int_{A_n} |f(x)| d\mu < \infty$$

then f is integrable on X and

$$\int_X f(x) d\mu = \sum_n \int_{A_n} f(x) d\mu$$

Sketch of Proof. First, we must prove this for simple functions $f(x) = c_i$ for $x \in B_i$. Then, let

$$A_{n,i} = A_n \cap B_i$$

so that

$$\int_{A_n} |f(x)| d\mu = \sum_i |c_i| \cdot \mu(A_{n,i})$$

We note that $\cup_i B_i = X$, so that

$$\begin{aligned}\sum_n \sum_i |c_i| \cdot \mu(A_{n,i}) &= \sum_i \sum_n |c_i| \cdot \mu(A_n \cap B_i) \\ &= \sum_i |c_i| \cdot \mu(B_i) < \infty\end{aligned}$$

where the second equality comes from the assumption of the theorem. Thus,

$$\sum_n \int_{A_n} |f(x)| d\mu < \infty$$

which gives

$$\int_X f(x) d\mu = \sum_i c_i \mu(B_i)$$

Now, for a general function g, there exists a simple function f such that $|g - f| < \epsilon$. Then

$$\int_{A_n} |f(x)| d\mu \leq \int_{A_n} |g(x)| d\mu + \epsilon \cdot \mu(A_n)$$

so that

$$\sum_n \int_{A_n} |f(x)| d\mu \leq \sum_n \int_{A_n} |g(x)| d\mu + \epsilon \sum_n \mu(A_n)$$

where the first RHS term converges by assumption and the second converges as well since $\sum_n \mu(A_n) = \mu(X)$ must be finite. Thus, if f is integrable, then so is g. □

SECTION 5.3

COMPARING RIEMANN & LEBESGUE INTEGRALS

The reason why we need the Lebesgue integral is because the Riemann integral falls short in many applications (particularly in probability theory), so this tool is useful for us.

THEOREM 68. *If the Riemann integral*

$$J = (R) \int_a^b f(x) dx$$

exists, then $f(x)$ is Lebesgue integrable on $[a, b]$ and

$$\int_{[a,b]} f(x) d\mu = J$$

Proof. Consider the partition of $[a, b]$ into 2^n subintervals by the points

$$x_k = a + \frac{k}{2^n}(b - a)$$

and the Darboux sums

$$\overline{S}_n = \frac{b-a}{2^n} \sum_{k=1}^{2^n} M_{nk}$$

$$\underline{S}_n = \frac{b-a}{2^n} \sum_{k=1}^{2^n} m_{nk}$$

where M_{nk} is the supremum of $f(x)$ on the interval $[x_{k-1}, x_k]$ and m_{nk} is the infimum of $f(x)$ on the same interval. Now, by definition, the Riemann integral is

$$J = \lim_{n \to \infty} \overline{S}_n = \lim_{n \to \infty} \underline{S}_n$$

Now, we define

$$\overline{f}_n(x) = M_{nk} \qquad x_{k-1} \leq x \leq x_k$$

$$\underline{f}_n(x) = m_{nk} \qquad x_{k-1} \leq x \leq x_k$$

The functions $\overline{f}_n$ and $\underline{f}_n$ can be extended to the point $x = b$ arbitrarily. Clearly, we have

$$\int_{[a,b]} \overline{f}_n(x) d\mu = \overline{S}_n$$

$$\int_{[a,b]} \underline{f}_n(x)d\mu = \underline{S}_n$$

Now, since the sequence $\overline{f}_n$ is a nonincreasing sequence and the sequence $\underline{f}_n$ is a nondecreasing sequence, we get that

$$\overline{f}_n(x) \to \overline{f}(x) \geq f(x) \;\; a.e.$$
$$\underline{f}_n(x) \to \underline{f}(x) \leq f(x) \;\; a.e.$$

By Theorem 64, we get that

$$\int_{[a,b]} \overline{f}(x)d\mu = \lim_{n\to\infty} \overline{S}_n = J = \lim_{n\to\infty} \underline{S}_n = \int_{[a,b]} \underline{f}(x)d\mu$$

and thus

$$\int_{[a,b]} |\overline{f}(x) - \underline{f}(x)|d\mu = \int_{[a,b]} (\overline{f}(x) - \underline{f}(x))d\mu = 0$$

and thus

$$\overline{f}(x) = \underline{f}(x) = f(x) \;\; a.e.$$

Finally, we arrive at

$$\int_{[a,b]} f(x)d\mu = \int_{[a,b]} \overline{f}(x)d\mu = J$$

as needed. □

Remark. We note that the function f is continuous at x if and only if

$$\overline{f}(x) = \underline{f}(x)$$

Section 5.4

Midterm Review

The following chosen homework problems are given as theorems.

Theorem 69 (Inclusion-Exclusion Principle). *Suppose that $\mathcal{R}$ is a ring of measurable subsets of X with measure μ, then*

(i) If $A, B \in \mathcal{R}$, then

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

(ii) If $A, B, C \in \mathcal{R}$, then

$$\mu(A \cup B \cup C) = \mu(A) + \mu(B) + \mu(C) - \mu(A \cap B) - \mu(A \cap C) - \mu(B \cap C) + \mu(A \cap B \cap C)$$

(iii) If $A_1, \dots, A_n \in \mathcal{R}$, then

$$\begin{aligned}\mu\left(\bigcup_{k=1}^n A_k\right) = \sum_{k=1}^n \mu(A_k) - \sum_{1\leq j<k\leq n} \mu(A_j \cap A_k) + \sum_{1\leq i<j<k\leq n} \mu(A_i \cap A_j \cap A_k) \\ - \cdots + (-1)^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n} \mu\left(\bigcap_{j=1}^k A_{i_j}\right) \\ + \cdots + (-1)^{n-1}\mu\left(\bigcap_{k=1}^n A_k\right)\end{aligned}$$

Example 5. *We wish to compute the probability that two numbers are relatively prime. Thus, we want*

$$\mathbb{P}\left[GCD(m,n)=0\right]=\mathbb{P}\left[m \text{ and } n \text{ are not both divisible by some prime } p\right]$$

Thus, the event where $2|m$ and $2|n$ is independent of $3|m, 3|n$. Thus, E_1, E_2 are independent, so

$$\mathbb{P}\left[E_1\cap E_2\right]=\mathbb{P}\left[E_1\right]\mathbb{P}\left[E_2\right]$$

Now, it is a fact that if $\{E_1,\dots,E_n\}$ are independent, then so are $\{E_1^C,\dots,E_n^C\}$.

We claim that the events $p|m, p|n$ are independent for different primes p, and from here, we wish to find the probability of the union of the complements of these events. So, we have that

$$\mathbb{P}\left[\{p|m,p|n\}^C\right]=\left(1-\frac{1}{p^2}\right)$$

and now

$$\mathbb{P}\left[m \text{ and } n \text{ are relatively prime}\right]=\prod_{p \text{ prime}}\left(1-\frac{1}{p^2}\right)$$

so we write

$$\prod_p\frac{1}{\left(1-\frac{1}{p^2}\right)}=\prod_p\left(1+\frac{1}{p^2}+\cdots+\frac{1}{p^{2n}}+\cdots\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)$$

and since

$$\frac{1}{\mathbb{P}\left[\ m,n \text{ are relatively prime}\right]}=\zeta(2)=\frac{\pi^2}{6}$$

we have

$$\mathbb{P}\left[GCD(m,n)=1\right]=\frac{1}{\zeta(2)}=\frac{6}{\pi^2}$$

Now, let's try to find $\mathbb{P}\left[GCD(n_1,n_2,n_3)=1\right]$. We write

$$\begin{aligned}\frac{1}{\mathbb{P}\left[GCD(n_1,n_2,n_3)=1\right]}&=\frac{1}{\prod_p\left(1-\frac{1}{p^3}\right)}=\prod_p\left(1\frac{1}{p^3}+\cdots+\frac{1}{p^{3n}}+\cdots\right)\\&=\sum_{n=1}^{n}\frac{1}{n^3}=\zeta(3)\end{aligned}$$

so that

$$\mathbb{P}\left[GCD(n_1,n_2,n_3)=1\right]=\frac{1}{\zeta(3)}$$

From here, it's fairly easy to see that

$$\mathbb{P}\left[GCD(n_1,\dots,n_k)=1\right]=\frac{1}{\zeta(k)}$$

Proposition 70. *Consider a function $f:X\to[0,\infty)$, and suppose that*

$$\int_X f(x)d\mu=C<\infty$$

Then we have

$$\lim_{n\to\infty}\int_X n\ln\left(1+\left(\frac{f}{n}\right)^{\alpha}\right)d\mu=\begin{cases}C, & \alpha=1\\0, & 1<\alpha<\infty\\\infty, & 0<\alpha<1\end{cases}$$

Proof. Let

$$f_n(x) = n \ln\left(1 + \left(\frac{f}{n}\right)^{\alpha}\right) = \ln\left[\left(1 + \left(\frac{f}{n}\right)^{\alpha}\right)^{n}\right]$$

Now, n^{α} is increasing, and so we have that

$$n^{\alpha-1} f_n = \ln\left[\left(1 + \left(\frac{f}{n}\right)^{\alpha}\right)^{n^{\alpha}}\right]$$

$\square$

Products of Sets & Measures

We denote the set Z defined as the sets of ordered sequences $(x_1, \dots, x_n)$ where $x_1 \in X_1, \dots, x_n \in X_n$ as

$$Z = \times_{i=1}^n X_i$$

and if $X_1 = \cdots = X_n = X$, then $Z = X^n$. The same reasoning applies to collections of sets. We wish to define the collection of subsets of $Z = \times_{i=1}^n X_i$ representable as

$$A = \times_{i=1}^n A_i$$

where $A_i \subseteq X_i$ for each i and $\mathcal{F}_i$ is the collection of subsets of X_i. Then we denote this collection as

$$\mathcal{F} = \times_{i=1}^n \mathcal{F}_i$$

Theorem 71. *Suppose that $\mathcal{R}_1, \dots, \mathcal{R}_n$ are semi-rings, then $\mathcal{R} = \times_{i=1}^n \mathcal{R}_i$ is semi-ring.*

Proof. To satisfy the definition of a semi-ring, we must show that $A, B \in \mathcal{R}$ implies that $A \cap B \in \mathcal{R}$ and also that $B \subseteq A$ implies that $A = \cup_{i=1}^m C_i$ where $C_1 = B, C_i \cap C_j = \emptyset$ for $i \neq j$ and $C_i \in \mathcal{R}$ for each i. We will prove this for $n = 2$.

Suppose that $A, B \in \mathcal{R}_1 \times \mathcal{R}_2$. Then $A = A_1 \times A_2$ and $B = B_1 \times B_2$ and thus, $A \cap B = (A_1 \cap B_1) \times (A_2 \cap B_2)$ and since $\mathcal{R}_1$ and $\mathcal{R}_2$ are both semi-rings, we get $A \cap B \in \mathcal{R}$.

Now, suppose that $A, B \in \mathcal{R}$ and that $B \subseteq A$. Thus, $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$. Now, since $\mathcal{R}_1$ and $\mathcal{R}_2$ are both semi-rings, it follows that

$$A_i = B_i \cup B_i^{(1)} \cup \cdots \cup B_i^{(k_i)}$$

for $i = 1, 2$. We then get that

$$\begin{aligned}
A &= A_1 \times A_2 \\
&= (B_1 \times B_2) \cup (B_1 \times B_2^{(1)} \cup \cdots \cup (B_1 \times B_2^{(k_2)}) \\
&\quad \cup \cdots \cup (B_1^{(k_1)} \times B_2) \cup (B_1^{(k_2)} \times B_2^{(1)}) \cup \cdots \cup (B_1^{(k_2)} \times B_2^{(k_2)})
\end{aligned}$$

Here, the first term is $B_1 \times B_2 = B$ and all the rest are contained in $\mathcal{R}_1 \times \mathcal{R}_2$, which yields the claim. $\square$

From here, we now define the product of measures $\mu_1, \dots, \mu_n$ defined on $\mathcal{F}_1, \dots, \mathcal{F}_n$ to be

$$\times_{i=1}^n \mu_i$$

defined on $\times_{i=1}^n \mathcal{F}_i$. It is also true that if $A = \times_{i=1}^n A_i$ then

$$\mu(A) = \prod_{i=1}^n \mu_i(A_i)$$

It is straightforward to prove that $\times_{i=1}^n \mu_i$ is an additive measure.

THEOREM 72. *If the measures $\mu_1, \ldots, \mu_n$ are σ-additive, then the measure $\mu = \times_{i=1}^n \mu_i$ is also σ-additive.*

Proof. We will prove this for $n = 2$. Let λ_1 be the Lebesgue extension of μ_1. Let $C = \cup_{n=1}^{\infty} C_n$ where C and C_n are contained in $\mathcal{R}_1 \times \mathcal{R}_2$, so that $C = A \times B$, $C_n = A_n \times B_n$ where $A, A_n \in \mathcal{R}$ and $B, B_n \in \mathcal{R}$. Now, for $x \in A$, we let

$$f_n(x) = \begin{cases} \mu_2(B_n) & x \in A_n \\ 0 & x \notin A_n \end{cases}$$

It is now easy to see that $x \in A$ implies that

$$\sum_n f_n(x) = \mu_2(B)$$

Thus, it follows from Theorem 67 that

$$\sum_n \int_A f_n(x) d\lambda_1 = \int_A \mu_2(B) d\mu_1(A) = \mu(C)$$

but then

$$\int_A f_n(x) d\lambda_1 = \mu_2(B_n)\mu_1(A_n) = \mu(C_n)$$

so that

$$\sum_n \mu(C_n) = \mu(C)$$

as needed. □

Now, the Lebesgue extension of the measure $\times_{i=1}^n \mu_i$ will be called the **Product of Measures** and will be denoted as

$$\bigotimes_k \mu_k$$

and if $\mu_1 = \cdots = \mu_n$, then as before, we have $\mu^n = \otimes_k \mu_k$.

SECTION 6.1

GEOMETRIC DEFINITION OF THE LEBESGUE INTEGRAL

Let G be the region in the plane bounded in x by $a < b$ and y between $\phi(x)$ and $\psi(x)$. The area of this region is given by

$$V(G) = \int_a^b (\phi(x) - \psi(x)) dx$$

What we wish to do here is to extend this method of measuring areas to an arbitrary product-measure $\mu = \mu_x \otimes \mu_y$ where μ_x and μ_y defined on Borel algebras with units X and Y respectively are σ additive and complete. Now, let

$$A_y = \{y : (x, y) \in A\} \qquad A_x = \{x : (x, y) \in A\}$$

If $X \times Y$ is the plane, then A_{x_0} is the projection on the Y axis of the section of the set A with vertical $x = x_0$.

Lemma 73. *Suppose that A is a μ-measurable set and there exists a set B such that*

$$B = \bigcap_n B_n, \qquad B_1 \supseteq \cdots \supseteq B_n \supseteq \cdots$$

and

$$B_n = \bigcup_k B_{nk}, \qquad B_{n1} \subseteq \cdots \subseteq B_{nk} \subseteq \cdots$$

where the sets B_{nk} are elements of $\mathcal{R}(S_m)$, $A \subseteq B$ and $\mu(A) = \mu(B)$.

Proof. We first recall the fact that by measurability, for arbitrary n, the set A can be included in a union

$$C_n = \bigcup_r \Delta_{nr}$$

of sets Δ_{nr} of S_m such that $\mu(C_n) - \mu(A) < \frac{1}{n}$.
We let

$$B_n = \bigcap_{n=1}^{n} C_k$$

and see that the sets B_n will have the form

$$B_n = \bigcup_s \delta_{ns}$$

where the sets δ_{ns} are elements of S_m. Finally, by letting

$$B_{nk} = \bigcup_{s=1}^{k} \delta_{ns}$$

we obtain the sets required by the lemma. □

Theorem 74. *With the above notation, and where A_x and A_y are μ_y and μ_x measurable respectively for almost all x, y, we have*

$$\mu(A) = \int_X \mu_y(A_x) d\mu_x = \int_Y \mu_x(A_y) d\mu_y$$

for an arbitrary μ-measurable set A.

Proof. By symmetry, it is sufficient to prove that

$$\mu(A) = \int_X \phi_A(x) d\mu_x$$

where $\phi_A(x) = \mu_y(A_x)$. By measurability, the function $\phi_A(x)$ is μ_x is μ_x-measurable. If this were not so, ten the integral would be meaningless. The Lebesgue extension μ of $m = \mu_x \times \mu_y$ is defined on the collection S_m of sets of the form $A = A_{y_0} \times A_{x_0}$ where both sets are μ_x and μ_y measurable respectively. The result makes sense for these sets since

$$\phi_A(x) = \begin{cases} \mu_y(A_{x_0}) & x \in A_{y_0} \\ 0 & \text{otherwise} \end{cases}$$

and the result can be extended to sets of $\mathcal{R}(S_m)$, which is the set of finite unions of sets of S_m. We can extend the result here to the sets B and B_n from the lemma above by using the sets $B_{nk} \in \mathcal{R}(S_m)$ by means of the Monotone Convergence Theorem, since

$$\phi_{B_n}(x) = \lim_{k\to\infty} \phi_{B_{nk}}(x), \qquad \phi_{B_{n1}} \leq \phi_{B_{n2}} \leq \cdots$$

$$\phi_B(x) = \lim_{n\to\infty} \phi_{B_n}(x), \qquad \phi_{B_1} \leq \phi_{B_2} \leq \cdots$$

Now, if $\mu(A) = 0$ and $\mu(B) = 0$ and $\phi_B(x) = \mu_y(B_x) = 0$ a.e.
Since $A_x \subseteq B_x$, we know that A_x is measurable for almost all x and that $\phi_A(x) = \mu_y(A_x) = 0$ and

$$\int \phi_A(x) d\mu_x = 0 = \mu(A)$$

Thus, the result holds for sets A such that $\mu(A) = 0$. If A is arbitrary, then we write it as $A = B \backslash C$ where in view of the lemma above, we have $\mu(C) = 0$. Finally, since the result holds for B and C, it must also hold for A. □

We now let $Y = \mathbb{R}$, μ_y be linear Lebesgue measure and let

$$A = \{(x, y) : x \in M, 0 \leq y \leq f(x)\}$$

where M is a μ_x measurable set and $f(x)$ is an integrable nonnegative function.

Theorem 75. *The Lebesgue integral of a nonnegative integrable function f is equal to the measure $\mu = \mu_x \otimes \mu_y$ of the set A defined above.*

Section 6.2

Fubini's Theorem

We begin by considering the product

$$U = X \times Y \times Z = X \times (Y \times Z) = (X \times Y) \times X$$

and we define the measures μ_x, μ_y, μ_z on X, Y, Z respectively. Then we define the measure μ_u to be

$$\mu_u = \mu_x \otimes \mu_y \otimes \mu_z$$

This leads us to Fubini's Theorem.

Theorem 76 (Fubini's Theorem). *Suppose that the measures μ_x, μ_y defined on Borel algebras with units X, Y respectively are σ-additive and complete. Now, define*

$$\mu = \mu_x \otimes \mu_y$$

and suppose that the function $f(x, y)$ is μ-integrable on $A = A_{y_0} \times A_{x_0}$ where $A_x = \{y : (x, y) \in A\}$ adn $A_y = \{x : (x, y) \in A\}$. Then

$$\int_A f(x, y) d\mu = \int_X \int_{A_x} f(x, y) d\mu_y d\mu_x = \int_Y \int_{A_y} f(x, y) d\mu_x d\mu_y$$

where the integrals above are assumed to exist for almost all x, y contained in whichever sets apply to them in each integral.

Proof. We will prove this for the case where $f(x, y) \geq 0$. Consider the triple product

$$U = X \times Y \times \mathbb{R}$$

and consider the product measure

$$\lambda = \mu_x \otimes \mu_y \otimes \mu^1 = \mu \otimes \mu^1$$

where μ^1 is linear Lebesgue measure. Now define the subset $W \subseteq U$ to be

$$W = \{(x, y, z) \in U : x \in A_{y_0}, y \in A_{x_0}, 0 \leq z \leq f(x, y)\}$$

Now, from Theorem 75, we have that

$$\lambda(W) = \int_A f(x, y) d\mu$$

but by Theorem 74, we also have that

$$\lambda(W) = \int_X \xi(W_x) d\mu_x$$

where $\xi = \mu_y \otimes \mu^1$ and $W_x = \{(y, z) : (x, y, z) \in W\}$. Now, another application of Theorem 75 yields

$$\xi(W_x) = \int_{A_x} f(x, y) d\mu_y$$

and putting the pieces together, we see that

$$\int_A f(x, y) d\mu = \int_X \int_{A_x} f(x, y) d\mu_y d\mu_x$$

and we obtain the second equality trivially by symmetry.
All that remains now is to generalize this proof to any function f. We know that this is now true for $f(x, y) \geq 0$, but a general function f is reduced to this case by means of the relation

$$f(x, y) = f_+(x, y) = f_-(x, y)$$

where

$$f_+(x, y) = \frac{1}{2}(|f(x, y)| + f(x, y))$$

and

$$f_-(x, y) = \frac{1}{2}(|f(x, y)| - f(x, y))$$

as needed. □

Remark. It can also be shown that if $f(x, y)$ is μ-measurable and if

$$\int_X \int_{A_x} f(x, y) d\mu_y d\mu_x$$

exists, then so too must

$$\int_A f(x, y) d\mu$$

exist.

We will now look at a few examples of some violations of the theorem.

To construct the first example, let $A = [-1,1]^2$ and let

$$f(x,y) = \frac{xy}{(x^2+y^2)^2}$$

This gives

$$\int_{-1}^{1} f(x,y)dx = 0 \quad \text{and} \quad \int_{-1}^{1} f(x,y)dy = 0$$

assuming that $y \neq 0$ for the first equality and that $x \neq 0$ for the second equality. Thus, we have

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy = \int_{-1}^{1}\int_{-1}^{1} f(x,y)dydx = 0$$

but the Lebesgue double integral over A does not exist since

$$\int_{-1}^{1}\int_{-1}^{1} |f(x,y)|dxdy \geq \int_{0}^{1} dr \int_{0}^{2\pi} \frac{\sin(\phi)\cos(\phi)}{r} d\phi = 2\int_{0}^{1} \frac{dr}{r} = \infty$$

which shows that this does not follow Fubini's theorem.

The second example begins by letting $A = [0,1]^2$ and letting

$$f(x,y) = \begin{cases} 2^{2n} & (x,y) \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]^2 \\ -2^{2n+1} & x \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], y \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \\ 0 & \text{otherwise} \end{cases}$$

and this yields

$$\int_{0}^{1}\int_{0}^{1} f(x,y)dxdy = 0 \quad \text{and} \quad \int_{0}^{1}\int_{0}^{1} f(x,y)dydx = 1$$

which is another violation of the result from Fubini's theorem.

Chapter 7

Absolute Continuity

We begin with a definition.

Definition 29. *We say that the measure λ on the σ-algebra $\mathcal{A}$ is* ***Absolutely Continuous*** *with respect to the measure μ on $\mathcal{A}$ if $\lambda(E) = 0$ for every $E \in \mathcal{A}$ for which $\mu(E) = 0$. That is to say that they share sets of measure zero.*

Definition 30. *We say that a measure λ is* ***Concentrated*** *on a set $A \in \mathcal{A}$ if A is such that $\lambda(E) = \lambda(A \cap E)$ for every $E \in \mathcal{A}$.*

Definition 31. *Let λ_1 and λ_2 be measures on $\mathcal{A}$, and suppose that there exists a pair of disjoint sets A and B such that λ_1 is concentrated on A and λ_2 is concentrated on B. We then say that λ_1 and λ_2 are* ***Mutually Singular****.*

Definition 32. *We define the set $L^1(\mu)$ to be the collection of functions f on X such that*

$$\int_X |f| d\mu < \infty$$

which is to say that it is the collection of all absolutely Lebesgue integrable functions.

Section 7.1

The Radon-Nikodym Theorem

Before tackling the proof of the following theorem, we will need a lemma.

Lemma 77. *Let μ be a σ-additive measure on a σ-algebra $\mathcal{A}$ in a set X, then there is a function w contained in $L^1(\mu)$ such that $w(x) \in (0, 1)$ for every $x \in X$.*

Proof. Since μ is σ-additive, we know that X is the union of countably many sets $E_n \in \mathcal{A}$ for which $\mu(E_n) < \infty$. Let $w_n(x) = 0$ whenever $x \in X \backslash E_n$ and let

$$w_n(x) = \frac{1}{2^n(1 + \mu(E_n))}$$

whenever $x \in E_n$. Then we know that

$$w = \sum_{n=1}^{\infty} w_n$$

has the required properties as needed.

$\square$

The point of that was to show that the measure μ can be replaced by a finite measure $\hat{\mu}$ such that $d\mu = wd\hat{\mu}$ which, because of the strict positivity of w shares all sets of measure zero with μ.

THEOREM 78 (Radon-Nikodym Theorem). *Let μ be a σ-additive measure on a σ-algebra $\mathcal{A}$ in a set X and let λ be a measure on $\mathcal{A}$. Then*

(i) There is a unique pair of measures λ_a and λ_s on $\mathcal{A}$ such that

$$\lambda = \lambda_a + \lambda_s$$

and λ_a is absolutely continuous with respect to μ and λ_s and μ are mutually singular. This is known as the ***Lebesgue Decomposition*** *of λ.*

(ii) There is a unique function $h \in L^1(\mu)$ such that

$$\lambda_a(E) = \int_E h d\mu$$

for every set $E \in \mathcal{A}$.

Proof. The uniqueness of the pairs in (i) is clear since if (λ'_a, λ'_s) were another pair which satisfies the conditions in (i), then

$$\lambda'_a - \lambda_a = \lambda_s - \lambda' s$$

and the first difference is then absolutely continuous with respect to μ and the second difference is mutually singular with μ and thus, both sides must be 0.

Also, the uniqueness of the function h from (ii) is trivial in view of the fact that for a function $h \in L^1(\mu)$ such that

$$\int_E f d\mu = 0$$

for every $E \in \mathcal{A}$, we have that $f = 0$ a.e. and so it follows from letting h' be another such function and then integrating $h' - h$.

From here, we will prove the existence portions of both parts. Suppose that λ is a bounded measure on $\mathcal{A}$. Let w be the same as in Lemma 77. Then we define $d\phi = d\lambda + wd\mu$ which is also a bounded measure on $\mathcal{A}$. By the definition of these measures, we get

$$\int_X f d\phi = \int_X f d\lambda + \int_X f w d\mu$$

for $f = \chi_E$, and hence for simple f and further for nonnegative measurable functions f. If $f \in L^2(\phi)$, then the Cauchy-Schwarz inequality gives us that

$$\left| \int_X f d\lambda \right| \leq \int_X |f| d\lambda \leq \int_X |f| d\phi \leq \left(\int_X |f|^2 \right)^{\frac{1}{2}} \phi(X)^{\frac{1}{2}}$$

and since $\phi(X) < \infty$, we see that

$$f \to \int_X f d\lambda$$

is a bounded linear functional on $L^2(\phi)$. We already know that every bounded linear functional on a Hilbert space H is given by an inner product with an element of H. Thus, there exists $g \in L^2(\phi)$ such that

$$\int_X f d\lambda = \int_X f g d\phi$$

for every $f \in L^2(\phi)$. The existence of g comes from the completeness of $L^2(\phi)$.

Now, let $f = \chi_E$ in the above equality and for every $E \in \mathcal{A}$ with $\phi(E) > 0$, the LHS of that equality is then $\lambda(E)$ and since $0 \leq \lambda \leq \phi$, we get that

$$0 \leq \frac{1}{\phi(E)} \int_E g d\phi = \frac{\lambda(E)}{\phi(E)} \leq 1$$

so that $g \in [0,1]$ for almost every x. We can now rewrite the equality in question as

$$\int_X (1-g) f d\lambda = \int_X f g w d\mu$$

Now, put

$$A = \{x : g \in [0,1)\}, \quad B = \{x : g(x) = 1\}$$

and define measures λ_a and λ_s by

$$\lambda_a(E) = \lambda(A \cap E), \quad \lambda_s(E) = \lambda(B \cap E)$$

for each $E \in \mathcal{A}$. Now, if $f = \chi_B$ in the most recent integral equality, then the left side is 0, the right side is

$$\int_B w d\mu$$

Since $w(x) > 0$ for all x by definition, we can conlcude that $\mu(B) = 0$. Thus, λ_s is mutually singular with μ. Since g is bounded, that equation holds if f is also replaced by

$$(1 + g + \cdots + g^n)\chi_E$$

for any n and for any $E \in \mathcal{A}$. This replacement yields

$$\int_E (1 - g^{n+1}) d\lambda = \int_E g(1 + \cdots + g^n) w d\mu$$

Now, at every point of B, we have $g(x) = 1$ so that $1 - g^{n+1} = 0$. At every point of A, we have that $g^{n+1}(x) \to 0$ as $n \to \infty$ monotonically. The left side of the now most recent integral equality then converges to $\lambda(A \cap E) = \lambda_a(E)$ as $n \to \infty$.

The integrands on the RHS increase monotonically to a nonnegative measurable limit h and the Monotone Convergence Theorem tells us tha the RHS tends to

$$\int_E h d\mu$$

as $n \to \infty$. We have just proved that (i) holds for every $E \in \mathcal{A}$ and taking $E = X$, we see that $h \in L^1(\mu)$ since $\lambda_a(X) < \infty$. Finally, (ii) shows that λ_a is absolutely continuous with respect to μ and the proof is complete. $\square$

Consequences of the Radon-Nikodym Theorem

We begin with a theorem that we will use to prove the Hahn Decomposition Theorem.

Theorem 79. *Let μ be a measure on a σ-algebra $\mathcal{A}$ in X. Then there is a measurable function h such that $|h(x)| = 1$ for all $x \in X$ and such that*

$$d\mu = hd|\mu|$$

*where the above is called the **Polar Decomposition** of μ.*

Proof. It is trivial that μ is absolutely continuous with respect to $|\mu|$ so that the Radon-Nikodym Theorem guarantees the existence of $h \in L^1(\mu)$ which satisfies the polar decomposition.
Now, let $A_r = \{x : |h(x)| < r\}$, where $r > 0$ and let $\{E_j\}$ be a partition of A_r. Thus,

$$\sum_j |\mu(E_j)| = \sum_j \left| \int_{E_j} hd|\mu| \right| \leq \sum_j r|\mu|(E_j) = r|\mu|(A_r)$$

so that $|\mu|(A_r) \leq r|\mu|(A_r)$. Now, if $r < 1$, then this forces $|\mu|(A_r) = 0$ which implies that $|h| \geq 1$ a.e.
On the other hand, if $|\mu|(E) > 0$, the decomposition shows that

$$\left| \frac{1}{|\mu|(E)} \int_E hd|\mu| \right| = \frac{|\mu(E)|}{|\mu|(E)} \leq 1$$

and it now follows that $|h| \leq 1$ a.e.
Finally, let $B = \{x \in X : |h(x)| \neq 1\}$. We have shown that $|\mu|(B) = 0$ and if we redefine h on B so that $h(x) = 1$ on B, we obtain the desired function. □

Theorem 80 (Hahn-Decomposition Theorem). *Let μ be a real measure on a σ-algebra $\mathcal{A}$ in a set X. Then there exists sets $A, B \in \mathcal{A}$ such that $A \cup B = X$ and $A \cap B = \emptyset$, and such that the positive and negative variations μ_+ and μ_- of μ satisify*

$$\mu_+(E) = \mu(A \cap E) \quad \textit{and} \quad \mu_-(E) = -\mu(B \cap E)$$

for $E \in \mathcal{A}$.

*That is to say that X is the union of two disjoint measurable sets A and B such that A carries all the positive mass of μ and B carries all the negative mass of μ and the pair (A, B) is called the **Hahn-Decomposition** of X induced by μ.*

Proof. Theorem 79 gives us that $d\mu = hd|\mu|$ where $|h| = 1$. Now it follows that $h = \pm 1$. Put

$$A = \{x : h(x) = 1\} \quad \text{and} \quad B = \{x : h(x) = -1\}$$

By definition, we have that

$$\mu_+ = \frac{1}{2}(|\mu| + \mu)$$

and since

$$\frac{1}{2}(1 + h) = \begin{cases} h & \text{on } A \\ 0 & \text{on } B \end{cases}$$

we have for any $E \in \mathcal{A}$ that

$$\mu_+(E) = \frac{1}{2}\int_E (1+h)d|\mu| = \int_{E\cap A} hd|\mu| = \mu(E \cap A)$$

Finally, since $\mu(E) = \mu(E \cap A) + \mu(E \cap B)$ and since $\mu = \mu_+ - \mu_-$, the second half of the result of the theorem follows from the first. $\square$

Corollary 81. *If $\mu = \lambda_1 - \lambda_2$ where λ_1 and λ_2 are positive measures, then $\lambda_1 \geq \mu_+$ and $\lambda_2 \geq \mu_-$.*

Proof. Since $\mu \leq \lambda_1$, we have

$$\mu_+(E) = \mu(E \cap A) \leq \lambda_1(E \cap A) \leq \lambda_1(E)$$

as needed. $\square$

Chapter 8

Square Integrable Functions

Section 8.1

The Space L_2

We begin by prescribing a measure μ on a space X such that $\mu(X) < \infty$. The functions f defined on this space are measurable and defined a.e. on X.

Definition 33. *We say that a function $f(x)$, defined on X, is **Square Integrable** if and only if*

$$\int_X f^2(x) d\mu < \infty$$

and we say that f is an L_2 function.

Theorem 82. *The product of two square integrable functions is an integrable function.*

Proof. This result follows from the definition of the Lebesgue integral paired with the fact that

$$|f(x)g(x)| \leq \frac{1}{2}(f(x)^2 + g(x)^2)$$

which is trivial to deduce. □

Corollary 83. *A square integrable function is integrable.*

Theorem 84. *If $f, g \in L_2$ then $f + g \in L_2$.*

Proof. Indeed we have that

$$(f(x) + g(x))^2 \leq f(x)^2 + 2|f(x)|g(x)| + g(x)^2$$

and the result follows from Theorem 1 applied to the LHS. □

Theorem 85. *If $f \in L_2$ and $\alpha \in \mathbb{R}$, then $\alpha f(x) \in L_2$.*

It is quite clear by the above two theorems that L_2 satisfies the 8 conditions of a linear space and thus, we can conclude that L_2 is linear.

Definition 34. *We define the **Inner Product** (f, g) on L_2 to be*

$$(f, g) = \int_X f(x)g(x)d\mu$$

and we note that this satisfies

(i) $(f,g)=(g,f)$

(ii) $(f_1+f_2,g)=(f_1,g)+(f_2,g)$

(iii) $(\lambda f,g)=\lambda(f,g)$

(iv) $(f,f)>0$ *for* $f\neq 0$

Before proceeding, we list a few useful properties of this inner product. The first is the Schwarz inequality

$$\left(\int_X f(x)g(x)d\mu\right)^2 \leq \left(\int_X f(x)^2 d\mu\right)\left(\int_X g(x)^2 d\mu\right)$$

which is satisfied in L_2 as it is in any Euclidean space. We also have the Triangle intequality given by

$$\left(\int_X (f(x)+g(x))^2 d\mu\right)^{\frac{1}{2}} \leq \left(\int_X f(x)^2 d\mu\right)^{\frac{1}{2}}\left(\int_X g(x)^2 d\mu\right)^{\frac{1}{2}}$$

and in particular, the Schwarz inequality yields the following useful inequality which is

$$\left(\int_X f(x)d\mu\right)^2 \leq \mu(X)\int_X f(x)^2 d\mu$$

and finally we will introduce the norm on L_2 to be

$$\|f\| = (f,f)^{\frac{1}{2}} = \left(\int_X f(x)^2 d\mu\right)^{\frac{1}{2}}$$

and this brings us to one of the most important theorems thus far.

Theorem 86. *The space L_2 is complete.*

Proof. See Kolmogorov & Fomin. □

Section 8.2
Mean Convergence & Dense Subsets of L_2

As we have now defined a norm (and thus an induced metric) on the space L_2, it is natural to talk about convergence of sequences of elements of L_2, namely, the square integrable functions.

Definition 35. *We say that the sequence f_n is* ***Mean Convergent*** *to f or* ***Mean Square Convergent*** *to f if and only if*

$$\int_X (f_n(x)-f(x))^2 d\mu \xrightarrow[n\to\infty]{} 0$$

or simply that $\|f_n - f\| \xrightarrow[n\to\infty]{} 0$.

Theorem 87. *If a sequence f_n of L_2 functions converges uniformly to $f(x)$, then $f(x)\in L_2$ and the sequence f_n is also mean convergent to f.*

Proof. The proof can be found in Kolmogorov & Fomin on page 84 in §51. □

THEOREM 88. *Let X be a metric space with a measure satisfying the property that all open and closed sets of X are measurable and*

$$\mu^*(M) = \inf_{G \supseteq M} \{\mu(G) : M \subseteq G\}$$

for every $M \subseteq X$. Given this space X, the set of all continuous functions on X is dense in L_2.

Proof. The proof can be found in Kolmogorov & Fomin on page 85 in §51. □

THEOREM 89. *If a sequence f_n converges to f in the mean, then it contains a subsequence f_{n_k} which converges to f a.e.*

Proof. The proof can be found in Kolmogorov & Fomin on page 86 in §51. □

There is also a useful diagram on page 87 which illustrates the relationships (if any) between the four relevant types of convergence covered in this course.

SECTION 8.3

ORTHOGONAL SETS OF FUNCTIONS & ORTHOGONALIZATION

We begin with a definition.

DEFINITION 36. *We say that a set of functions $\Phi = \{\phi_1, \ldots, \phi_n\}$ defined on a set X are* ***Linearly Dependent*** *if and only if there exists constants $c_1, \ldots, c_n$ not all equal to zero such that*

$$(*) \qquad c_1\phi_1 + \cdots + c_n\phi_n = 0 \quad a.e.$$

on X. On the other hand, we say that the functions of Φ are ***Linearly Independent*** *if and only if $(*)$ implies that*

$$c_1 = \cdots = c_n = 0$$

DEFINITION 37. *An infinite sequence of functions $\Phi = \{\phi_1, \ldots, \phi_n, \ldots\}$ is said to be* ***Linearly Independent*** *if every finite subset of Φ is linearly independent. Also, denote the set of all finite linear combinations of functions of Φ by*

$$M = M(\phi_1, \ldots, \phi_n, \ldots) = M(\{\phi_k\})$$

and we say that M is the ***Linear Hull*** *of Φ or the* ***Linear Manifold Generated by*** *$\{\phi_k\}$. Also, we say that the closure $\overline{M}$ of M is called the* ***Closed Linear Hull*** *or* ***Subspace Generated by*** *$\{\phi_k\}$.*

DEFINITION 38. *The set Φ of functions is said to be* ***Complete*** *or* ***Closed*** *if and only if*

$$\overline{M} = L_2$$

Fact. If L_2 contains a finite complete set $\phi_1, \ldots, \phi_n$ of linearly independent functions, then $L_2 = \overline{M}(\{\phi_k\}) = M(\{\phi_k\})$ is isomorphic to Euclidean n-space. If not, then we say that L_2 is **Infinite Dimensional**. It is also important to note that L_2 in infinite-dimensional in all interesting cases in analysis.

DEFINITION 39. *We say that two functions f, g of L_2 are **Orthogonal** if*

$$(f, g) = \int_X f(x)g(x)d\mu = 0$$

*and if $\Phi = \{\phi_1, \ldots, \phi_n, \ldots\}$ is pairwise orthogonal, then we will say that Φ is an **Orthogonal Set** and if $\|\phi_n\| = 1$ for any n, then we say further that Φ is an **Orthonormal** set.*

THEOREM 90. *Suppose that the set of functions $f_1, \ldots, f_n, \ldots$ is linearly independent. Then there exists a set of functions $\Phi = \{\phi_1, \ldots, \phi_n, \ldots\}$ such that*

(i) The set Φ is orthonormal.

(ii) Every function ϕ_n is a linear combination of the functions $f_1, \ldots, f_n$ so that

$$\phi_n = a_{n,1}f_1 + \cdots + a_{n,n}f_n$$

with $a_{n,n} \neq 0$.

(iii) Every function f_n is a linear combination of the functions $\phi_1, \ldots, \phi_n$ so that

$$f_n = b_{n,1}\phi_1 + \cdots + b_{n,n}\phi_n$$

with $b_{n,n} \neq 0$.

Also, every function of Φ is uniquely (up to sign) determined by the conditions above.

Proof. The proof can be found in Kolmogorov & Fomin on page 94 in §53. □

SECTION 8.4

FOURIER SEIRES & THE RIESZ-FISCHER THEOREM

Let $\phi_1, \ldots, \phi_k, \ldots$ be an orthonormal set in some L^2 space H. We have $f \in L^2$ and we want to take a linear combination

$$S_n = \sum_{k=1}^{n} \alpha_k \phi_k$$

and minimize the distance

$$
\begin{aligned}
\|F-S_n\|_2^2 &= \left(f-\sum_{k=1}^{n}\alpha_k\phi_k,\, f-\sum_{k=1}^{n}\alpha_k\phi_k\right)\\
&= (f,f)-2\sum_{k=1}^{n}\alpha_k(f,\phi_k)+\sum_{k=1}^{n}\alpha_k^2\|\phi_k\|_2^2\\
&= \|f\|_2^2-2\sum_{k=1}^{n}\alpha_k c_k+\sum_{k=1}^{n}\alpha_k^2\\
&= \|f\|_2^2-2\sum_{k=1}^{n}\alpha_k c_k+\sum_{k=1}^{n}\alpha_k^2\pm\sum_{k=1}^{n}c_k^2\\
&= \|f\|_2^2-2\sum_{k=1}^{n}c_k^2+\sum_{k=1}^{n}(c_k-\alpha_k)^2
\end{aligned}
$$

where $(f,\phi_k)=c_k$ and we use the fact that $\|\phi_k\|_2=1$ and $(\phi_j,\phi_k)=0$ for $j\neq k$. Now choose $\alpha_k=c_k$ so that

$$\|F-S_n\|_2^2=\|f\|_2^2-\sum_{k=1}^{n}c_k^2$$

Definition 40. *The above quantity $c_k=(f,\phi_k)$ is called the k^{th} **Fourier Coefficient** of f with respect to $\{\phi_k\}$.*

Now, we see that

$$S_n=\sum_{k=1}^{n}(f,\phi_k)\phi_k$$

is the n^{th} partial sum of the **Fourier Series**.

Remark. Notice that $f-S_n$ is orthogonal to each of $\phi_1,\dots,\phi_n$.

Now, look at

$$\|f\|_2^2-\sum_{k=1}^{n}c_k^2=\|f-S_n\|_2^2\Longrightarrow\|f\|_2^2-\|f-S_n\|_2^2=\sum_{k=1}^{n}c_k^2$$

so that we arrive at the so-called **Bessel Inequality** which says that since, $\|f-S_n\|_2^2\geq 0$, we get

$$\sum_{k=1}^{n}c_k^2\leq\|f\|_2^2$$

Definition 41. *We say that the set $\{\phi_k\}$ is **Complete** (or **Closed**) if and only if*

$$\sum_{k=1}^{\infty}c_k^2=\|f\|_2^2$$

for each $f \in L^2$ or equivalently, if

$$\iff \sum_{k=1}^{n} (f, \phi_k)\phi_k \to f$$

in L_2.

THEOREM 91. *In L_2, every complete orthonormal system is closed.*

Proof. See Kolmogorov & Fomin for the proof of this. □

Remark. Notice that if $f \in L_2$, and $c_k = (f, \phi_k)$, then

$$\sum_{k=1}^{\infty} c_k^2 \leq \|f\|_2^2 < \infty$$

and so the sequence $\{c_k\}$ is contained in l_2.

THEOREM 92 (Riesz-Fischer). *if $\{\phi_k\}$ is an orthonormal system in L_2 and if the sequence $c_k \in l_2$, then there exists a function $f \in L_2$ such that $(f, \phi_k) = c_k$. Moreover,*

$$\|f\|_2^2 = \sum_{k=1}^{\infty} c_k^2$$

*which is called **Parseval's Identity**.*

Sketch of Proof. Let

$$f_n = \sum_{k=1}^{n} c_k \phi_k$$

and we want that $\{f_n\}$ is Cauchy. So we write

$$\|f_{n+m} - f_n\|_2^2 = \sum_{k=n+1}^{n+m} c_k^2$$

which is simply the tail of a convergent series. Now, as $n \to \infty$, this difference $\|f_{m+n} - f_n\|$ will become uniformly small in m which shows that f_n is Cauchy. For the fine details of the proof, see Kolmogorov & Fomin. □

THEOREM 93 (Criterion for Completeness). *Let $\Phi = \{\phi_1, \ldots, \phi_n, \ldots\}$ be an orthonormal system. Then Φ is complete if and only if $\psi \in L_2$ and $(\phi_k, \psi) = 0$ for each k implies that $\phi \equiv 0$.*

Proof. See Kolmogorov & Fomin for full proof. □

Definition 42. *We say that two Euclidean spaces U, V are isomorphic if and only if there is a one-to-one correspondance between their elements such that*

$$x \leftrightarrow x' \qquad y \leftrightarrow y'$$

implies that

(i) $x + y \leftrightarrow x' + y'$.

(ii) $\alpha x \leftrightarrow \alpha x'$

(iii) $(x, y) \leftrightarrow (x', y')$

Theorem 94. *The space L_2 is isomorphic to the space l_2.*

Proof. See Kolmogorov & Fomin for full proof. □

Remark. The real valued inner product is

$$(f, g) = \int_X f(x)g(x)d\mu_x$$

and the complex valued one is

$$(f, g) = \int_X f(x)\overline{g}(x)d\mu_x$$

where $\overline{g}$ denotes the complex conjugate of g.

Fact. It is true that $L_2(\mathbb{R}/2\phi\mathbb{Z}) \cong L_2([0, 2\pi])$.

The system of functions $\{e^{inx} : n \in \mathbb{Z}\}$ is complete in $L_2([0, 2\pi])$ and by Euler's formula, we have

$$e^{inx} = \cos(nx) + i\sin(nx)$$

Note that Tchebyshev polynomials have basis

$$\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\}$$

and we know that

$$\cos(n\theta) = T_n(\cos(\theta)) = T_n(x)$$

which is the polynomial of the first kind and

$$\frac{\sin(n\theta)}{\sin(\theta)} = U_n(\cos(\theta)) = U_n(x)$$

is the polynomial of the second kind.
Now, let $\mathbb{T}^n \cong [0, 2\pi]^n$ and the basis of $L_2(\mathbb{T}^n)$ is obtained using vectors $[x_1, \ldots, x_n]$.

Example 6. *Let $n = 2$ and we have*

$$e^{i(k_1,k_2)\cdot(x_1,x_2)} = e^{i(k_1x_1+k_2x_2)}$$

where $k_1, k_2 \in \mathbb{Z}^2$ and

$$\{e^{i(k_1x_1+k_2x_2)} : k_1, k_2 \in \mathbb{Z}\}$$

forms an orthonormal basis for $L_2(\mathbb{T}^2)$*. Now,*

$$\{e^{i(k_1x_1+\cdots+k_nx_n)} : k_j \in \mathbb{Z}\}$$

forms an orthonormal basis for $L_2(\mathbb{T}^n)$ *so that* $\phi_k(x) = e^{i(k,x)}$ *where* $k \in \mathbb{Z}^n$*.*

Section 8.5

Applications

We will look at several partial differential equations.

(i) We look at the heat equation

$$\frac{\partial}{\partial t}u = \nabla^2 u \qquad \text{on } \mathbb{T}^n$$

and we assume that

$$u(x,t) = f(x)g(t)$$

and see that

$$f(x) = \phi_k(x) = e^{i(k,x)}$$

and so

$$\nabla^2\phi_k = -(k_1^2 + \cdots + k_n^2)e^{i(k,x)} = -|k|^2\phi_k$$

and we say that $\phi_k(x)$ is an **Eigenfunction** of ∇^2 with eigenvalue $-|k|^2$. Now, we have

$$u_k(x,t) = \phi_k(x)g_k(t)$$

and

$$\frac{\partial}{\partial t}g_k(t)\phi_k(x) = -|k|^2\phi_k(x)g_k(t)$$

$$\Longrightarrow \frac{\partial}{\partial t}g_k(t) = -|k|^2 g_k(t)$$

$$\Longrightarrow g_k(t) = e^{-|k|^2 t}$$

so that

$$u_k(t) = \phi_k(t)e^{-|k|^2 t}$$

where u_0 is the temerature at time $t = 0$ which gives

$$u_0(x) = \sum_{k\in\mathbb{Z}^n} c_k\phi_k(x)$$

where $x \in \mathbb{T}^n$. Now,

$$u(x,t) = \sum_{k\in\mathbb{Z}^n} c_k\phi_k(x)e^{-|k|^2 t}$$

(ii) We will look at a Hyperbolic Equation, namely the Wave Equation given by

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

on $\mathbb{T}^n$. We look at

$$u_k(x,t) = g_k(t)\phi_k(x)$$

where $\phi_k(x) = e^{i(k,x)}$ so that $\nabla^2 \phi_k = -|k|^2 \phi_k$ and we write

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = g_k''(t)\phi_k(x) \\ \nabla^2 u_k = -|k|^2 g_k(t)\phi_k(x) \end{cases}$$

and we have that the wave equation is satisfied if and only if

$$g_k''(t) = -|k|^2 g_k(t)$$

and thus

$$g_k(t) = e^{\pm i|k|t}$$

and so

$$u_k(x,t) = e^{\pm i|k|t} e^{i(k,x)}$$

as needed.

(iii) We now turn to Schrodinger's Equation. We will suppose that the vacuum potential is identically zero and we have

$$\frac{1}{i}\frac{\partial u}{\partial t} = \nabla^2 u$$

and once again assume that

$$u_k(x,t) = g_k(t)\phi_k(x)$$

so that

$$\frac{1}{i}\frac{\partial}{\partial t} g_k(t) = -|k|^2 g_k(t)$$

and thus

$$g_k(t) = e^{-i|k|^2 t}$$

so that

$$u_k(x,t) = e^{-i|k|t} e^{i(k,x)}$$

as needed.

(iv) Finally, we look at Euler's Equation for fluids. First notice that this problem is nonlinear and Fourier expansions are thus, less useful here. Let's work only on $\mathbb{T}^3$. Now, suppose that the fluid in question is incompressible and has no viscosity which is reasonable for water, but not for wax. Then the equation is given by

$$\frac{\partial \vec{u}}{\partial t} - \vec{u} \times (\mathrm{curl}(\vec{u})) + \nabla\left(\rho + \frac{\|\vec{u}\|^2}{2}\right) = 0$$

where ρ is the pressure function, and sometimes we make the extra assumption that $\vec{\omega}$ =curl($\vec{u}$) and so

$$\frac{\partial \vec{\omega}}{\partial t} - \mathrm{curl}(\vec{u} \times \vec{\omega}) = 0$$

Now, for Force-Free solutions (Beltrami flows), suppose that u satisfies $\vec{\omega}$ =curl$(\vec{u}) = \mu\vec{u}$ where μ is a scalar function of space or maybe just a constant. Then,

$$\vec{u} \times \vec{\omega} = \mu \cdot (\vec{u} \times \vec{u}) = 0$$

We continue with an example of ABC-Flows given by

$$\vec{u}(\vec{x}) = a \begin{bmatrix} 0 \\ \cos(x) \\ \sin(x) \end{bmatrix} + b \begin{bmatrix} \sin(y) \\ 0 \\ \cos(y) \end{bmatrix} + c \begin{bmatrix} \cos(z) \\ \sin(z) \\ 0 \end{bmatrix}$$

where $\vec{x} = [x, y, z]^T$. Now, chekc that $\text{curl}(\vec{u}) = \mu\vec{u}$ for some μ. Now, with

$$\rho = -\frac{\|\vec{u}\|^2}{2}$$

the Euler equation becomes

$$\frac{\partial \vec{u}}{\partial t} = 0$$

and thus the fluid moves with constant velocity. Also, if we solve the Navier-Stokes equation with initial conditions as ABC-Flows, then the solution has exponentially decaying velocity.

(v) Another physical problem where Beltrami fields appear is in Plasma Physics.

Chapter 9

Abstract Hilbert Spaces

Section 9.1

Basics & Definitions

Definition 43. *A set H of arbitrary elements $f, g, \ldots, h, \ldots$ is said to be a **Hilbert Space** if and only if*

(i) H is a linear space.

(ii) An inner product is defined in H. That is, every pair of elements $f, g \in H$ is assigned a real number (f, g) such that

(a) $(f, g) = (g, f)$

(b) $(\alpha f, g) = \alpha(f, g)$

(c) $(f_1 + f_2, g) = (f_1, g) + (f_2, g)$

(d) $(f, f) > 0$ whenever $f \neq 0$.

*That is, H is a Euclidean space and $\|f\| = \sqrt{(f, f))}$ is called the **Norm** of f in H.*

(iii) The space H is complete in the metric $\rho(f, g) = \|f - g\|$.

(iv) H is infinite-dimensional. That is, for every natural number n, H contains n linearly independent vectors.

(v) H is separable (which is usually an optional condition) so that H has a countable dense set.

Example 7. *The space l_2 is a Hilbert space. It has already been proven that this is true as l_2 is finite dimensional, complete, separable and Euclidean. Also, since L_2, the space of square-integrable functions is isomorphic to l_2, we can conclude that L_2 is also a Hilbert space.*

Proposition 95. *All Hilbert spaces are isomorphic.*

Proof. To this end, it is sufficient to show that all Hilbert spaces are isomorphic to l_2.
Choose in H a countable dense set and apply it to the process of orthogonalization described for L_2 and we will construct in H a complete orthonormal set

$$h_1, \ldots, h_n, \ldots$$

satisfying

(i) For $h_i, h_j \in H$, we have

$$(h_i, h_j) = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$

(ii) The linear combinations of the elements $h_1, \dots, h_n, \dots$ are dense in H.

Now, let $f \in H$ be an arbitrary element. Let $c_k = (f, h_k)$. Then the series

$$\sum_k c_k^2 < \infty$$

and also

$$\sum_k c_k^2 = (f, f)$$

for an arbitrary complete orthonormal set $\{h_k\}$ and $f \in H$. Now, suppose that $\{h_k\}$ is complete and orthonormal in H. If $c_1, \dots, c_n, \dots$ is a sequence of numbers such that

$$\sum_k c_k^2 < \infty$$

then there exists $f \in H$ such that $c_k = (f, h_k)$ and $\sum_k c_k^2 = (f, f)$. It is clear from what we have said that an isomorphism between H and l_2 can be realizing by associating to each f the sequence

$$(c_1, \dots, c_n, \dots)$$

where $c_k = (f, h_k)$ and the set $h_1, \dots, h_n, \dots$ is arbitrary, orthonormal and complete in H. □

Section 9.2

Subspaces, Orthogonal Complements & Direct Sums

As we've seen before, a **Linear Manifold** in a Hilbert space H is a subset L of H such that if $f, g \in L$ then $\alpha f + \beta g \in L$ for any real α and β. A **Subspace** of H is a closed linear manifold.

Lemma 96. *If a metric space R contains a countable dense set, then every subspace R' of R contains such a set.*

Proof. The proof of this lemma can be found in §57 of Kolmogorov & Fomin. □

Theorem 97. *Every subspace of a Hilbert space H is either a finite-dimensional Euclidean space or itself a Hilbert space.*

Proof. For the first three axioms of Hilbert spaces, this result is clear. The fifth axiom follows from the above lemma. □

Theorem 98. *Every subspace M of a Hilbert space H contains an orthogonal set $\{\phi_n\}$ whose linear closure coincides with M. That is,*

$$M = \overline{M}(\phi_1, \dots, \phi_n, \dots)$$

so that M is closed.

Proof. Let M be a subspace of H. Now, let $M' = H \ominus M = \{g \in H : (g, f) = 0 \; \forall \; f \in M\}$ be the so called **Orthogonal Complement** of M. We claim that M' is a subspace of H. M' is clearly linear, since $(g_1, f) = (g_2, f) = 0$ which implies that $(\alpha_1 g_1 + \alpha_2 g_2, f) = 0$. Now, for closure, suppose that $g_n \in M'$ and that $g_n \to g$ Then

$$(g, f) = \lim_{n\to\infty} (g_n, f) = 0$$

for any $f \in M$ and thus $g \in M'$ and so M' is closed. $\square$

THEOREM 99. *If M is a subspace of H, then every $f \in H$ is uniquely representable in the form $f = h + h'$ where $h \in M$ and $h' \in M'$.*

Proof. To prove existence, choose in M a complete orthonormal set $\{\phi_n\}$ such that $M = \overline{M}$ and set

$$h = \sum_{n=1}^{\infty} c_n \phi_n$$

where $c_n = (f, \phi_n)$. Now, since the square sum of c_n converges, h exists and is an element of M. Now, set

$$h' = f - h$$

then it follows that $(h', \phi_n) = 0$ for any n. Now, since an arbitrary element ζ of M can be written as

$$\zeta = \sum_n a_n \phi_n$$

it follows that

$$(h', \zeta) = \sum_n a_n (h', \phi_n) = 0$$

which proves existence.

Now, suppose that there exists another decomposition $f = h_1 + h_1'$. Then we have

$$(h_1, \phi_n) = (f, \phi_n) = c_n$$

so that $h_1 = h$ and $h_1' = h'$ which yields uniqueness. $\square$

COROLLARY 100. *The orthogonal complement of the orthogonal complement of a subspace M coincides with M.*

COROLLARY 101. *Every orthonormal set Φ can be extended to a complete set in H.*

COROLLARY 102. *If an orthogonal set Φ is finite, then the number of its terms is its dimension and its **Deficiency** is the number of terms in M'. The orthogonal complement of a subspace of finite dimension n has deficiency n and conversely.*

DEFINITION 44. *If every vector $f \in H$ is represented in the form $f = h + h'$ for $h' \in M$, then we say that H is the **Direct Sum** of the orthogonal subspaces M and M' and we write*

$$H = M \oplus M'$$

We can also extend this by saying that H is the direct sum of a countable number of subspaces, then H is the direct sum of subspaces $M_1, \ldots, M_n, \ldots$ and

$$H = M_1 \oplus \cdots \oplus M_n \oplus \cdots$$

if

(i) The subspaces M_i are pairwise orthogonal.

(ii) Every $f \in H$ can be written in the form

$$f = h_1 + \cdots + h_n + \cdots$$

where

$$\sum_n \|h_n\|^2 < \infty$$

must hold.

We also note that if H_1, H_2 are Hilbert spaces, then $H = H_1 \oplus H_2$ is the collection of all possible pairs (h_1, h_2) where $h_1 \in H_1, h_2 \in H_2$ and the inner product of such pairs is

$$((h_1, h_2), (h_1', h_2')) = (h_1', h_1) + (h_2', h_2)$$

This can be extended to any countable sum of Hilbert spaces written as

$$\bigoplus_{k=1}^{\infty} H_k$$

which is defined as all possible sequences $h = (h_1, \ldots, h_n, \ldots)$ such that

$$\sum_{n=1}^{\infty} \|h_n\|^2 < \infty$$

The inner products must satisfy

$$(g, h) = \sum_{n=1}^{\infty} (g_n, h_n)$$

where $g, h \in H$.

Section 9.3

Linear & Bilinear Functionals in Hilbert Spaces

Proposition 103. *The sequence x_n converges to x weakly in X, a Banach space, if*

(i) $\|x_n\| \leq M$ for any n which really isn't necessary.

(ii) For any $F \in X^*$, we have $F(x_n) \to F(x)$.

Proof. It is enough to check (ii) on a dense set in X^*. Thus, suppose that Δ is a set whose linear hull is X^*. Now suppose that (ii) holds for any $F \in \Delta$ and we claim that (ii) holds for any $F \in X^*$.

To this end, if (ii) holds for any $F \in \Delta$, then it also holds for their finite linear combinations.
Now, by assumption, these linear combinations are dense in X^*. Let $F \in X^*$ and let $F_k \to F$ in X^*. F_k is a finite linear combination of functions from Δ. Next, let $x_n \to x$ weakly where $\|x_n\| \leq M$ then $\|x\| \leq M$. Now, there exists k_0 such taht for all $k \geq k_0$ such that

$$\|F_k - F\|_{X^*} < \epsilon$$

and

$$\begin{aligned} |F(x) - F(x_n)| &\leq |F(x_n) - F_k(x_n)| + |F_k(x_n) - F_k(x)| + |F_k(x) - F(x)| \\ &\leq |F_k(x_n) - F_k(x)| + 2\epsilon M \end{aligned}$$

and letting $k \to \infty$ yields

$$|F_k(x_n) - F_k(x)| \to 0$$

so that

$$|F(x) - F(x_n)| < 2\epsilon M$$

as needed. □

PROPOSITION 104. *If $x_n \to x$ strongly, that is, if $\|x_n - x\|_X \to 0$, then $x_n \to x$ weakly. We note that the converse is not true in general. An example will follow.*

Proof. For the proof, see Kolmogorov & Fomin or Rudin's Real & Complex Analysis (his proof is nicer). □

EXAMPLE 8. *Take $e_j = (0, \ldots, 0, 1, 0, \ldots) \in l_2$. We claim that $e_j \to 0$ weakly.*

Proof. For every $F \in l_2^* \cong l_2$, there exists $a \in l_2$ such that $F(x) = (x, a)$. Now, $(e_j, a) = a_j$ where $a = (a_1, \ldots, a_j, \ldots)$ and

$$\sum_j a_j^2 < \infty$$

which implies that $a_j \to 0$ as $j \to \infty$. □

Remark. In finite-dimensional spaces, strong convergence is equivalent to weak convergence.

On l_2, weak convergence is equivalent to convergence of coordinates.

EXAMPLE 9. *Let $X^* = (l_2)^* = l_2$ and let $\Delta = (e_1^*, \ldots, e_j^*, \ldots)$.*

EXAMPLE 10. *In $L_2([0, 2\pi])$, weak convergence is equivalent to convergence of sequences of Fourier coefficients.*

Example 11. *On spaces of continuous functions like $C([a,b])$, we can show that it suffices to take*

$$\Delta = \{\delta_{x_0} : x_0 \in [a,b]\}$$

so let

$$\delta_{x_0}(f) = f(x_0)$$

Here, weak convergence is equivalent to pointwise convergence. Thus, $f_n \to f$ weakly if and only if

$$\delta_{x_0}(f_n) \to \delta_{x_0}(f)$$

for every $x_0 \in [a,b]$ which is the same as

$$f_n(x_0) \to f(x_0)$$

for every $x_0 \in [a,b]$.

Definition 45. *We say that a sequence of linear functionals in X^* converges weakly to F if and only if*

(i) $\|F_n\| \le M < \infty$ *for any n.*

(ii) $F_n(x) \to F(x)$ *for all $x \in X$.*

Example 12 (δ Functions). *Let $\phi_n(t) = 0$ for $|t| \ge \frac{1}{n}$ and $\phi_n(t) \ge 0$ otherwise and suppose that ϕ_n must be continuous in t on $[-1,1]$ and that*

$$\int_{-1}^{1} \phi_n(t)dt = 1$$

for any n. Now, let $h(t) \in C([-1,1])$ and so

$$\int_{-1}^{1} h(t)\phi_n(t)dt = h(\xi_n)\int_{-1}^{1} \phi_n(t)dt$$

where ξ_n is contained in the compact support of ϕ_n which is simply $\left[\frac{-1}{n}, \frac{1}{n}\right]$. Now, let $n \to 0$ so that $\xi_n \to 0$ and thus

$$\int h(t)\phi_n(t)dt \to h(0) = \delta_0(h)$$

Theorem 105. *Suppose that X is a separable normed linear space. Now let F_n be a bounded sequence in X^*. Then there exists a subsequence F_{n_k} of F_n such that $F_{n_k} \to G \in X^*$.*

We recall that every linear functional on H has the form $F_g(x) = (x,y)$ for every fixed $y \in H$. A corollary to this fact is that a sequence $F_{y_n} \to F_z$ if and only if $y_n \to z$ in H. Now, if $\phi_1, \dots, \phi_n, \dots$ is an orthonormal sequence in H, then F_{ϕ_n} converges weakly to 0 as $n \to \infty$. For all $h \in H$, we have

$$\sum_{k=1}^{\infty} (f, \phi_k)^2 \le \|h\|_H^2$$

so that $(h, \phi_k) = F_{\phi_k}(h) \to 0$.

Corollary 106. *Let $f \in L_2([-\pi,\pi])$, then $(\sin(kx), \cos(kx))$ forms an orthonormal sequence in $L_2([-\pi,\pi], cdx)$. Then*

$$\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \int_{-\pi}^{\pi} f(x)\cos(kx)dx \to 0$$

as $k \to \infty$.

Chapter 10

Introductory Harmonic Analysis

Section 10.1

Fourier Series

Theorem 107 (Stone-Weierstrass). *The set $\{e^{inx}\}$ is dense in $C(\mathbb{T})$.*

Definition 46. *The Fourier series of a function f is*

$$S[f] \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{int}$$

Proposition 108. *The following properties of the Fourier series hold.*

(i)

$$\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$$

(ii)

$$\widehat{(cf)}(n) = c\hat{f}(n)$$

(iii) If f is complex valued, then

$$(\overline{f})(n) = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(t)}e^{-int}dt$$

Corollary 109. *Suppose that $f \in L_1(\mathbb{T})$ and that $\|f_j - f_0\| \to 0$ as $j \to \infty$. Then $\|\hat{f}_j(n) - \hat{f}_0(n)\| \to 0$ uniformly in n.*

Section 10.2

Convolutions

Suppose that $f, g \in L_1(\mathbb{T})$, then for almost every τ, the function $f(t-\tau)$ is integrable as a function of τ. Now, if we let

$$h(t) = \frac{1}{2\pi}\int f(t-\tau)g(\tau)d\tau$$

then h is also integrable and we say that $h = f * g$ is the **Convolution** of f and g.

Proposition 110. *The following properties of convolutions hold.*

(i)

$$\|h\|_1 \leq \|f\| \cdot \|g\|_1$$

(ii)

$$\hat{h}(n) = \hat{f}(n) \cdot \hat{g}(n)$$

(iii) $f * g = g * f$.

(iv) $f * (g * h) = (f * g) * h$.

(v) $f * (g + h) = f * g + f * h$.

LEMMA 111. *Let $\phi_n(t) = e^{int}$. Then,*

$$(\phi_n * f) = \hat{f}(n)e^{int}$$

SECTION 10.3

SUMMABILITY KERNELS

Let $\{k_n\}$ be a sequence of functions such that

(i) $k_n \in C(\mathbb{T})$ so that $k_n(t \pm 2\pi) = k_n(t)$.

(ii)

$$\frac{1}{2\pi} \int_0^{2\pi} k_n(t) dt = 1$$

(iii)

$$\frac{1}{2\pi} \int_0^{2\pi} |k_n(t)| dt \leq C$$

where C is independent of n.

(iv)

$$\lim_{n \to \infty} \int_\delta^{2\pi - \delta} |k_n(t)| dt = 0$$

PROPOSITION 112. *Let $\phi \in C(\mathbb{T})$. Then*

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} k_n(\tau)\phi(\tau) d\tau = \phi(c)$$

THEOREM 113. *Let $f \in L_1(\mathbb{T})$ and define $k_n(t)$ as usual and let $t \in [0, 2\pi]$ which is our summability kernel. Then*

$$\|f - f * k_n\|_1 \to 0$$

as $n \to \infty$. Thus,

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} k_n(\tau) f(t - \tau) d\tau = f$$

in $L_1(\mathbb{T})$.

Lemma 114. *Let $k \in C(\mathbb{T})$ and let $f \in L_1$. Then*

$$(k * f)^{(t)} = \frac{1}{2\pi} \int k(\tau) f_\tau(t) d\tau$$

where $f_\tau(t) = f(t - \tau)$.

Definition 47. *We say that*

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}$$

*is the n^{th} **Dirichlet Kernel**.*

Lemma 115. *Let*

$$P(t) = \sum_{k=-M}^{M} a_k e^{ikt}$$

Then, we have

$$(f * P)(t) = \sum_{k=-M}^{M} \hat{f}(k) a_k e^{ikt}$$

Section 10.4

Fejer Kernels

Definition 48. *We say that*

$$\sigma_n(f;t) = \frac{1}{n+1} \sum_{k=1}^{n} S_k(f;t) = (f * k_n)(t)$$

*is the n^{th} **Fejer Sum** and that $k_n(t)$ is the n^{th} **Fejer Kernel** where k_n is defined as*

$$k_n(t) = \frac{1}{n+1} \sum_{k=1}^{n} D_n(t)$$

It is also true that

$$k_n(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt}$$

Proposition 116. *k_n is a summability kernel.*

Lemma 117. *It is true that*

$$k_n(t) = \frac{1}{n+1} \left(\frac{\sin\left(\frac{n+1}{2}\right)}{\sin\left(\frac{1}{2}t\right)} \right)^2$$

Proposition 118. *Let $f \to f_\tau$ be a continuous function of τ on $L_p(\mathbb{T})$ where $1 \le p < \infty$. Then*

$$\lim_{\tau \to 0} \|f - f_\tau\|_{L_1} = 0$$

Proof. There exists $g \in C(\Pi)$ such that $\|f - g\|_1 < \frac{\epsilon}{3}$. Now, let

$$f_\tau(t) = f(t - \tau)$$

and we have

$$\|f - f_\tau\|_1 \le \|f - g\|_1 + \|g - g_\tau\| + \|f_\tau - g_\tau\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

as needed. □

Theorem 119. *Let $f \in L_1$ be defined as*

$$f = \lim_{n \to \infty} \frac{1}{2\pi} \int k_n(\tau) f_\tau d\tau$$

then

$$\lim_{n \to \infty} \left\| \frac{1}{2\pi} \int k_n(\tau) f_\tau d\tau - f \right\|_1 = 0$$

Let

$$K_n = \frac{1}{n+1} \frac{\sin^2\left(\frac{n+1}{2} \cdot t\right)}{\sin^2\left(\frac{t}{2}\right)}$$

and we see that $K_n \ge 0$. Also,

$$\frac{1}{2\pi} \int_0^{2\pi} |K_n(t)| dt = 1$$

and

$$\lim_{n \to \infty} \int_\delta^{2\pi - \delta} |K_n(t)| dt = 0$$

Now,

$$\int_\delta^{2\pi - \delta} K_n(t) dt \le \int_\delta^{2\pi - \delta} \frac{1}{(n+1)\sin^2\left(\frac{\delta}{2}\right)} \to 0$$

as $n \to \infty$.

Corollary 120 (Uniqueness). *If $f \in L_1(\mathbb{T})$ adn $\hat{f}(n) = 0$ for all n, then $f \equiv 0$.*

Proof. If $\hat{f}(n) = 0$, for any n, then

$$\sigma_n(f; t) = f * K_n = \frac{1}{n+1} \sum_{k=-n}^{n} \hat{f}(f) \left(1 - \frac{|k|}{n+1}\right) e^{ikt} \equiv 0$$

and by a previous result, we have $f * K_n \to f$ in L_1 so that $f \equiv 0$. □

Theorem 121 (Riemann-Lebesgue). *Let $f \in L_1(\mathbb{T})$. Then*

$$\lim_{n \to \infty} \hat{f}(n) = 0$$

Proof. Trigonometric polynomials are dense in $L_1(\mathbb{T})$. We have

$$P = \sum_{k=-N}^{N} a_k e^{ikt} \Longrightarrow \hat{P}(n) = 0$$

for $|n| > N$. Now, for any $\epsilon > 0$, and for all $f \in L_1$, w choose P such that

$$\|f - P\|_1 < \epsilon$$

and we have

$$|\hat{f}(n)| \leq |\hat{P}(n)| + |\hat{f}(n) - \hat{P}(n)| < \epsilon$$

as needed. □

THEOREM 122 (Fejer). *Let $f \in L_1(\mathbb{T})$ and assume that the limit*

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h)}{2}$$

exists and is equal to $f(x)$ if f is continuous at x. Then

$$\sigma_n(f;x) = f * K_n \to \tilde{f}(x)$$

as $n \to \infty$. Also, if f is continuous at x_0, then $\sigma_n(f;x_0) \to f(x_0)$. Finally, if $m \leq f \leq M$, then $m \leq \sigma_n(f;x) \leq M$.

COROLLARY 123. *If $S_n(f) = (f * D)$ converges at x_0, and if f is continuous at x_0, then $S_n(f;x_0) \to f(x_0)$.*

Proof. If $S_n(f;x_0) \to A$, then $\sigma_n(f;x_0) \to A$. Thus, averaging improves convergence. But then, $A = f(x_0)$. □

COROLLARY 124 (Lebesgue). *Let $f \in L_1$, then $\sigma_n(f;x) \to f(x)$ for a.a. $x \in [0, 2\pi]$.*

Proof. This follows from the fact that if $f \in L_1(\mathbb{T})$, then

$$\lim_{h \to 0} \frac{1}{h} \int_{x_0 - h}^{x_0 + h} \left| \frac{f(x_0+h) + f(x_0-h)}{2} - f(x_0) \right| dh = 0$$

for a.a. $x_0 \in [0, 2\pi]$. □

COROLLARY 125. *If $S_n(f;x)$ converges for $x \in E$ with $\mu(E) > 0$, then $S_n(f;x) \to f(x)$ for a.a. $x \in E$. In particular, if $S_n(f;x) \to 0$ a.e. on E, then $\hat{f}(n) = 0$ for all n so that $f \equiv 0$.*

THEOREM 126. *Suppose that $f \in C^k([0, 2\pi])$ then*

$$|\hat{f}(n)| < \frac{C}{|n|^k}$$

Proof. We know that

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$$

where we let $u = f(x)$ and $dv = e^{-inx}$ in the integration by parts formula. Thus,

$$\int_a^b udv = [uv]_a^b - \int_a^b vdu$$

and so

$$dv = e^{-inx}dx \qquad e^{-inx}dx = d\left(\frac{e^{-inx}}{-in}\right)$$

which gives

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\left(\left[f(x)\frac{e^{-inx}}{-in}\right]_0^{2\pi} - \int_0^{2\pi}\frac{e^{-inx}}{-in}(f'(x))dx\right) \\ &- \frac{1}{2\pi in}\int_0^{2\pi} f'(x)e^{-inx}dx\end{aligned}$$

and thus

$$in\hat{f}(n) = \hat{f}'(n)$$

where $f' \in C([0,2\pi]) \subseteq L_1([0,2\pi])$ so that

$$|\hat{f}(n)| \to 0$$

as $|n| \to \infty$. Now, there exists C such that $|\hat{f}'(n)| < C$. Now, applying the argument k times, we get that for all $1 \leq j \leq k$, we have

$$|n|^j|\hat{f}(n)| = |\hat{f}^{(j)}(n)| \leq \|f^{(j)}\|_1$$

Now, we have

$$|\hat{f}(n) \leq \min_{1\leq j\leq k}\frac{\|f^{(j)}\|_1}{|n|^k}$$

If $j = k$, then we have

$$|\hat{f}(n) < \frac{C}{|n|^k}$$

for all $n \neq 0$. □

Remark. If

$$|\hat{f}(n)| < \frac{C}{|n|^k}$$

then it doesn't necessarly follow that $f \in C^k$.

THEOREM 127. *If $f \in L_2([0,2\pi])$, then $\hat{f} \in L_2$ and*

$$\|f\|_{L_2}^2 = \|\hat{f}\|_{L_2}^2$$

THEOREM 128. *Let $1 < p \leq 2$ and suppose that $f \in L_p([0,2\pi])$. Then let*

$$q = \frac{p}{p-1} \Longrightarrow \frac{1}{p} + \frac{1}{q} = 1$$

Then

$$\sum_n ||\hat{f}(n)|^q < \infty$$

That is, $\hat{f} \in l_q$.

Theorem 129. *Let* $a_n \geq 0$ *and suppose that* $a_n \to 0$ *and also that*

$$a_{n-1} + a_{n+1} - 2a_n \geq 0$$

so that

$$\frac{a_{n-1} + a_{n+1}}{2} \geq a_n$$

which implies convexity. Then there exists $f \in L_1(\mathbb{T})$ *with* $f \geq 0$ *such that* $\hat{f}(n) = a_n$.

Proposition 130. *Let* $a_n > 0$ *and suppose that*

$$\sum_n \frac{a_n}{n} = 0$$

Then

$$\sum_{n=1}^{\infty} a_n \sin(nt)$$

is NOT a Fourier series.

Chapter 11

Things To Know For The Final (Tentative)

- Measurable Functions
- Monotone Convergence
- Dominated Convergence
- Fatou's Lemma
- Simple Functions
- Measurable Sets
- Lusin's Theorem
- Chebychev Inequality
- $L_p, p \geq 1$
- Lebesgue Integration
- Orthogonal Compliment
- Subspaces Of L_2
- Basis
- Parseval
- Bessel
- Fourier Coefficients
- Rieman-Lebesgue
- Convolutions
- $$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2} \qquad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}$$